CERTAIN CLASSES OF FINITE SUMS THAT INVOLVE GENERALIZED FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

The beautiful identity

$$\sum_{n=1}^{j} F_n^2 = F_j F_{j+1}$$  \hspace{1cm} (1.1)

was the inspiration for [2], in which analogous sums involving cubes of Fibonacci numbers were developed. In turn, [2] was the motivation for [5], [6], and [7]. In the present paper, where we restrict ourselves to summands that consist of products of at most two terms (as in (1.1)), our motivation has again been to find sums where the right side has a pleasing form. We have found it profitable to consider non-alternating sums, alternating sums, and sums that alternate according to $(-1)^{n(n+1)}$. Fibonacci-related sums of the latter type are almost non-existent in the literature.

During the discovery process we became aware of numerous connections that exist between various groups of sums. So, rather than merely to present a collection of sums, a priority of ours has been to highlight the strong thread of unity that exists. Another priority has been to achieve a balance between elegance and generality, and, to achieve this, experimentation has led us to employ the four sequences that we now define.

We define the sequence $\{W_n\}$, for all integers $n$, by

$$W_n = pW_{n-1} + W_{n-2}, \quad W_0 = a, \quad W_1 = b,$$  \hspace{1cm} (1.2)

where $a$, $b$, and $p$ are assumed to be arbitrary complex numbers with $p(p^2 + 2)(p^2 + 4) \neq 0$. Then, since $\Delta = p^2 + 4 \neq 0$, the roots $\alpha$ and $\beta$ of $x^2 - px - 1 = 0$ are distinct. Hence the Binet form (see [3]) for $W_n$ is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$ \hspace{1cm} (1.3)

where $A = b - a\beta$ and $B = b - a\alpha$. The Binet form gives $W_n$ for all integers $n$.

We define another sequence $\{X_n\}$ by $X_n = W_{n+1} + W_{n-1}$, and, with the use of (1.3), we find that

$$X_n = A\alpha^n + B\beta^n$$  \hspace{1cm} (1.4)

for $(a,b) = (0,1)$ write $W_n = U_n$ and $X_n = V_n$. Thus $W_n$ and $X_n$ generalize $U_n$ and $V_n$, respectively, which in turn generalize $F_n$ and $L_n$, respectively. Aspects of $\{W_n\}$ and $\{X_n\}$ have been treated, for example, in [1], [4], and [11], and more recently in [8].

In Sections 2, 3, and 4 we present our results, and then conclude by giving a sample proof. Since we have decided upon a limited focus, we do not claim that our results are exhaustive. Indeed, we expect that there is scope for further research along the lines that we set forth.
CERTAIN CLASSES OF FINITE SUMS THAT INVOLVE GENERALIZED FIBONACCI AND LUCAS NUMBERS

In each of our sums the lower limit is allowed to vary. Accordingly, in keeping with convention, we always assume the upper limit to be greater than the lower limit, and that either limit may be negative.

2. THE FIRST SET OF SUMS

In this section, and the sections that follow, we systematically consider non-alternating sums, alternating sums, and sums that alternate according to \((-1)^{\frac{n(n+1)}{2}}\). We begin with the following three results.

\[
\sum_{n=i}^{j} W_n = \begin{cases} 
\frac{1}{p} V_{(j-i+1)/2}(W_{(j+i+1)/2} + W_{(j+i-1)/2}) & \text{if } j-i \equiv 1(\text{mod } 4) \\
\frac{1}{p} U_{(j-i+1)/2}(X_{(j+i+1)/2} + X_{(j+i-1)/2}) & \text{if } j-i \equiv 3(\text{mod } 4),
\end{cases}
\]

(2.1)

\[
\sum_{n=i}^{j} (-1)^n W_n = \begin{cases} 
(-1)^{\frac{j-i}{2}} V_{(j-i+1)/2}(W_{(j+i+1)/2} - W_{(j+i-1)/2}) & \text{if } j-i \equiv 1(\text{mod } 4) \\
(-1)^{\frac{j-i}{2}} U_{(j-i+1)/2}(X_{(j+i+1)/2} - X_{(j+i-1)/2}) & \text{if } j-i \equiv 3(\text{mod } 4),
\end{cases}
\]

(2.2)

and

\[
\sum_{n=i}^{j} (-1)^{\frac{n(n+1)}{2}} W_n = \begin{cases} 
(-1)^{\frac{j-i}{2}} W_{(j-i+1)/2}(U_{(j-i+2)/2} + (-1)^j U_{(j-i)/2}) & \text{if } j-i \equiv 0(\text{mod } 2) \\
(-1)^{\frac{j-i}{2}} W_{(j-i+1)/2}(W_{(j+i+1)/2} + (-1)^j W_{(j+i-1)/2}) & \text{if } j-i \equiv 1(\text{mod } 2).
\end{cases}
\]

(2.3)

Now suppose that, in the summands of (2.1) and (2.2), \(W\) is replaced by \(X\). Then if \(j-i \equiv 3 \pmod{4}\) we modify the right sides by multiplying with \(\Delta\) and replacing each occurrence of \(X\) by \(W\). It is now clear what (2.1)-(2.3) become when any of \(U_n, V_n,\) or \(X_n\) is substituted for \(W_n\) in the summand.

3. THE SECOND SET OF SUMS

In this section we consider sums similar to (2.1)-(2.3) in which \(W_n\) is replaced by \(W_{2n}\).

\[
\sum_{n=i}^{j} W_{2n} = \begin{cases} 
\frac{1}{p} V_{j-i+1} W_{j+i} & \text{if } j-i \equiv 0(\text{mod } 2) \\
\frac{1}{p} U_{j-i+1} X_{j+i} & \text{if } j-i \equiv 1(\text{mod } 2),
\end{cases}
\]

(3.1)

and

\[
\sum_{n=i}^{j} X_{2n} = \frac{\Delta}{p} U_{j-i+1} W_{j+i} & \text{if } j-i \equiv 1 \pmod{2}.
\]

(3.2)

In the alternating sum that follows the parity of the limits is not an issue.

\[
\sum_{n=i}^{j} (-1)^n W_{2n} = (-1)^j U_{j-i+1} W_{j+i}.
\]

(3.3)
For the following group of sums the upper and lower limits are specified as belonging to certain residue classes modulo 4.

\[ \sum_{n=4i+1}^{4j} (-1)^{n(n+1)/2} W_{2n} = \frac{\Delta}{\Delta - 2} U_{4j-4i} W_{4j+4i+1}, \quad (3.4) \]

\[ \sum_{n=4i+3}^{4j} (-1)^{n(n+1)/2} W_{2n} = \frac{1}{\Delta - 2} V_{4j-4i-2} X_{4j+4i+3}, \quad (3.5) \]

\[ \sum_{n=4i}^{4j+3} (-1)^{n(n+1)/2} W_{2n} = \frac{p}{\Delta - 2} U_{4j-4i+4} X_{4j+4i+3}, \quad (3.6) \]

\[ \sum_{n=4i+2}^{4j+3} (-1)^{n(n+1)/2} W_{2n} = \frac{p}{\Delta - 2} V_{4j-4i+2} W_{4j+4i+5}. \quad (3.7) \]

We now make an observation concerning the subscripts on the right sides of (3.1)-(3.7). The subscripts of \( U \) and \( V \) are (upper limit-lower limit+1), while the subscripts of \( W \) and \( X \) are (upper limit + lower limit). The reader can observe that this also applies to most of the sums in this paper.

Notice that, in (3.4)-(3.7), one limit of summation is even while the other is odd. Accordingly, we have observed that each of (3.4)-(3.7) has a dual sum that is obtained with the use of the rule below. We highlight this rule since it also applies to certain groups of sums in Section 4.

**Rule for the Formation of the Dual Sum**

- Replace the even limit by the even limit corresponding to the other residue class modulo 4 and the odd limit by the odd limit corresponding to the other residue class modulo 4.
- Calculate the subscripts on the right in accordance with the paragraph following (3.7).
- Multiply the right side by \(-1\).

For example, the dual of (3.7) is

\[ \sum_{n=4i+1}^{4j+1} (-1)^{n(n+1)/2} W_{2n} = \frac{-p}{\Delta - 2} V_{4j-4i+2} W_{4j+4i+1}. \quad (3.8) \]

We also remark that if, in (3.5) and (3.6), \( W \) in the summand is replaced by \( X \), then on the right we replace \( X \) by \( W \) and multiply by \( \Delta \). This also applies to the duals of (3.5) and (3.6). It is now clear what the summation identities of this section become when \( W \) in the summand is replaced by either \( U \), \( V \), or \( X \).
CERTAIN CLASSES OF FINITE SUMS THAT INVOLVE GENERALIZED FIBONACCI AND LUCAS NUMBERS

Finally if, in each sum of this section, \( W_{2n} \) is replaced by \( W_{2n+k} \), the subscript of \( W \) or \( X \) on the right side is simply increased by \( k \). The same applies to each sum where \( X_{2n} \) occurs in the summand.

4. THE THIRD SET OF SUMS

In this section we consider sums where the summand contains a second order product. Since, for example, \( U_n V_n = U_{2n} \), sums that involve the product \( U_n V_n \) follow from Section 3. However, sums that involve \( U_n^2 \) or \( V_n^2 \) do not follow from anything we have done so far. To remedy this, and to achieve more generality, we next consider sums in which the summand contains \( U_n W_n, U_n X_n, V_n W_n, \) or \( V_n X_n \).

In the three sums that follow \( i \) and \( j \) are assumed to have different parities. Under this assumption we have

\[
\sum_{n=i}^{j} U_n W_n = \frac{1}{p} U_{j-i+1} W_{j+i}, \tag{4.1}
\]

\[
\sum_{n=i}^{j} V_n W_n = \frac{1}{p} U_{j-i+1} X_{j+i}, \tag{4.2}
\]

and

\[
\sum_{n=i}^{j} V_n X_n = \Delta \frac{1}{p} U_{j-i+1} W_{j+i}, \tag{4.3}
\]

It is clear that the corresponding sum involving the summand \( U_n X_n \) can be obtained from (4.1).

Next we consider the associated alternating sums, where, unfortunately, the right sides do not factorise nicely. Nevertheless, for the sake of completeness, we have managed to salvage something in the spirit of our previous results. Without any assumptions on the parities of the limits we have

\[
\sum_{n=i}^{j} (-1)^n U_n W_n = \frac{1}{\Delta} \left( (-1)^j U_{j-i+1} X_{j+i} - (j - i + 1) X_0 \right), \tag{4.4}
\]

\[
\sum_{n=i}^{j} (-1)^n U_n X_n = (-1)^j U_{j-i+1} W_{j+i} - (j - i + 1) W_0. \tag{4.5}
\]

In (4.4), if \( U_n W_n \) is replaced with \( V_n X_n \), the right side is multiplied by \( \Delta \) and the sign of the coefficient of \( X_0 \) is changed. In (4.5), if \( U_n X_n \) is replaced by \( V_n W_n \), then only the sign of the coefficient of \( W_0 \) is changed on the right side.

The next group of sums has \( (-1)^{\frac{n(n+1)}{2}} \) in the summand.

\[
\sum_{n=4i+1}^{4j} (-1)^{\frac{n(n+1)}{2}} U_n X_n = \Delta \frac{1}{\Delta - 2} U_{4j-4i} W_{4j+4i+1}, \tag{4.6}
\]
CERTAIN CLASSES OF FINITE SUMS THAT INVOLVE GENERALIZED FIBONACCI AND LUCAS NUMBERS

The right sides of these four sums should be compared, respectively, with the right sides of (3.4)-(3.7). The only difference occurs in (4.9) where the term $2W_0$ is added. Furthermore, each of (4.6)-(4.9) has a dual sum that is obtained with the use of the rule in Section 3. And from this total of eight sums we obtain a further eight sums if, in each summand, we replace $U_nX_n$ by $V_nW_n$. In each case the right side remains unchanged, except for (4.9) and its dual, where the coefficient of $W_0$ undergoes a change in sign.

To conclude the list of sums in this paper we list four more, and, as in the previous paragraph, we describe how to obtain an additional twelve. We have found the following.

$$\sum_{n=4i+3}^{4j} (-1)^{\frac{n(n+1)}{2}} U_n X_n = \frac{1}{\Delta - 2} V_{4j-4i-2} X_{4j+4i+3}, \quad (4.7)$$

$$\sum_{n=4i}^{4j+3} (-1)^{\frac{n(n+1)}{2}} U_n X_n = \frac{p}{\Delta - 2} U_{4j-4i+4} X_{4j+4i+3}, \quad (4.8)$$

$$\sum_{n=4i+2}^{4j+3} (-1)^{\frac{n(n+1)}{2}} U_n X_n = \frac{p}{\Delta - 2} V_{4j-4i+2} W_{4j+4i+5} + 2W_0. \quad (4.9)$$

Each of (4.10)-(4.13) has a dual sum that is obtained with the use of the rule in Section 3. And from this total of eight sums we obtain a further eight sums if, in each summand, we replace $U_nW_n$ by $V_nX_n$ and multiply the right side by $\Delta$.

Multiplication by $\Delta$ is the only change we are required to make to the right side, except for (4.13) and its dual, where, in addition, we must change the sign of the coefficient of $X_0$. For example, the dual of (4.13) is

$$\sum_{n=4i}^{4j+1} (-1)^{\frac{n(n+1)}{2}} U_n W_n = -\frac{1}{\Delta} \left( \frac{p}{\Delta - 2} V_{4j-4i+2} X_{4j+4i+5} + 2X_0 \right). \quad (4.13)$$

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and an additional sum is

\[ \sum_{n=4i}^{4i+1} (-1)^{n(n+1)/2} V_n X_n = - \left( \frac{p}{\Delta - 2} V_{4j-4i+2} X_{4j+4i+1} - 2X_0 \right). \]  

(4.15)

By way of example, if we put \( W_n = F_n \) then (3.3), (3.5), and (4.12) become, respectively,

\[ \sum_{n=i}^{j} (-1)^n F_{2n} = (-1)^j F_{j-i+1} F_{j+1}, \]  

(4.16)

\[ \sum_{n=4i+3}^{4j} (-1)^{n(n+1)/2} F_{2n} = \frac{1}{3} L_{4j-4i-2} L_{4j+4i+3}, \]  

(4.17)

and

\[ \sum_{n=4i}^{4i+3} (-1)^{n(n+1)/2} F_n^2 = \frac{1}{3} F_{4j-4i+4} F_{4j+4i+3}. \]  

(4.18)

5. A SAMPLE PROOF

Each of our sums can be proved by induction, and we illustrate by proving (4.12). We require the following result, which is a special case of (79) in [1]:

\[ X_{n+k} - X_{n-k} = \Delta U_k W_n, \quad k \text{ even.} \]  

(5.1)

By our earlier assumption on the limits, the smallest allowable value of \( j \) is \( i \). Thus, we begin our inductive proof by verifying that

\[ \sum_{n=4i}^{4i+3} (-1)^{n(n+1)/2} U_n W_n = \frac{p}{p^2 + 2} U_i W_{8i+3}. \]  

(5.2)

We recall that \( \Delta = p^2 + 4 = (\alpha - \beta)^2 \), replace \( U_n \) and \( W_n \) with their Binet forms, and expand to obtain

\[ \text{LHS} = \frac{1}{\Delta} \left( (X_{8i+6} - X_{8i+4}) - (X_{8i+2} - X_{8i}) \right) \]

\[ = \frac{p}{\Delta} (X_{8i+5} - X_{8i+1}) \quad \text{(from the recurrence for } X_n) \]

\[ = \frac{p}{\Delta} (X_{8i+3+2} - X_{8i+3-2}) \]

\[ = p^2 W_{8i+3} \quad \text{(from (5.1))} \]
which is the right side of (5.2) since \( U_4 = p^3 + 2p \).

Next, if (4.12) is true for the parameter \( j \), then

\[ \sum_{n=4j}^{4j+3} (-1)^{n(n+1)/2} U_n W_n = \frac{p}{p^2 + 2} U_{4j-4i+4} W_{4j+4i+3} + \sum_{n=4j+4}^{4j+7} (-1)^{n(n+1)/2} U_n W_n, \quad (5.3) \]

and we are required to prove that the right side of (5.3) is equal to \( \frac{p}{p^2 + 2} U_{4j-4i+8} W_{4j+4i+7} \). Equivalently, we are required to prove that

\[ \sum_{n=4j+4}^{4j+7} (-1)^{n(n+1)/2} U_n W_n = \frac{p}{p^2 + 2} (U_{4j-4i+8} W_{4j+4i+7} - U_{4j-4i+4} W_{4j+4i+3}). \]

Proceeding as before, we find

\[
LHS = \frac{1}{\Delta} (X_{8j+14} - X_{8j+12} - (X_{8j+10} - X_{8j+8})) \\
= \frac{p}{\Delta} (X_{8j+13} - X_{8j+9}) \\
= p^2 W_{8j+11} \quad \text{(from (5.1))}. \text{ Similarly}
\]

\[
RHS = \frac{p}{(p^2 + 2)\Delta} (X_{8j+15} - X_{8j+7}) = \frac{p}{(p^2 + 2)\Delta} \times \frac{\Delta U_4 W_{8j+11}}{1} = LHS,
\]

and the proof is complete.

6. CONCLUDING COMMENTS

In [9] Russell considers the sums \( \sum_{i=0}^j R_n \) and \( \sum_{n=0}^j R_n S_n \) with no restrictions on the limits of summation. Here \( \{R_n\} \) and \( \{S_n\} \) are sequences generated by the recurrence \( W_n = pW_{n-1} + qW_{n-2} \) with \( p \) and \( q \) real. In each of these sums several cases are given depending on the values of \( p \) and \( q \). Indeed, there are three cases for the first sum and seven cases for the second. The character of Russell’s sums and ours is quite different, since our motivation has been to present sums in which the right side has a pleasing form. In a subsequent paper, Russell [10] gives finite sums in which each summand consists of products of up to three terms, where each term is generated by \( W_n = W_{n-1} + W_{n-2} \). Neither of Russell’s papers contains alternating sums or sums that alternate according to \( (-1)^n \). However, they are the only papers we have seen that contain finite sums where each summand consists of terms (or products of terms) generated by second-order linear recurrences, and where the lower limit of summation is allowed to vary.

We discovered each of our results numerically by first considering the Fibonacci and Lucas sequences. Initially all lower limits were taken to be one, and in each case we varied the upper limit until the sum could be expressed as a product of Fibonacci and/or Lucas numbers. We then decided to vary the upper and lower limits simultaneously and then found that our
CERTAIN CLASSES OF FINITE SUMS THAT INVOLVE GENERALIZED FIBONACCI AND LUCAS NUMBERS

results could be translated to the more general sequences defined herein. The process was quite tedious, and we gratefully acknowledge our use of the computer algebra package Mathematica 3.0.

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# TABLE OF CONTENTS

Referees ................................................................................................................................. 2

Featured Article ....................................................................................................................... 3

Edited by Andrew Granville

Fibonacci and Lucas Numbers Through the Action of the Modular Group
on Real Quadratic Fields .................................................. Nguyen-Huu-Bong and Qaiser Mushtaq 20

The Eleventh International Conference on Fibonacci Numbers and Their Applications ..... 27

Orderings of Products of Fibonacci Numbers .................................................. Clark Kimberling 28

Generating Solutions for a Special Class of Diophantine
Equations II ................................................................. Pasquale J. Arpaia 36

On Pascal’s Triangle Modulo 2 in Fibonacci Representation ................. Antti Karttunen 38

Certain Classes on Finite Sums that Involve Generalized Fibonacci
and Lucas Numbers ........................................................................................................... R.S. Melham 47

Extending the Bernoulli-Euler Method for Finding Zeros of Holomorphic
Functions ...................................................... Benaissa Bemoussi, Mustapha Rachidi and Osamu Saeki 55

A Note on Summation of Certain Reciprocal Series Involving the Generalized
Fibonacci and Lucas Functions .......................... Feng-Zhen Zhao and Tianming Wang 66

Self-Similarity and Symmetries of Pascal’s Triangles and Simplices
Mod p ................................................................................. Richard P. Kubelka 70

Intrinsic Palindromes ................................................................. Antonio J. Di Scala and Martin Sombra 76

Proceedings of the Tenth International Research Conference on Fibonacci
Numbers and Their Applications ............................................................... 81

“Fibonacci’s Liber Abaci”: A Translation into Modern English
of Leonardo Pisano’s Book of Calculation by L.E. Sigler .......................... A.F. Horadam 82

Elementary Problems and Solutions .............................. Edited by Russ Euler and Jawad Sadek 86

Advanced Problems and Solutions ............................................ Edited by Florian Luca 92