The Definition of Optimal Solution and an Extended Kuhn-Tucker Approach for Fuzzy Linear Bilevel Programming

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***Abstract*— Organizational belivel decision-making, such as planning of land-use, transportation and water resource, all may involve uncertain factors. The parameters shown in a bileved programming model, either in the objective functions or constraints, are thus often imprecise, which is called fuzzy parameter bilevel programming (FPBLP) problem. Following our previous work [1, 2]. This study first proposes a model of FPBLP. It then gives the definition of solution for FPBLP problem. Based on the definition of solution and related theorems, this study develops a fuzzy number based Kuhn-Tucher approach to solve the proposed FPBLP problem. Finally, an example further illustrates the power of the fuzzy number based Kuhn-Tucher approach.**

***Index Terms*— Linear bilevel programming, Kuhn-Tucker approach, Fuzzy set, Optimization.**

I. INTRODUCTION

THE bilevel programming (BLP) problem, introduced by Von Stackelberg [3] in the context of unbalanced economic markets, is a hierarchical optimization problem where a subset of the variables is constrained to be a solution of a given optimization problem parameterized by the remaining variables [4, 5]. In a BLP problem, each decision maker (leader or follower) tries to optimize his/her own objective function with partially or without considering the objective of the other level, but the decision of each level affects the objective optimization of the other level [6].

The vast majority of research on bilevel programming has centered on the linear version of the problem. There have been nearly two dozen algorithms [5, 7-10] proposed for solving linear BLP problems since the field caught the attention of researchers in the mid-1970s [11-19]. Although linear BLP theory and technology have applied with remarkable success in different domains [20-22], its theoretical foundation remains to a large extent unsatisfactory and incomplete. Existing bilevel programming solving approaches mainly suppose the situation in which the objective functions and constraints are characterized with precise parameters. Therefore, the parameters are required to be fixed at some values in an experimental and/or subjective manner through the experts' understanding of the nature of the parameters in the problem-formulation process. It has been observed that, in most real-world situations, particularly in critical resource planning, the possible values of these parameters are often only imprecisely or ambiguously known to the experts, such as planning of land-use, transportation and water resource. It results in a difficulty to use parameters in the objective functions or constraints of a bilevel programming model. With this observation, it would be certainly more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy sets of the real line known as fuzzy set theory [23]. A bilevel programming problem in which the parameters either in objective function or in constrains is called fuzzy parameter bilevel programming (FPBLP) problem in the study.

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The FPBLP problem was first researched by Sakawa et al. in 2000 [24]. Sakawa et al. formulates bilevel programming problems with fuzzy parameters from the perspective of experts' imprecision and proposes a fuzzy programming method for fuzzy bilevel programming problems. However, Sakawa's work is mainly based on the definition of solution for bilevel programming proposed by Bard [5, 15]. One deficiency of Bard's linear BLP theory is that it could not well solve a linear bilevel programming problem when the upper-level constraint functions are of arbitrary linear form. Our recent research work has extended Bard's theory of bilevel programming by proposing a new definition of solution for linear bilevel programming which can overcome the arbitrary linear form problem indicated above [1]. We then proposed an extended Kuhn-Tucher approach, based on our definition of solution, for solving linear bilevel problems [2].

This study will follow our previous research results shown in [1, 2] and aims at solving a FPBLP problem by transferring it into a non-fuzzy bilevel programming problem. This paper first proposes a model of FPBLP problem which have an extension of the existing bilevel programming model in fuzziness of parameters. It then gives a definition of the optimal solution for the FPBLP problem. Based on the definition and related theorems, this study develops a fuzzy number based Kuhn-Tucher approach to solve the proposed FPBLP problems. As this paper only deals with linear bilevel problem, so bilevel programming means linear bilevel programming in the paper.

Following the introduction, Section 2 reviews related definitions, theorem and properties of fuzzy number, linear BLP solution and an extended Kuhn-Tucher approach for solving linear BLP problems. A definition of solution and a fuzzy number based Kuhn-Tucher approach for solving FPBLP problems are presented in Section 3. A numeral example is shown in Section 4 for illustrating the proposed fuzzy number based Kuhn-Tucher approach. Conclusion and further study are discussed in Section 5.

II. Preliminaries

A. Fuzzy Numbers

In this section, we present some basic concepts, definitions and theorems that are to be used in the subsequent sections. The work presented in this section can also be found from our recent paper in [25].

Let R be the set of all real numbers, R^n be n-dimensional Euclidean space, and $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ be any two vectors, where $x_i, y_i \in R$, $i = 1,2, \ldots, n$ and T denotes the transpose of the vector. Then we denote the inner product of x and y by $\langle x, y\rangle$. For any two vectors $x, y \in R^n$, we write $x \geqq y$ iff $x_i \geqq y_i, \forall\, i = 1,2,\cdots,n;$ $x \geq y$ iff $x \geqq y$ and $x \neq y$; $x > y$ iff $x_i > y_i$, $\forall\, i = 1,2, \ldots, n$.

Definition 2.1 A fuzzy number $\tilde{a}$ is defined as a fuzzy set on R, whose membership function $\mu_{\tilde{a}}$ satisfies the following conditions:

1. $\mu_{\tilde{a}}$ is a mapping from R to the closed interval [0, 1];
2. it is normal, i.e., there exists $x \in R$ such that $\mu_{\tilde{a}}(x) = 1$;
3. for any $\lambda \in (0, 1]$, $a_\lambda = \{x;\ \mu_{\tilde{a}}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^L, a_\lambda^R]$.

Let $F(R)$ be the set of all fuzzy numbers. By the decomposition theorem of fuzzy set, we have

$$\tilde{a} = \bigcup_{\lambda\in[0,1]} \lambda[a_\lambda^L, a_\lambda^R], \tag{2.1}$$

for every $\tilde{a} \in F(R)$.

Let $F^*(R)$ be the set of all finite fuzzy numbers on R.

Theorem 2.1 *Let* $\tilde{a}$ *be a fuzzy set on R, then* $\tilde{a} \in F(R)$ *if and only if* $\mu_{\tilde{a}}$ *satisfies*

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 & x \in [m,n] \\ L(x) & x < m \\ R(x) & x > n \end{cases},$$

where $L(x)$ *is the right-continuous monotone increasing function,* $0 \leqq L(x) < 1$ *and* $\lim_{x\to-\infty} L(x) = 0$, $R(x)$ *is the left-continuous monotone decreasing function,* $0 \leqq R(x) < 1$ *and* $\lim_{x\to\infty} R(x) = 0$.

Corollary 2.1 *For every* $\tilde{a} \in F(R)$ *and* $\lambda_1, \lambda_2 \in [0, 1]$, *if* $\lambda_1 \leqq \lambda_2$, *then* $a_{\lambda_2} \subset a_{\lambda_1}$.

Definition 2.2 For any $\tilde{a}, \tilde{b} \in F(R)$ and $0 \leqq \lambda \in R$, the sum of $\tilde{a}$ and $\tilde{b}$ and the scalar product of λ and $\tilde{a}$ are defined by the membership functions

$$\mu_{\tilde{a}+\tilde{b}}(t) = \sup_{t=u+v} \min\{\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)\}, \tag{2.2}$$

$$\mu_{\tilde{a}-\tilde{b}}(t) = \sup_{t=u-v} \min\{\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)\}, \tag{2.2}$$

$$\mu_{\lambda\tilde{a}}(t) = \sup_{t=\lambda u} \mu_{\tilde{a}}(u). \tag{2.3}$$

Theorem 2.2 *For any* $\tilde{a}, \tilde{b} \in F(R)$ *and* $0 \leqq \alpha \in R$,

$$\tilde{a} + \tilde{b} = \bigcup_{\lambda\in[0,1]} \lambda[a_\lambda^L + b_\lambda^L,\ a_\lambda^R + b_\lambda^R],$$

$$\tilde{a} - \tilde{b} = \tilde{a} + \left(-\tilde{b}\right) = \bigcup_{\lambda\in[0,1]} \lambda[a_\lambda^L - b_\lambda^R,\ a_\lambda^R - b_\lambda^L],$$

$$\alpha\tilde{a} = \bigcup_{\lambda\in[0,1]} \lambda[\alpha a_\lambda^L,\ \alpha a_\lambda^R].$$

Definition 2.3 Let $\tilde{a}_i \in F(R), i = 1,2,\cdots,n.$ We define $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$

$$\mu_{\tilde{a}} : R^n \to [0,1]$$

$$x \mapsto \bigwedge_{i=1}^{n} \mu_{\tilde{a}_i}(x_i),$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, and $\tilde{a}$ is called an n-dimensional fuzzy number on R^n. If $\tilde{a}_i \in F^*(R), i = 1,2,\cdots,n,$ $\tilde{a}$ is called an n-dimensional finite fuzzy number on R^n.

Let $F(R^n)$ and $F^*(R^n)$ be the set of all n-dimensional fuzzy numbers and the set of all n-dimensional finite fuzzy numbers on R^n respectively.

Proposition 2.1 *For every* $\tilde{a} \in F(R^n)$, $\tilde{a}$ *is normal.*

Proposition 2.2 *For every* $\tilde{a} \in F(R^n)$, *the* λ*-section of* $\tilde{a}$ *is an n-dimensional closed rectangular region for any* $\lambda \in [0,1]$.

Proposition 2.3 *For every* $\tilde{a} \in F(R^n)$ *and* $\lambda_1, \lambda_2 \in [0,1]$, *if* $\lambda_1 \leqq \lambda_2$, *then* $a_{\lambda_2} \subset a_{\lambda_1}$.

Definition 2.4 *For any n-dimensional fuzzy numbers* $\tilde{a}, \tilde{b} \in F(R^n)$, *we define*

1. $\tilde{a} \succeq \tilde{b}$ iff $a_{i\lambda}^L \geqq b_{i\lambda}^L$ and $a_{i\lambda}^R \geqq b_{i\lambda}^R$, $i = 1,2,\cdots,n,\ \lambda \in (0,1]$;
2. $\tilde{a} \succeq \tilde{b}$ iff $a_{i\lambda}^L \geq b_{i\lambda}^L$ and $a_{i\lambda}^R \geq b_{i\lambda}^R$, $i = 1,2,\cdots,n,\ \lambda \in (0,1]$;
3. $\tilde{a} \succ \tilde{b}$ iff $a_{i\lambda}^L > b_{i\lambda}^L$ and $a_{i\lambda}^R > b_{i\lambda}^R$, $i = 1,2,\cdots,n,\ \lambda \in (0,1]$.

We call the binary relations $\succeq, \succeq$ and $\succ$ a fuzzy max order, a strict fuzzy max order and a strong fuzzy max order, respectively.

B. The Extended Kuhn-Tucker Approach for Linear Bilevel Programming

Let write a linear programming (LP) as follows.

$$\begin{aligned} &\min f(x) = cx \\ &\text{subject to } Ax \leqq b \\ &\qquad x \geqq 0, \end{aligned}$$

where c is an n-dimensional row vector, b an m-dimensional column vector, A an $m \times n$ matrix with $m \le n$, and $x \in R^n$.

Let $\lambda \in R^m$ and $\mu \in R^n$ be the dual variables associated with constraints $Ax \geqq b$ and $x \geq 0$, respectively. Bard [5] gave the following proposition.

Proposition 2.4 [5] *A necessary and sufficient condition that* (x^*) *solves above LP is that there exist (row) vectors* λ^*, μ^* *such that* (x^*, λ^*, μ^*) *solves:*

$$\lambda A - \mu = -c$$
$$Ax - b \geqq 0$$
$$\lambda(Ax - b) = 0$$
$$\mu x = 0$$
$$x \geqq 0, \lambda \geqq 0, \mu \geqq 0$$

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F: X \times Y \to R^1$, and $f: X \times Y \to R^1$, a linear BLP problem is given by Bard [4]:

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y \quad (2.5a)$$
$$\text{subject to } A_1 x + B_1 y \leqq b_1 \quad (2.5b)$$
$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y \quad (2.5c)$$
$$\text{subject to } A_2 x + B_2 y \leqq b_2 \quad (2.5d)$$

where c_1, $c_2 \in R^n$, d_1, $d_2 \in R^m$, $b_1 \in R^p$, $b_2 \in R^q$, $A_1 \in R^{p \times n}$, $B_1 \in R^{p \times m}$, $A_2 \in R^{q \times n}$, $B_2 \in R^{q \times m}$.

Definition 2.5 [1]

(a)Constraint region of the linear BLP problem:
$S = \{(x, y) : x \in X, y \in Y, A_1 x + B_1 y \leqq b_1, A_2 x + B_2 y \leqq b_2\}$

(b) Feasible set for the follower for each fixed $x \in X$:
$S(X) = \{x \in X : \exists y \in Y, A_1 x + B_1 y \leqq b_1, A_2 x + B_2 y \leqq b_2\}$

(c)Projection of S onto the leader's decision space:
$S(X) = \{x \in X : \exists y \in Y, A_1 x + B_1 y \leqq b_1, A_2 x + B_2 y \leqq b_2\}$

(d) Follower's rational reaction set for $x \in S(X)$:
$P(x) = \{y \in Y : y \in \arg\min[f(x, \hat{y}) : \hat{y} \in S(x)]\}$
where
$\arg\min[f(x, \hat{y}) : \hat{y} \in S(x)] = \{y \in S(x) : f(x, y) \leqq f(x, \hat{y}), \hat{y} \in S(x)\}$

(e)Inducible region:
$IR = \{(x, y) : (x, y) \in S, y \in P(x)\}$

Definition 2.6 [1] (x^*, y^*) is said to be a complete optimal solution, if and only if there exists $(x^*, y^*) \in S$ such that $F(x^*, y^*) \leqq F(x, y)$ and $f(x^*, y^*) \leqq f(x, y)$ for all $(x, y) \in S$.

However, in general, such a complete optimal solution that simultaneously minimizes both the leader' and follower's objective functions does not always exist. Instead of a complete optimal solution, a new solution concept, called Pareto optimality, is introduced in linear BLP.

Definition 2.7 [1] (x^*, y^*) is said to be a Pareto optimal solution, if and only if there does not exist $(x, y) \in S$ such that $F(x, y) \leqq F(x^*, y^*)$, $f(x, y) \leqq f(x^*, y^*)$ and $F(x, y) \neq F(x^*, y^*)$ or $f(x, y) \neq f(x^*, y^*)$.

Definition 2.8 A topological space is compact if every open cover of the entire space has a finite subcover. For example, $[a, b]$ is compact in R (the Heine-Borel theorem) [26].

To ensure that (2.5) has a Pareto optimal solution, Bard gave the following assumption.

Assumption 2.1

(a) S is nonempty and compact.

(b) For decisions taken by the leader, the follower has some rooms to respond; i.e, $P(x) \neq \phi$.

(c) $P(x)$ is a point-to-point map.

To ensure that (2.5) is well posed we assume that S is nonempty and compact, and that $P(x)$ is a point-to-point map. The rational reaction set $P(x)$ defines the response while the inducible region IR represents the set over which the leader may optimize his objective. Thus in terms of the above notations, the linear BLP problem can be written as

$$\min\{F(x, y) : (x, y) \in IR\} \quad (2.6)$$

We also present the following theorem to characterize the condition under which there is a Pareto optimal solution for a linear BLP problem.

Theorem 2.3 [1] *If* S *is nonempty and compact, there exists a Pareto optimal solution for a linear BLP problem*

Theorem 2.4 [2] [Extended Kuhn-Tucher Theorem] *A necessary and sufficient condition that* (x^*, y^*) *solves the linear BLP problem (2.5) is that there exist (row) vectors* u^*, v^* *and* w^* *such that* $(x^*, y^*, u^*, v^*, w^*)$ *solves:*

$$\min F(x, y) = c_1 x + d_1 y \quad (2.7a)$$
$$\text{subject to } A_1 x + B_1 y \leq b_1 \quad (2.7b)$$
$$A_2 x + B_2 y \leq b_2 \quad (2.7c)$$
$$u B_1 + v B_2 - w = -d_2 \quad (2.7d)$$
$$u(b_1 - A_1 x - B_1 y) + v(b_2 - A_2 x - B_2 y) + wy = 0 \quad (2.7e)$$
$$x \geq 0, y \geq 0, u \geq 0, v \geq 0, w \geq 0 \quad (2.7f)$$

III. Fuzzy Parameter Linear Bilevel Programming Problem

Consider the following fuzzy parameter linear bilevel programming (FPBLP) problem:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F: X \times Y \to F^*(R)$, and $f: X \times Y \to F^*(R)$,

$$\min_{x \in X} F(x, y) = \tilde{c}_1 x + \tilde{d}_1 y \quad (3.1a)$$
$$\text{subject to } \tilde{A}_1 x + \tilde{B}_1 y \precsim \tilde{b}_1 \quad (3.1b)$$
$$\min_{y \in Y} f(x, y) = \tilde{c}_2 x + \tilde{d}_2 y \quad (3.1c)$$
$$\text{subject to } \tilde{A}_2 x + \tilde{B}_2 y \precsim \tilde{b}_2 \quad (3.5d)$$

where $\tilde{c}_1, \tilde{c}_2 \in F^*(R^n)$, $\tilde{d}_1, \tilde{d}_2 \in F^*(R^m)$, $\tilde{b}_1 \in F^*(R^p)$, $\tilde{b}_2 \in F^*(R^q)$, $\tilde{A}_1 = (\tilde{a}_{ij})_{p \times n}$, $\tilde{a}_{ij} \in F^*(R)$, $\tilde{B}_1 = (\tilde{b}_{ij})_{p \times m}$, $\tilde{b}_{ij} \in F^*(R)$, $\tilde{A}_2 = (\tilde{e}_{ij})_{q \times n}$, $\tilde{e}_{ij} \in F^*(R)$, $\tilde{B}_2 = (\tilde{s}_{ij})_{q \times m}$, $\tilde{s}_{ij} \in F^*(R)$.

Associated with the FPBLP problem, we now consider the following linear multi-objective multi-follower bilevel programming (LMMBLP) problem:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F: X \times Y \to F^*(R)$, and $f: X \times Y \to F^*(R)$,

$$\min_{x\in X}\left(F(x,y)\right)_{\lambda}^{L} = c_{1\lambda}^{L}x + d_{1\lambda}^{L}y, \quad \lambda\in[0,1]$$ (3.2a)
$$\min_{x\in X}\left(F(x,y)\right)_{\lambda}^{R} = c_{1\lambda}^{R}x + d_{1\lambda}^{R}y, \quad \lambda\in[0,1]$$

subject to $A_{1\lambda}^{L}x + B_{1\lambda}^{L}y \leqq b_{1\lambda}^{L},\ A_{1\lambda}^{R}x + B_{1\lambda}^{R}y \leqq b_{1\lambda}^{R}, \lambda\in[0,1]$ (3.2b)

$$\min_{y\in Y}\left(f(x,y)\right)_{\lambda}^{L} = c_{2\lambda}^{L}x + d_{2\lambda}^{L}y, \quad \lambda\in[0,1]$$ (3.2c)
$$\min_{y\in Y}\left(f(x,y)\right)_{\lambda}^{R} = c_{2\lambda}^{R}x + d_{2\lambda}^{R}y, \quad \lambda\in[0,1]$$

subject to $A_{2\lambda}^{L}x + B_{2\lambda}^{L}y \leqq b_{2\lambda}^{L},\ A_{2\lambda}^{R}x + B_{2\lambda}^{R}y \leqq b_{2\lambda}^{R}, \lambda\in[0,1]$ (3.5d)

where $c_{1\lambda}^{L}, c_{1\lambda}^{R}$, $c_{2\lambda}^{L}, c_{2\lambda}^{R} \in R^{n}$, $d_{1\lambda}^{L}, d_{1\lambda}^{R}$, $d_{2\lambda}^{L}, d_{2\lambda}^{R}\in R^{m}$, $b_{1\lambda}^{L}, b_{1\lambda}^{R}\in R^{p}, b_{2\lambda}^{L}, b_{2\lambda}^{R}\in R^{q}$, $A_{1\lambda}^{L} = \left(a_{ij\lambda}^{L}\right), A_{1\lambda}^{R} = \left(a_{ij\lambda}^{R}\right)\in R^{p\times n}$, $B_{1\lambda}^{L} = \left(b_{ij\lambda}^{L}\right), B_{1\lambda}^{R} = \left(b_{ij\lambda}^{R}\right)\in R^{p\times m}$, $R^{q\times n}, B_{2\lambda}^{L} = \left(s_{ij\lambda}^{L}\right), B_{2\lambda}^{R} = \left(s_{ij\lambda}^{R}\right)\in R^{q\times m}$.

Theorem 3.1 *Let* (x^*, y^*) *be the solution of the LMMBLP problem* (3.2). *Then it is also a solution of the FPBLP problem defined by* (3.1).

Proof. The proof is obvious from Definition 2.4.

Lemma 3.1 *If there is* (x^*, y^*) *such that* $cx + dy \geqq cx^* + dy^*$, $c_0^{L}x + d_0^{L}y \geqq c_0^{L}x^* + d_0^{L}y^*$ *and* $c_0^{R}x + d_0^{R}y \geqq c_0^{R}x^* + d_0^{R}y^*$, *for any* (x, y) *and isosceles triangle fuzzy numbers* $\tilde{c}$ *and* $\tilde{d}$, *then*

$$c_{\lambda}^{L}x + d_{\lambda}^{L}y \geqq c_{\lambda}^{L}x^* + d_{\lambda}^{L}y^*,$$
$$c_{\lambda}^{R}x + d_{\lambda}^{R}y \geqq c_{\lambda}^{R}x^* + d_{\lambda}^{R}y^*,$$

for any $\lambda\in(0,1)$, *where c and d are the centre of* $\tilde{c}$ *and* $\tilde{d}$ *respectively.*

Proof. As λ-section of isosceles triangle fuzzy numbers $\tilde{c}$ and $\tilde{d}$ are

$$c_{\lambda}^{L} = c_0^{L}(1-\lambda) + c\lambda \text{ and } c_{\lambda}^{R} = c_0^{R}(1-\lambda) + c\lambda$$
$$d_{\lambda}^{L} = d_0^{L}(1-\lambda) + d\lambda \text{ and } d_{\lambda}^{R} = d_0^{R}(1-\lambda) + d\lambda .$$

Therefore, we have

$$\begin{aligned} c_{\lambda}^{L}x + d_{\lambda}^{L}y &= c_0^{L}(1-\lambda)x + c\lambda x + d_0^{L}(1-\lambda)y + d\lambda y \\ &= (c_0^{L}x + d_0^{L}y)(1-\lambda) + (cx+dy)\lambda \\ &\geqq (c_0^{L}x^* + d_0^{L}y^*)(1-\lambda) + (cx^*+dy^*)\lambda \\ &= c_{\lambda}^{L}x^* + d_{\lambda}^{L}y^*, \end{aligned}$$

from $cx + dy \geqq cx^* + dy^*$ and $c_0^{L}x + d_0^{L}y \geqq c_0^{L}x^* + d_0^{L}y^*.$, we can prove $c_{\lambda}^{R}x + d_{\lambda}^{R}y \geqq c_{\lambda}^{R}x^* + d_{\lambda}^{R}y^*$ from similar reason.

Theorem 3.2 *For* $x\in X\subset R^{n}$, $y\in Y\subset R^{m}$, *If all the fuzzy coefficients* $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij}, \tilde{c}_i$ *and* $\tilde{d}_i$ *have triangle membership functions of the FPBLP problem* (3.1).

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z_0^{L} \\ \dfrac{t - z_0^{L}}{z - z_0^{L}} & z_0^{L} \leqq t < z \\ \dfrac{-t + z_0^{R}}{z_0^{R} - z} & z \leqq t < z_0^{R} \\ 0 & z_0^{R} \leqq t \end{cases}, \tag{3.3}$$

where $\tilde{z}$ *denotes* $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij}, \tilde{c}_i$ *and* $\tilde{d}_i$ *and z are the centre of* $\tilde{z}$ *respectively. Then, it is the solution of the problem (3.1) that* $(x^*, y^*)\in R^{n}\times R^{m}$ *satisfying*

$$\min_{x\in X}\left(F(x,y)\right)_{c} = c_1x + d_1y,$$
$$\min_{x\in X}\left(F(x,y)\right)_{0}^{L} = c_{10}^{L}x + d_{10}^{L}y,$$ (3.4a)
$$\min_{x\in X}\left(F(x,y)\right)_{0}^{R} = c_{10}^{R}x + d_{10}^{R}y,$$

subject to $A_1x + B_1y \leqq b_1,$

$$A_{10}^{L}x + B_{10}^{L}y \leqq b_{10}^{L},$$ (3.4b)
$$A_{10}^{R}x + B_{10}^{R}y \leqq b_{10}^{R},$$

$$\min_{y\in Y}\left(f(x,y)\right)_{c} = c_2x + d_2y,$$
$$\min_{y\in Y}\left(f(x,y)\right)_{0}^{L} = c_{20}^{L}x + d_{20}^{L}y,$$ (3.4c)
$$\min_{y\in Y}\left(f(x,y)\right)_{\lambda}^{R} = c_{20}^{R}x + d_{20}^{R}y,$$

subject to $A_2x + B_2y \leqq b_2,$

$$A_{20}^{L}x + B_{20}^{L}y \leqq b_{20}^{L},$$ (3.4d)
$$A_{20}^{R}x + B_{20}^{R}y \leqq b_{20}^{R}.$$

Proof. From Lemma 3.1, if (x^*, y^*) satisfies (3.4a) and (3.4c), then it satisfies (3.2a) and (3.2c). Then we need only prove, if (x^*, y^*) satisfies (3.4b) and (3.4d), then it satisfies (3.2b) and (3.2d). In fact, for any $\lambda\in(0,1)$,

$$a_{ij\lambda}^{L} = a_{ij}\lambda + a_{ij0}^{L}(1-\lambda),$$
$$b_{ij\lambda}^{L} = b_{ij}\lambda + b_{ij0}^{L}(1-\lambda) \quad \text{and}$$
$$b_{1\lambda}^{L} = b_1\lambda + b_{10}^{L}(1-\lambda),$$

we have

$$\begin{aligned} A_{1\lambda}^{L}x^* + B_{1\lambda}^{L}y^* &= (a_{ij\lambda}^{L})x^* + (b_{ij\lambda}^{L})y^* \\ &= \left(a_{ij}\lambda + a_{ij0}^{L}(1-\lambda)\right)x^* + \left(b_{ij}\lambda + b_{ij0}^{L}(1-\lambda)\right)y^* \\ &= \left(a_{ij}\right)x^*\lambda + \left(a_{ij0}^{L}\right)x^*(1-\lambda) + \left(b_{ij}\right)y^*\lambda + \left(b_{ij0}^{L}\right)y^*(1-\lambda) \\ &= \left(\left(a_{ij}\right)x^* + \left(b_{ij}\right)y^*\right)\lambda + \left(\left(a_{ij0}^{L}\right)x^* + \left(b_{ij0}^{L}\right)y^*\right)(1-\lambda) \\ &= \left(A_1x^* + B_1y^*\right)\lambda + \left(A_{10}^{L}x^* + B_{10}^{L}y^*\right)(1-\lambda) \\ &\leqq b_1\lambda + b_{10}^{L}(1-\lambda) = b_{1\lambda}^{L}, \end{aligned}$$

from (3.4b). Similarly, we can prove

$$A_{1\lambda}^{R}x^* + B_{1\lambda}^{R}y^* \leqq b_{1\lambda}^{R},$$
$$A_{2\lambda}^{L}x^* + B_{2\lambda}^{L}y^* \leqq b_{2\lambda}^{L},$$
$$A_{2\lambda}^{R}x^* + B_{2\lambda}^{R}y^* \leqq b_{2\lambda}^{R},$$

for any $\lambda\in(0,1)$ from (3.4b) and (3.4d). The proof is complete.

Theorem 3.3 [Extended Kuhn-Tucher Theorem] *A necessary and sufficient condition that* (x^*, y^*) *solves the FPBLP problem* (3.1) *with triangle fuzzy numbers is that there exist* (*row*) *vectors* u^*, v^* *and* w^* *such that* $(x^*, y^*, u^*, v^*, w^*)$ *solves:*

$$\min_{x\in X}\left(F(x,y)\right) = \left(c_1x + d_1y\right) + \left(c_{10}^{L}x + d_{10}^{L}y\right) + \left(c_{10}^{R}x + d_{10}^{R}y\right)$$ (3.5a)

subject to $A_1x + B_1y \leqq b_1,$

$$A_{10}^{L}x + B_{10}^{L}y \leqq b_{10}^{L},$$ (3.5b)
$$A_{10}^{R}x + B_{10}^{R}y \leqq b_{10}^{R},$$
$$A_2x + B_2y \leqq b_2,$$
$$A_{20}^{L}x + B_{20}^{L}y \leqq b_{20}^{L},$$ (3.5c)
$$A_{20}^{L}x + B_{20}^{L}y \leqq b_{20}^{L},$$

$$u_1B_1 + u_2B_{10}^L + u_3B_{10}^R + v_1B_2 + v_2B_{20}^L + v_3B_{20}^R - w = -\left(d_2 + d_{20}^L + d_{20}^R\right) \quad (3.5d)$$

$$u_1\left(b_1 - A_1x - B_1y\right) + u_2\left(b_{10}^L - A_{10}^Lx - B_{10}^Ly\right) + u_3\left(b_{10}^R - A_{10}^Rx - B_{10}^Ry\right) + v_1(b_2 - -A_2x - B_2y) + v_2\left(b_{20}^L - A_{20}^Lx - B_{20}^Ly\right) + v_3\left(b_{20}^R - A_{20}^Rx - B_{20}^Ry\right) + wy = 0 \quad (3.5e)$$

$$x \geq 0, y \geq 0, u \geq 0, v \geq 0, w \geq 0 \quad (3.5f)$$

Proof: (1) From Theorem 3.2, we know that we need only to solve the problem (3.4). In fact, to solve the problem (3.4), we can use the method of weighting [27] to this problem, such that it is the following problem:

$$\min_{x\in X} \left(F(x,y)\right) = \left(c_1x + d_1y\right) + \left(c_{10}^Lx + d_{10}^Ly\right) + \left(c_{10}^Rx + d_{10}^Ry\right) \quad (3.6a)$$

subject to $A_1x + B_1y \leqq b_1$,
$A_{10}^Lx + B_{10}^Ly \leqq b_{10}^L$, (3.6b)
$A_{10}^Rx + B_{10}^Ry \leqq b_{10}^R$,

$$\min_{y\in Y}\left(f(x,y)\right) = c_2x + d_2y + c_{20}^Lx + d_{20}^Ly + c_{20}^Rx + d_{20}^Ry \quad (3.6c)$$

subject to $A_2x + B_2y \leqq b_2$,
$A_{20}^Lx + B_{20}^Ly \leqq b_{20}^L$, (3.6d)
$A_{20}^Rx + B_{20}^Ry \leqq b_{20}^R$.

Therefore, the linear BLP problem can be written as

$$\min\{F(x,y) : (x,y) \in IR\} \quad (3.7)$$

Let us get an explicit expression of (3.7) and rewrite (3.7) as follows:

$\min F(x,y)$
subject to $(x,y) \in IR$.

We have

$\min F(x,y)$
subject to $(x,y) \in S$
$y \in P(x)$

by Definition 2.5(e). Then, we have

$\min F(x,y)$
subject to $(x,y) \in S$
$y \in \arg\min[f(x,\hat{y}) : \hat{y} \in S(x)]$

by Definition 2.5(d). We rewrite it as:

$\min F(x,y)$
subject to $(x,y) \in S$
$\min f(x,y)$
subject to $y \in S(x)$.

We have

$\min F(x,y)$
subject to $(x,y) \in S$
$\min_{y\in Y} f(x,y)$
subject to $(x,y) \in S$,

by Definition 2.5(c). Consequently, we can have

$$\min_{x\in X} \left(F(x,y)\right) = \left(c_1x + d_1y\right) + \left(c_{10}^Lx + d_{10}^Ly\right) + \left(c_{10}^Rx + d_{10}^Ry\right) \quad (3.8a)$$

subject to $A_1x + B_1y \leqq b_1$,
$A_{10}^Lx + B_{10}^Ly \leqq b_{10}^L$,
$A_{10}^Rx + B_{10}^Ry \leqq b_{10}^R$, (3.8b)
$A_2x + B_2y \leqq b_2$,
$A_{20}^Lx + B_{20}^Ly \leqq b_{20}^L$,
$A_{20}^Rx + B_{20}^Ry \leqq b_{20}^R$.

$$\min_{y\in Y}\left(f(x,y)\right) = c_2x + d_2y + c_{20}^Lx + d_{20}^Ly + c_{20}^Rx + d_{20}^Ry \quad (3.8c)$$

subject to $A_1x + B_1y \leqq b_1$,
$A_{10}^Lx + B_{10}^Ly \leqq b_{10}^L$,
$A_{10}^Rx + B_{10}^Ry \leqq b_{10}^R$, (3.8d)
$A_2x + B_2y \leqq b_2$,
$A_{20}^Lx + B_{20}^Ly \leqq b_{20}^L$,
$A_{20}^Rx + B_{20}^Ry \leqq b_{20}^R$.

by Definition 2.5(a).

This simple transformation has shown that solving the fuzzy linear BLP (3.1) is equivalent to solving (3.8).

(2) Necessity is obvious from (3.8).

(3) Sufficiency. If (x^*, y^*) is the optimal solution of (3.6), we need to show that there exist (row) vectors u_1^*, u_2^*, u_3^*, v_1^*, v_2^*, v_3^* and w^* such that $(x^*, y^*, u_1^*, u_2^*, u_3^*, v_1^*, v_2^*, v_3^*, w^*)$ to solve (3.5). Going one step farther, we only need to prove that there exist (row) vectors u_1^*, u_2^*, u_3^*, v_1^*, v_2^*, v_3^* and w^* such that $(x^*, y^*, u_1^*, u_2^*, u_3^*, v_1^*, v_2^*, v_3^*, w^*)$ satisfy the follows

$$u_1B_1 + u_2B_{10}^L + u_3B_{10}^R + v_1B_2 + v_2B_{20}^L + v_3B_{20}^R - w = -(d_2 + d_{20}^L + d_{20}^R) \quad (3.9a)$$

$$u_1(b_1 - A_1x - B_1y) = 0 \quad (3.9b)$$

$$u_2(b_{10}^L - A_{10}^Lx - B_{10}^Ly) = 0 \quad (3.9c)$$

$$u_3(b_{10}^R - A_{10}^Rx - B_{10}^Ry) = 0 \quad (3.9d)$$

$$v_1(b_2 - A_2x - B_2y) = 0 \quad (3.9e)$$

$$v_2(b_{20}^L - A_{20}^Lx - B_{20}^Ly) = 0 \quad (3.9f)$$

$$v_3(b_{20}^R - A_{20}^Rx - B_{20}^Ry) = 0 \quad (3.9g)$$

$$wy = 0, \quad (3.9h)$$

where $u_1, u_2, u_3 \in R^p$, $v_1, v_2, v_3 \in R^q$, $w \in R^m$ and they are not negative variables.

Because (x^*, y^*) is the optimal solution of (3.6), we have

$$(x^*, y^*) \in IR,$$

by (3.7). Thus we have

$$y^* \in P(x^*),$$

by Definition 2.5(e). y^* is the optimal solution to the following problem

$$\min(f(x^*, y) : y \in S(x^*)), \quad (3.10)$$

by Definition 2.5(d). Rewrite (10) as follows

$\min f(x,y)$
subject to $y \in S(x)$
$x = x^*$.

From Definition 3.2(b), we have

$$\min_{y\in Y}\left(f(x,y)\right) = c_2x + d_2y + c_{20}^Lx + d_{20}^Ly + c_{20}^Rx + d_{20}^Ry \quad (3.11a)$$

subject to $A_1x + B_1y \leqq b_1$, (3.11b)

$A_{10}^{L}x + B_{10}^{L}y \leqq b_{10}^{L}$, (3.11c)

$A_{10}^{R}x + B_{10}^{R}y \leqq b_{10}^{R}$, (3.11d)

$A_2x + B_2y \leqq b_2$, (3.11e)

$A_{20}^{L}x + B_{20}^{L}y \leqq b_{20}^{L}$, (3.11f)

$A_{20}^{R}x + B_{20}^{R}y \leqq b_{20}^{R}$. (3.11g)

$x = x^*$ (3.11h)

$y \geqq 0$ (3.11i)

To simplify (3.11), we can have

$\min g(y) = (d_2 + d_{20}^{L} + d_{20}^{R})y$ (3.12a)

subject to $-B_1y \geqq -(b_1 - A_1x^*)$, (3.12b)

$-B_{10}^{L}y \geqq -(b_{10}^{L} - A_{10}^{L}x^*)$, (3.11c)

$-B_{10}^{R}y \geqq -(b_{10}^{R} - A_{10}^{R}x^*)$, (3.12d)

$-B_2y \geqq -(b_2 - A_2x^*)$, (3.12e)

$-B_{20}^{L}y \geqq -(b_{20}^{L} - A_{20}^{L}x^*)$, (3.12f)

$-B_{20}^{R}y \geqq -(b_{20}^{R} - A_{20}^{R}x^*)$, (3.12g)

$y \geqq 0$. (3.12h)

Let we note

$$B = \begin{pmatrix} B_1 \\ B_{10}^{L} \\ B_{10}^{R} \\ B_2 \\ B_{20}^{L} \\ B_{20}^{R} \end{pmatrix}, \quad A = \begin{pmatrix} A_1 \\ A_{10}^{L} \\ A_{10}^{R} \\ A_2 \\ A_{20}^{L} \\ A_{20}^{R} \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_{10}^{L} \\ b_{10}^{R} \\ b_2 \\ b_{20}^{L} \\ b_{20}^{R} \end{pmatrix}. \qquad (3.13)$$

We rewrite (3.12) by using (3.13) and we get

$\min g(y) = (d_2 + d_{20}^{L} + d_{20}^{R})y$ (3.14a)

subject to $-By \geqq -(b - Ax^*)$ (3.14b)

$y \geqq 0$. (3.14c)

Now we see that y^* is the optimal solution of (3.14) which is a LP problem. By Proposition 2, there exists vector λ^*, μ^*, such that (y^*, λ^*, μ^*) satisfy a system below

$\lambda B - \mu = -(d_2 + d_{20}^{L} + d_{20}^{R})$ (3.15a)

$-By + (b - Ax^*) \geqq 0$ (3.15b)

$\lambda(-By + (b - Ax^*)) = 0$ (3.15c)

$\mu y = 0$, (3.15d)

where $\lambda \in R^{3p+3q}$ and $\mu \in R^m$.

Let $u_1, u_2, u_3 \in R^p$, $v_1, v_2, v_3 \in R^q$ and $w \in R^m$ and define

$\lambda = (u_1, u_2, u_3, v_1, v_2, v_3)$

$w = \mu$.

Thus we have $(x^*, y^*, u_1^*, u_2^*, u_3^*, v_1^*, v_2^*, v_3^*, w^*)$ that satisfy (3.9). Our proof is completed.

Theorem 3.3 means that the most direct approach to solving (3.1) is to solve the equivalent mathematical program given in (3.5). One advantage that it offers is that it allows for a more robust model to be solved without introducing any new computational difficulties

IV. An Illustrative Example

Example 1 Consider the following FPBLP problem with $x \in R^1$, $y \in R^1$, and $X = \{x \geq 0\}$, $Y = \{y \geq 0\}$,

$\min_{x \in X} F(x, y) = \tilde{1}x - \tilde{2}y$ (4.1a)

subject to $-\tilde{1}x + \tilde{3}y \preceq \tilde{4}$ (4.1b)

$\min_{y \in Y} f_1(x, y) = \tilde{1}x + \tilde{1}y$ (4.1c)

subject to $\tilde{1}x - \tilde{1}y \preceq \tilde{0}$ (4.1d)

$-\tilde{1}x - \tilde{1}y \preceq \tilde{0}$ (4.1e)

where

$$\mu_{\tilde{1}}(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leqq t < 1 \\ 2 - t & 1 \leqq t < 2 \\ 0 & 2 \leqq t \end{cases},$$

$$\mu_{\tilde{2}}(t) = \begin{cases} 0 & t < 1 \\ t - 1 & 1 \leqq t < 2 \\ 3 - t & 2 \leqq t < 3 \\ 0 & 3 \leqq t \end{cases},$$

$$\mu_{\tilde{3}}(t) = \begin{cases} 0 & t < 2 \\ t - 2 & 2 \leqq t < 3 \\ 4 - t & 3 \leqq t < 4 \\ 0 & 4 \leqq t \end{cases},$$

$$\mu_{\tilde{4}}(t) = \begin{cases} 0 & t < 3 \\ t - 3 & 3 \leqq t < 4 \\ 5 - t & 4 \leqq t < 5 \\ 0 & 5 \leqq t \end{cases},$$

$$\mu_{\tilde{0}}(t) = \begin{cases} 0 & t < -1 \\ t + 1 & -1 \leqq t < 0 \\ 1 - t & 0 \leqq t < 1 \\ 0 & 1 \leqq t \end{cases}.$$

<u>*Step 1*</u> The problem is transferred to the following LMMBLP problem by using Theorem 3.2

$\min_{x \in X} (F(x, y))_c = 1x - 2y$

$\min_{x \in X} (F(x, y))_0^{L} = 0x - 3y$

$\min_{x \in X} (F(x, y))_0^{R} = 2x - 1y$

subject to $-1x + 3y \leq 4$

$-2x + 2y \leq 3$

$0x + 4y \leq 5$

$\min_{y \in Y} (f(x, y))_c = 1x + 1y$

$\min_{y \in Y} (f(x, y))_0^{L} = 0x + 0y$

$\min_{y \in Y} (f(x, y))_0^{R} = 2x + 2y$

subject to $1x - 1y \leq 0$

$0x - 2y \leq -1$

$2x - 0y \leq 1$

$-1x - 1y \leq 0$

$0x - 0y \leq 0$

$$-2x-2y\le -1$$

Step 2. The problem is transferred to the following linear BLP problem by using method of weighting [27].

$$\min_{x\in X} F(x,y)=3x-6y$$
subject to
$$\begin{aligned}&-1x+3y\le 4\\&-2x+2y\le 3\\&0x+4y\le 5\\&\min_{y\in Y} f(x,y)=3x+3y\\&\text{subject to } 1x-1y\le 0\\&\quad 0x-2y\le -1\\&\quad 2x-0y\le 1\\&\quad -1x-1y\le 0\\&\quad -2x-2y\le -1.\\&\quad 0x-0y\le 1\end{aligned}$$

Step 3 Solve this linear BLP problem

$$\min_{x\in X} F(x,y)=3x-6y$$
subject to
$$\begin{aligned}&-1x+3y\le 4\\&-2x+2y\le 3\\&0x+4y\le 5\\&1x-1y\le 0\\&0x-2y\le -1\\&2x-0y\le 1\\&-1x-1y\le 0\\&-2x-2y\le -1.\\&0x-0y\le 1\\&3u_1+2u_2+4u_3-u_4-2u_5-0u_6-u_7-2u_8-0u_9-u_{10}=-3\\&u_1(4+1x-3y)+u_2(3+2x-2y)+u_3(5-4y)+\\&u_4(-x+y)+u_5(-1+2y)+u_6(1-2x)+\\&u_7(x+y)+u_8(-1+2x+2y)+u_9+u_{10}y=0\\&x\ge 0, y\ge 0, u_1\ge 0,\dots,u_{10}\ge 0.\end{aligned}$$

Step 4 The result is

$$\min_{x\in X}\big(F(x,y)\big)_c=1x-2y=-1$$
$$\min_{x\in X}\big(F(x,y)\big)_0^L=0x-3y=-1.5$$
$$\min_{x\in X}\big(F(x,y)\big)_0^R=2x-1y=-0.5$$

and

$$\min_{y\in Y}\big(f(x,y)\big)_c=0.5$$
$$\min_{y\in Y}\big(f(x,y)\big)_0^L=0$$
$$\min_{y\in Y}\big(f(x,y)\big)_0^R=1$$
$$x=0, y=0.5$$

Consequently, we have the solution of the problem (4.1)

$$\min_{x\in X} F(x,y)=\tilde{1}x-\tilde{2}y=\tilde{c}$$
$$\min_{y\in Y} f_1(x,y)=\tilde{1}x+\tilde{1}y=\tilde{d}$$

and

$$x=0, y=0.5,$$

where

$$\mu_{\tilde{c}}(t)=\begin{cases}0 & t<-1.5\\ \dfrac{t+1.5}{0.5} & -1.5\le t<-1\\ \dfrac{-0.5-t}{0.5} & -1\le t<-0.5\\ 0 & -0.5\le t\end{cases},\quad \mu_{\tilde{d}}(t)=\begin{cases}0 & t<0\\ \dfrac{t}{0.5} & 0\le t<0.5\\ \dfrac{1-t}{0.5} & 0.5\le t<1\\ 0 & 1\le t\end{cases}.$$

V. Conclusion

Following our previous research [1, 2], this paper proposes the definition of solution and related theorems of optimal solution for fuzzy parameter based linear bilevel programming. By using the proposed definition and theorems, this study develops a fuzzy number based Kuhn-Tucher approach to solve proposed FPBLP problem. A numeral example is shown to illustrate the proposed fuzzy number based Kuhn-Tucher approach. Further study will include the development of fuzzy parameter based multi-follower and multi-objective bilevel programming problems.

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