A Boundary Meshless Method for Transient Eddy Current Problems

Yong Zhang, K.R. Shao
Huazhong Univ. of Sci. & Tech., Wuhan, 430074, P.R. China
Youguang Guo
University of Technology, Sydney, NSW 2007, Australia
J.D. Lavers

University of Toronto, Toronto, Ont. M5S 3G4, Canada

Abstract -- This paper presents a boundary meshless method (BMLM) for transient eddy current problems. With difference to the traditional boundary element method, the BMLM combines a point interpolation method (PIM) for construction of spatial interpolation functions with a boundary integral formulation for the governing equations. Thus the spatial interpolation functions satisfy the Kronecker delta function and the essential boundary condition can be directly imposed without any other procedure. Theoretical analysis in details is given and a transient eddy current example is also illustrated in this paper to prove the proposed theory.

I. Introduction

Boundary element methods (BEMs) are attractive and important computational techniques for reducing the dimensionality of the solving problems. The construction of interpolation functions and the discretization are two key steps of the BEMs. Using a scattered set of points instead of using elements or panels, the complex mesh generation process can be alleviated. Several boundary-type meshless methods have been developed for many potential and elastic problems, they need no discretization of the boundary and are proven as robust numerical methods. But very few of them are used to solve electromagnetic problems, say nothing of transient eddy current problems. This may be due to the fact that more modification is needed in transient analysis when

compared with static analysis. In this paper, a boundary meshless method (BMLM) for transient analysis is presented. With difference to traditional boundary element method, the BMLM combines a point interpolation method for construction of spatial shape functions for the governing equations, thus the spatial shape functions satisfy the Kronecker delta function and the essential boundary condition can be directly imposed on the boundary. Then the theoretical analysis is given in details, and an example is also illustrated to prove the proposed theory.

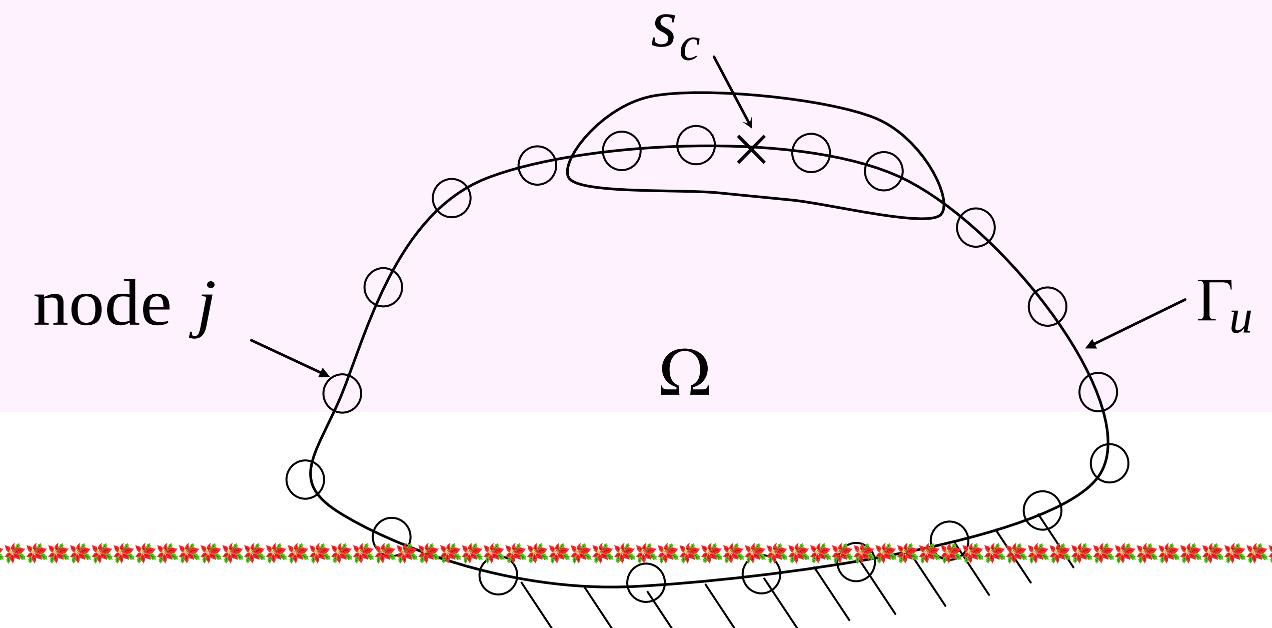
II. Point Interpolation on Curves

Consider a two-dimensional domain Ω with boundary Γ, as shown in Fig.1. In using boundary meshless method, only the boundary Γ of the problem domain is represented using nodes. The

point interpolation method (PIM) is constructed on the one-dimensional bounding curve Γ of two-dimensional domain Ω, using a set of discrete nodes on Γ. As in the conventional BEM method, u and q are constructed independently using PIM shape function as

$$u(s) = \sum_{j=1}^{m} p_j(s)\, a_j = \mathbf{p}^{\mathrm{T}}(s) \cdot \mathbf{a} \qquad \textbf{(1.a)}$$

$$q(s) = \sum_{j=1}^{m} p_j(s)\, b_j = \mathbf{p}^{\mathrm{T}}(s) \cdot \mathbf{b} \qquad \textbf{(1.b)}$$



where s is a curvilinear coordinate on Γ, m is the number of the nodes in the support domain of a point of interest s_c, which is often a quadrature point of integration, $p_j(s)$ is a basis function of a complete polynomial with $p_1 = 1$ and $p_j = s^{j-1}$, and a_j and b_j are the coefficients that change when s_c changes. In matrix form, there are

$$\mathbf{a} = [a_1, a_2, \cdots, a_m]^{\mathrm{T}} \qquad \textbf{(2.a)}$$

$$\mathbf{b} = [b_1, b_2, \cdots, b_m]^{\mathrm{T}} \tag{2.b}$$

$$\mathbf{p}(s) = [1, s, \cdots, s^{m-1}]^{\mathrm{T}} \tag{2.c}$$

The coefficient a_j and b_j in (2.a) and (2.b) can be determined by enforcing (1.a) and (1.b) to be satisfied at the m nodes surrounding the point s_c. Equation (1) can then be written in the following matrix form

$$\mathbf{u}_m = \mathbf{P}_c\, \mathbf{a} \tag{3.a}$$

$$\mathbf{q}_m = \mathbf{P}_c\, \mathbf{b} \tag{3.b}$$

In (3.a) and (3.b), we have

$$\mathbf{u}_m = [u_1, u_2, \cdots, u_m]^{\mathrm{T}} \tag{4.a}$$

$$\mathbf{q}_m = [q_1, q_2, \cdots, q_m]^{\mathrm{T}} \tag{4.b}$$

$$\mathbf{P}_c = [\mathbf{p}(s_1), \mathbf{p}(s_2), \cdots, \mathbf{p}(s_m)]^{\mathrm{T}} \tag{4.c}$$

Solving ${}_{a}$and ${}_{b}$from (3.a) and (3.b), then substituting them into (1.a) and (1.b), we can obtain

$$u(s) = \mathbf{\Phi}^{\mathrm{T}}(s)\,\mathbf{u}_m \tag{5.a}$$

$$q(s) = \mathbf{\Phi}^{\mathrm{T}}(s)\,\mathbf{q}_m \tag{5.b}$$

Where the matrix form of shape matrix $\mathbf{\Phi}(s)$ is defined by

$$\mathbf{\Phi}^{\mathrm{T}}(s) = \mathbf{p}^{\mathrm{T}}(s)\mathbf{P}_c^{-1} = [\phi_1(s), \phi_2(s), \cdots, \phi_m(s)] \tag{6}$$

Therefore equation (5) can be written as

$$u(s) = \sum_{j=1}^{m} \phi_j(s)\, u_j \tag{7.a}$$

$$q(s) = \sum_{j=1}^{m} \phi_j(s)\, q_j \tag{7.b}$$

The shape function $\phi(s)$ satisfies Kronecker delta function as

$$\phi_j(s_i) = \delta_{ji} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad , \quad \sum_{j=1}^{m} \phi_j(s) = 1 \tag{8}$$

Therefore, the shape functions constructed have the delta function property, and the essential boundary conditions can be easily imposed as in traditional BEM.

It should be pointed out that the accuracy of the interpolation depends on the number of nodes in the support domain of a quadrature point. Therefore, a suitable support domain should be chosen to ensure a proper area of coverage for interpolation. To

define the support domain for a point s_c, a curvilinear support domain is used. The arc-length of the curvilinear domain d_s is computed by

$$d_s = \alpha_s d_c \tag{9}$$

Where α_s is the dimensionless size of the support domain and d_c is a characteristic length related to the nodal spacing near the point at s_c. A suitable choice of α_s is that 3 to 5 points in the support domain for interpolation.

III. Transient Eddy Current Problem

The full set of equations for low-frequency electromagnetic field can be now written as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad , \quad \mathbf{B} = \mu \mathbf{H} \tag{10.a}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_e + \mathbf{J}_s \quad , \quad \mathbf{J}_e = \sigma \mathbf{E} \tag{10.b}$$

Where $\mathbf{J}_e$ and $\mathbf{J}_s$ are eddy current and source current respectively. For linear conductive medium, with Lorentz gauge, the equivalent form by using magnetic potential vector of equation (10) can be written as follows

$$\nabla^2 \mathbf{A} = \sigma \mu \frac{\partial \mathbf{A}}{\partial t} - \mu \mathbf{J}_s \tag{11}$$

Therefore, for two-dimensional problems, equation (11) can be reduced to a scalar diffusion equation as follows

$$\nabla^2 u = \sigma \mu \frac{\partial u}{\partial t} - P \tag{12}$$

Where u is one of the three components of vector potential $\mathbf{A}$, P is the supplied source. The initial and boundary problems for two-dimensional transient eddy current field is written as

$$\begin{cases} \nabla^2 u = \sigma\mu\dfrac{\partial u}{\partial t} - P & \text{in } \Omega \text{ at } 0 < t < t_n \\ u = \bar{u} & \text{on } \Gamma_u \text{ at } 0 < t < t_n \\ q = \bar{q} & \text{on } \Gamma_q \text{ at } 0 < t < t_n \\ u = u_0 & \text{in } \Omega \text{ at } t = 0 \end{cases} \quad (13)$$

Where Ω is the solver domain with boundary $\Gamma = \Gamma_u + \Gamma_q$, in which Γ_u and Γ_q are essential boundary and natural boundary respectively.

Ⅳ. Boundary Meshless Method

The well-known boundary integration equation for two-dimensional

linear medium problems is given by

$$c_i(u_i)_{t_n} + \frac{1}{\sigma\mu}\int_0^{t_n}\int_\Gamma q^* u \,\mathrm{d}\Gamma \mathrm{d}t = \frac{1}{\sigma\mu}\int_0^{t_n}\int_\Gamma u^* q \,\mathrm{d}\Gamma \mathrm{d}t$$

$$+ \frac{1}{\sigma\mu}\int_0^{t_n}\int_\Omega u^* P \,\mathrm{d}\Omega \mathrm{d}t + [\int_\Omega u^* u \mathrm{d}\Omega]_{t=0} \qquad \textbf{(14)}$$

Where

$$c_i = \begin{cases} 1 & \text{for point } i \text{ inside } \Omega \\ 0 & \text{for point } i \text{ outside } \Omega \\ \theta_i/(2\pi) & \text{for point } i \text{ on } \Gamma \end{cases} \qquad \textbf{(15)}$$

$$u^* = \frac{\sigma\mu}{4\pi(t_n - t)}\exp\left(-\frac{\sigma\mu r^2}{4\pi(t_n - t)}\right) \qquad \textbf{(16.a)}$$

$$q^* = -\frac{(\sigma\mu)^2}{8\pi(t_n - t)^2}\exp\left(-\frac{\sigma\mu r^2}{4\pi(t_n - t)}\right)\frac{\partial r}{\partial n} \qquad \textbf{(16.b)}$$

In (16) r is the distance between the field point $_i$ and the source point. To get the numerical solution of equation (14), time should be discrete into N time steps for computation. A time interpolative function is needed here, they are

$$u(t) = \sum_{k=1}^{l} M^k u^k = \mathbf{M}^{\mathrm{T}}\mathbf{u}^t \qquad \textbf{(17.a)}$$

$$q(t) = \sum_{k=1}^{l} M^k q^k = \mathbf{M}^{\mathrm{T}}\mathbf{q}^t \qquad \textbf{(17.b)}$$

Where u^k and q^k are the values of u and q at time $t = t_k$, $\mathbf{M}^{\mathrm{T}} = [M^1, M^2, \cdots, M^l]$ are the shape functions of u^k and q^k,

$\mathbf{u}^t = [u^1, u^2, \cdots, u^2]$ **and** $\mathbf{q}^t = [q^1, q^2, \cdots, q^2]$.

Substituting (7) and (17) into (14) yields the boundary meshless method for all nodes on the boundary of the problem domain

$$\mathbf{H}^k \mathbf{U}^k = \mathbf{G}^k \mathbf{Q}^k + \mathbf{F}^k \mathbf{P}^k + \mathbf{B}\overline{\mathbf{U}}^0 \qquad \textbf{(18)}$$

Where

$$\mathbf{H}^k = \mathbf{c}_i + \frac{1}{\sigma\mu}\int_{t_{k-1}}^{t_k}\int_\Gamma \mathbf{q}^* \mathbf{\Phi}^{\mathrm{T}}(s)\mathbf{M}^{\mathrm{T}}\, \mathrm{d}\Gamma \mathrm{d}t \qquad \textbf{(19.a)}$$

$$\mathbf{G}^k = \frac{1}{\sigma\mu}\int_{t_{k-1}}^{t_k}\int_\Gamma \mathbf{u}^* \mathbf{\Phi}^{\mathrm{T}}(s)\mathbf{M}^{\mathrm{T}}\, \mathrm{d}\Gamma \mathrm{d}t \qquad \textbf{(19.b)}$$

$$\mathbf{F}^k = \frac{1}{\sigma\mu}\int_{t_{k-1}}^{t_k}\int_\Omega \mathbf{u}^* \mathbf{\Phi}^{\mathrm{T}}(s)\mathbf{M}^{\mathrm{T}}\, \mathrm{d}\Omega \mathrm{d}t \qquad \textbf{(19.c)}$$

$$\mathbf{B} = \left[\int_\Omega \mathbf{u}^* \mathrm{d}\Omega\right]_{t=0} \qquad \textbf{(19.d)}$$

$$\mathbf{U}^k = [u_1, u_2, \cdots, u_n]_{t=t_k}^{\mathrm{T}} \quad \textbf{(19.e)}$$

$$\mathbf{Q}^k = [q_1, q_2, \cdots, q_n]_{t=t_k}^{\mathrm{T}} \quad \textbf{(19.f)}$$

$$\mathbf{P}^k = [p_1, p_2, \cdots, p_m]_{t=t_k}^{\mathrm{T}} \quad \textbf{(19.g)}$$

$$\overline{\mathbf{U}}^0 = [\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_m]_{t=0}^{\mathrm{T}} \quad \textbf{(19.h)}$$

Solving equation (18) step by step in time domain, the unknown value of u and q on boundary Γ at any time can be obtained. Equation (14) is used again with the same procedure as (19) and other unknown values of u and q in solving domain can also be obtained.

Ⅴ. MUMERICAL EXAMPLES

In order to verify the proposed method, a metal column with infinite length is magnetizing here, its cross section of one quadrant is shown as Fig.2. The parameters of the size and the medium type

are: $OA = 0.4$ m, $OB = 0.2$ m, $\mu_r = 5000$, $\sigma = 5200$ S/m. At time $t = 0$, a step magnetic field H_0 with direct $-z$ is imposed on the outer surface of the metal column. Points $O(0,0)$, $P(0.1,0.1)$ and $Q(0.3,0)$ are investigated here to compare the analytical solution and the numerical one by using the proposed method (BMLM). The corresponding initial-boundary problem is

$$\begin{cases} \nabla^2 H_z = \sigma\mu \dfrac{\partial H_z}{\partial t} & \text{in } \Omega \text{ at } t > 0 \\ H_z = H_0 & \text{on } AC, BC \text{ at } t > 0 \\ \dfrac{\partial H_z}{\partial n} = 0 & \text{on } OB, OA \text{ at } t > 0 \\ H_z = 0 & \text{in } \Omega \text{ at } t = 0 \end{cases} \qquad \textbf{(20)}$$

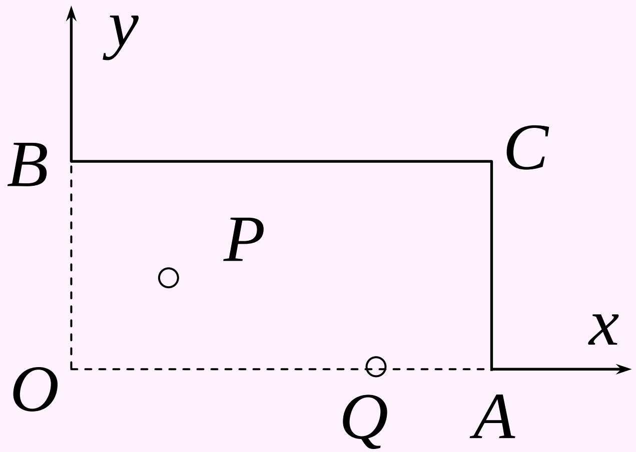
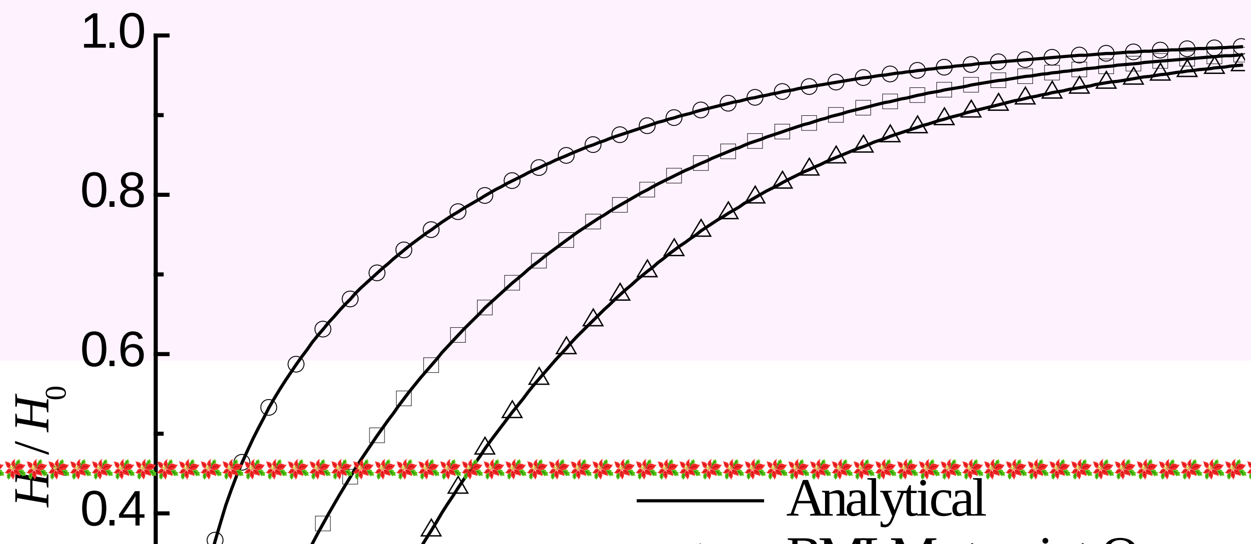


Fig.2 Cross section of one quadrant of the metal column



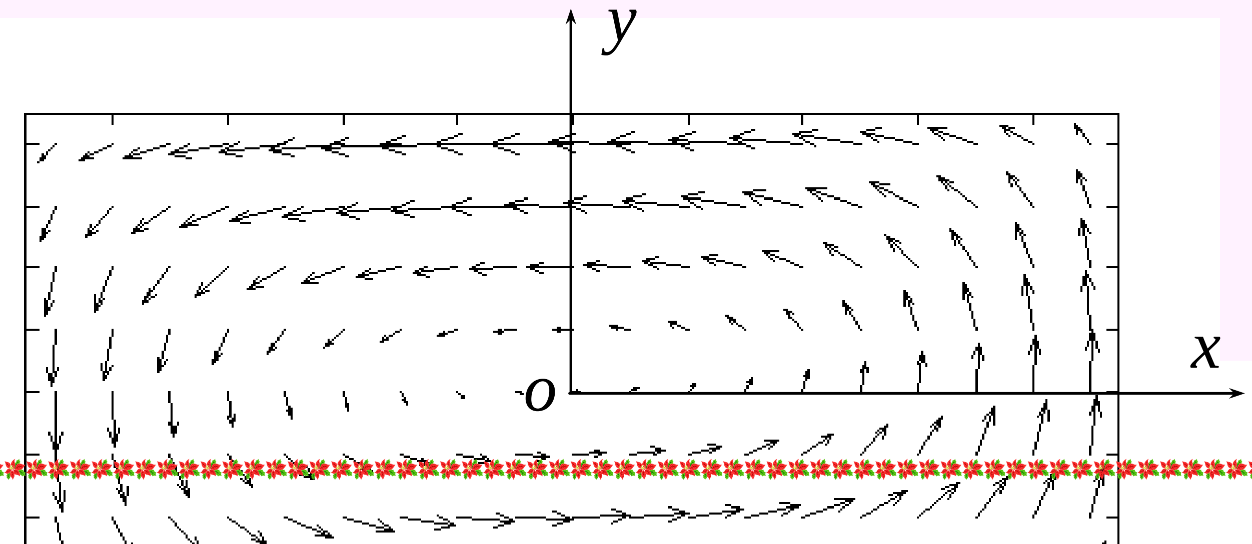
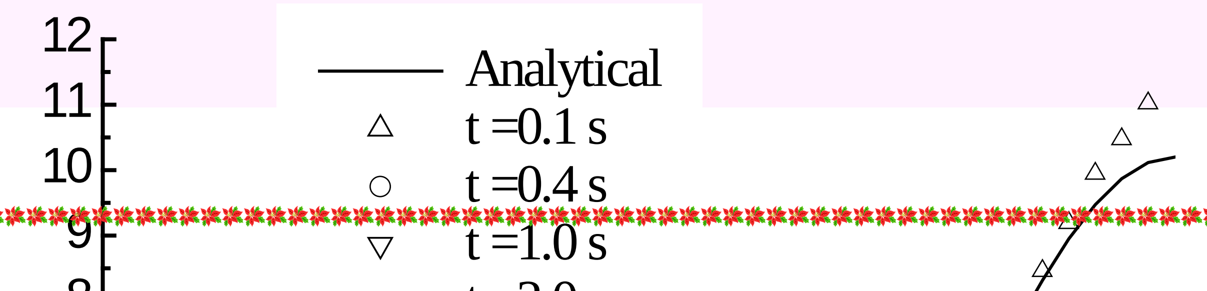

y
x
O

Fig.3 shows the numerical solution by BMLM and the analytical one, the results yielded by using the proposed method are found to be in good agreement with the analytical solution. Eddy current distribution in the whole metal column domain and the eddy current density on symmetry axis OA are also given by using of the proposed method.



These prove that BMLM is an effective technique to analysis and solve transient eddy current problems.

Ⅵ. CONCLUSIONS

A boundary meshless method (BMLM) for transient eddy current problems is presented in this paper. With difference to the traditional boundary element method, it combines a point interpolation method (PIM) for construction of spatial interpolation functions with a boundary integral formulation for the governing equations, and the spatial interpolation functions satisfy the Kronecker delta function. Theoretical analysis is given in details, and an example is presented as well. The good agreement between the numerical solution and the analytical one proves that BMLM is an effective technique to analysis and solve transient eddy current problems.