Minimal multirealisation of MIMO linear systems

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Abstract

This paper explores the minimal multirealisation problem, which is the determination of a minimal degree, parameter-dependent, state variable description to express a finite set of linear multivariable systems. The form of the parameter-dependent state variable description is selected as a feedback form to implement "state sharing" and "bumpless transfer", which are possible ways to improve poor transient responses for switching control. The problem is solved by finding a special kind of minimal multiplier for a finite set of polynomial matrices.

Index Terms – System multirealisation, linear multivariable systems, switching systems, Multiple Model Adaptive Control.

I. Introduction

As an extension of the concept of state variable realisation of a **single** transfer function, the **multirealisation** of linear systems deals with the task of finding a parameter-dependent state variable description to realise a finite set of linear systems. The **minimal** multirealisation problem is that of ensuring that the parameter-dependent state variable realisation is of minimal degree.

The motivation for investigating minimal multirealisation problems partially originiates from multiple model adaptive control (MMAC) algorithms [1] [2] [3] [4] [5] [6] [7]. Multirealisation

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is relevant in efficiently realising the multicontroller structure of MMAC. It is observed [5] that a MMAC system only needs to generate one control signal at any time, because only one of the constituent controllers is applied to the plant at any one instant of time. It may be possible to efficiently generate all control signals by just using a single system with adjustable parameters, an observation giving rise to the multirealisation problem. In this paper, the multicontroller structure is implemented by using a single stable linear system with adjustable parameters. The state of the stable linear system is shared by the family of controllers, and the bumpless switching of controllers is implemented by adjusting the parameter dependent feedback (see Definition 1). This implementation is termed a state sharing multirealisation using parameter dependent feedback. State sharing additionally has the potential to ameliorate the poor transient response problem that can arise due to controller switching, and also is efficient in avoiding use of unnecessarily many parameters. Our previous work [8] presented an algorithm for multirealisation that did not necessarily secure minimal degree. Minimal degree multirealisation is the focus of this paper.

The realisation of a **single** linear system involves more tools than just state-variable realisation themselves, especially in the MIMO case, with matrix fraction descriptions being valuable in connecting to canonical state variable realisations, see eg [9] [10] [11] [12] [13] [14] [15]. The same turns out to be true for the multirealisation problem, which has been much less studied: in the MIMO case, introducing matrix fraction descriptions is of great utility. Past work on the implementation of multicontroller structure includes that of Morse [12], in the context of examining MMAC for scalar plants, and forthcoming work [8] by the authors on MIMO multirealisation obtaining realisations which are minimal only generically so that the algorithm on occasions does not lead to a minimal multirealisation.

For different purposes, the form of realistion could be different. For studying uncertain systems, linear fractional transformation (LFT) form representations of systems with parametric [16] or structured [17] uncertainties have been developed for the establishment of a comprehensive theory of system analysis and synthesis [18] [19]. To efficiently realise multicontroller structure of switching control systems, this paper investigate two multirealisation forms $\{A_0 + B_0 K_i, B_0, C_i\}$ and $\{A_0 + F_i C_0, B_i, C_0\}$ (which are dual). The multirealisation form $\{A_0 + F_i C_0, B_i, C_0\}$ is prefered for the implementation of multicontroller structures, because it can ensure that the output of the switched system remains continuous across switching instants, provided its input is reasonably well behaved, e.g. is piecewise continuous, i.e. "**bumpless**" transfer [5] is achieved.

However, it is slightly more convenient to investigate the dual form $\{A_0+B_0K_i, B_0, C_i\}$ because for this multirealisation form we can directly lift known results on the invariant description of linear multivariable systems [11] [13]. Corresponding results for the multirealisation form $\{A_0 + F_iC_0, B_i, C_0\}$ can be easily achieved by using the duality relationship (e.g. see Method 2).

The definition of the concept of minimal stably based multirealisation is given as below:

Definition 1: Assume that there is given a number N of m-input p-output strictly proper real rational transfer function matrices P_i ($i \in \{1, \cdots, N\}$). A multirealisation of the set of systems P_i is a set of state variable realisations $\{A_0+B_0K_i, B_0, C_i\}$ (with the pair (A_0, B_0) being controllable and adjustable parameter matrices C_i and K_i) realising all the systems P_i ($i \in \{1, \cdots, N\}$). If all eigenvalues of A_0 are in the left half plane, $\{A_0 + B_0K_i, B_0, C_i\}$ is termed a stably based multirealisation of the set of systems P_i ($i \in \{1, \cdots, N\}$). Furthermore, if the dimension of A_0 is the smallest of all such stably based multirealisations, then we call $\{A_0 + B_0K_i, B_0, C_i\}$ a minimal stably based multirealisation of the set of systems P_i ($i \in \{1, \cdots, N\}$).

Because of the assumption of controllability of the pair (A_0, B_0), it is evident that the requirement that the multirealisation be stably based poses no extra theoretical challenge (If A_0 is not stable, find $\bar{K}$ so that $A_0+B_0\bar{K}$ is stable, and replace K_i by $K_i-\bar{K}$). It is important in implementation for a multirealisation to be stably based [4].

Standard concepts and notations, such as column reduced polynomial matrices, are defined as in [11]. A new operator ($\mathcal{D}_{hc}\{\cdot\}$) is introduced as below:

Definition 2: Given a polynomial matrix $D(s)$, it is always possible to write $D(s) = D^{hc}S(s)+ D_{lc}\Psi(s)$. Where, $S(s) \triangleq diag\{s^{k_1}, s^{k_2}, \cdots, s^{k_m}\}$, k_i is the degree of the i-th column [1] of $D(s)$, D^{hc} is a matrix formed from the coefficients of the highest degree polynomials in the columns of $D(s)$ (highest-degree-coefficient matrix),

$\Psi^T(s) \triangleq block\ diag\{[s^{k_1-1}, \cdots, s, 1], [s^{k_m-1}, \cdots, s, 1]\}$,

and D_{lc} is a matrix formed from the remaining coefficients of polynomials in the columns of $D(s)$ (lower-degree-coefficient matrix).

Define the operator $\mathcal{D}_{hc}(\cdot)$ as $\mathcal{D}_{hc}(D(s)) = D^{hc}S(s)$.

In the next section, necessary and sufficient conditions for the multirealisation of multivari-

[1]The i-th column degree of a polynomial matrix is the highest degree of all entries in the i-th column.

able systems are presented first. Then, the solution of the minimal multirealisation problem is provided.

II. Minimal stably based multirealisation for multivariable systems

A. Conditions for multirealisations

Necessary and sufficient conditions for any given set of linear systems with compatible input and output dimensions are presented in the following theorem.

Theorem 1: Consider a set of m-input p-output strictly proper systems $H_i(s)$ ($i \in \{1, 2, \cdots, N\}$). The following two statements 1 and 2 are equivalent.

1. There exists a controllable pair (A_0, B_0) ($dim\{A_0\} = n$), and appropriately dimensioned real matrices C_i and K_i (for $i \in \{1, 2, \ldots, N\}$) such that A_0 is stable, and $\{A_0 + B_0K_i, B_0, C_i\}$ is a controllable realisation of system $H_i(s)$, (for $i \in \{1, 2, \ldots, N\}$).

And

2. There exists a right polynomial MFD for each system $H_i(s)$ described by $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$ (where $D_{Ei}(s)$ is a Popov polynomial matrix [11] [13] with degree n, i.e. $deg\{D_{Ei}(s)\}$ $= n, \forall i \in \{1, 2, \cdots, N\}$) such that
   - i) $k_{il} = k_{jl}$ for $i, j \in \{1, 2, \ldots, N\}$ and $l \in \{1, 2, \ldots, m\}$, where k_{ij} is the j-th column degree of the matrix $D_{Ei}(s)$, and
   - ii) the matrix D_{Ei}^{hc}, which is the highest-degree-coefficient matrix of the $D_{Ei}(s)$, is identical for $i \in \{1, 2, \ldots, N\}$.

Proof: The theorem can be proved by using the relationship between invariant Popov parameters of a controllable pair (A, B) and the coefficients in a Popov form matrix $D_E(s)$ [11] [13]. The detailed proof can be seen in [8] or [20]. ∎

From Theorem 1, we can derive that the minimal degree of the multirealisation of SISO systems is equal to the maximum McMillan degree of any of the $H_i(s)$. For the MIMO systems, Theorem 1 gives no guidance as to the minimal dimension, it may turn out that it is not possible to obtain a dimension as low as the maximum McMillan degree of any of the $H_i(s)$, due to varying possible column degrees. In the next subsection, we provide a method to find the minimal degree of the multirealisation of MIMO systems.

B. Minimal multirealisation

In order to simplify our discussion, we present a problem that is equivalent to the minimal stably based multirealisation problem. We call it the "minimal common hc- (highest column degree) multiplier problem" for a set of polynomial matrices.

Problem 1: Given a finite set of square ($m \times m$) column-reduced polynomial matrices $D_i(s)$, find nonsingular stable polynomial matrices $X_i(s)$ (that is, the zeros of $det(X_i(s))$ lie in the left half plane $Re(s) < 0$) such that there exists a column-reduced polynomial matrix $D_{min}(s)$ with the property that

$$\mathcal{D}_{hc}[D_i(s)X_i(s)] = D_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\}, \tag{1}$$

and $D_{min}(s)$ has the lowest possible degree.

Although the minimal common hc- (highest column degree) multiplier problem is actually equivalent to the minimal stably based multirealisation problem, here we are only particularly interested in whether it is possible to construct the minimal stably based multirealisation from the solution of the minimal common hc- (highest column degree) multiplier problem.

Theorem 2: Consider a set of m-input p-output strictly proper systems $H_i(s)$ ($i \in \{1, 2, \cdots, N\}$) described by right polynomial MFDs, i.e. $H_i(s) = N_i(s)D_i^{-1}(s)$, and $(N_i(s), D_i(s))$ are right coprime polynomial matrices. If for the set of polynomial matrices $D_i(s)$, one can find a minimal common hc-multiplier (as stated in Problem 1) $D_{min}(s)$, i.e. the column reduced polynomial matrix $D_{min}(s)$ satisfies equation (1) with the lowest possible degree, then, a minimal stably-based multirealisation $\{A_0 + B_0 K_{q_i}, B_0, C_{q_i}\}$ with $dim\{A_0\} = deg\{D_{min}\}$ for the set of systems $H_i(s)$ can be constructed.

Proof: The proof is based on the following two steps. The first step is to construct a stably based multirealisation with $\{A_0 + B_0 K_{q_i}, B_0, C_{q_i}\}$ with $dim\{A_0\} = deg\{D_{min}\}$. The second step is to prove by contradiction based on the results of Theorem 1 that this multirealisation is minimal. ■

In order to solve Problem 1, we introduce a new concept, $hc-(highest\ column\ degree)$ dependence on a set of polynomial vectors.

Definition 3: A polynomial vector $d_e(s)_{n\times 1}$ is hc-(highest column degree) dependent on a collection of polynomial vectors $d_i(s)_{n\times 1}$, $i = 1, 2, \cdots, m$ if there exists a set of scalar polynomials

$r_i(s)$ such that

$$\mathcal{D}_{hc}\{d_e(s)\} = \mathcal{D}_{hc}\{\sum_1^m r_i(s)d_i(s)\}.$$

In Problem 1, it can be seen that each column of the minimal polynomial matrix $D_{min}(s)$ must be hc-dependent on the columns of $D_i(s)$ for each $i \in \{1, 2, \cdots, N\}$. The following theorem provides necessary and sufficient condition for $hc-(highest\ column\ degree)$ dependence.

Theorem 3: Assume there is given a collection of polynomial vectors $d_i(s)_{n\times 1}$, $i = 1, 2, \cdots, m$, such that their column degrees, k_i, are ordered as

$$k_1 \leq k_2 \leq \cdots k_m.$$

Assume further that the matrix $[d_1(s)\, d_2(s) \cdots\, d_m(s)]$ is such that $D^{hc} = [d_1^{hc}\ \ d_2^{hc} \ldots\ d_m^{hc}]$ has full column rank. Then a given polynomial vector $d_e(s)_{n\times 1}$ (with column degree k_e) is hc-dependent on the collection of polynomial vectors $d_i(s)$, $i = 1, 2, \cdots, m$ if and only if the real vector d_e^{hc} (the highest-(column)degree-coefficient vector of $d_e(s)$) is a linear combination of real vectors d_1^{hc}, d_2^{hc}, $\cdots$, d_l^{hc} where $l = \max_i\{\arg_i\{k_i \leq k_e\}\}$.

Proof: (Forward Implication) If $\mathcal{D}_{hc}\{d_e(s)\} = \mathcal{D}_{hc}\{\sum_1^m r_i(s)d_i(s)\}$, for some polynomial $r_i(s)$, then $d_e(s) + g(s) = \sum_1^m r_i(s)d_i(s)$, where $g(s)$ is a polynomial vector with column degree less than k_e. According to Theorem 6.3-13 in pp387 of [11], if $k_i > k_e$, we must have $r_i(s) = 0$, and the ordering of k_i and the definition of l imply that

$$d_e(s) + g(s) = \sum_1^l r_i(s)d_i(s). \tag{2}$$

If d_e^{hc} is not a linear combination of real vectors d_1^{hc}, d_2^{hc}, $\cdots$, d_l^{hc}, then $d_e(s), d_1(s), \cdots, d_l(s)$ are linearly independent. Considering that the column degree of $g(s)$ is less than k_e, equation (2) is impossible. Then, the necessity is proved.

(Reverse Implication) If the real vector d_e^{hc} is a linear combination of real vectors d_1^{hc}, d_2^{hc}, $\cdots$, d_l^{hc}, then

$$d_e^{hc} = \Sigma_{i=1}^l r_i d_i^{hc},$$

where r_i, for $i \in \{1, \cdots, l\}$ are real numbers.

It follows that

$$d_e^{hc} s^{k_e} = \Sigma_{i=1}^l r_i s^{k_e - k_i} d_i^{hc} s^{k_i} = \mathcal{D}_{hc}\{\Sigma_{i=1}^l r_i s^{k_e - k_i} d_i^{hc} s^{k_i}\}.$$

Therefore, setting $r_i(s) = r_i s^{k_e - k_i}$, we have

$$\mathcal{D}_{hc}\{d_e(s)\} = d_e^{hc} s^{k_e} = \mathcal{D}_{hc}\{\sum_1^m r_i(s) d_i(s)\}.$$

■

Let us first indicate two simplifications to Problem 1. If in the problem statement any $D_i(s)$ is replaced by $\tilde{D}_i(s) = D_i(s)U_i(s)$ where $U_i(s)$ is unimodular, but otherwise arbitrary, then the problem is effectively unchanged. In fact, the solution $X_i(s)$ is just replaced by $U_i^{-1}(s)X_i(s)$. Second, if $D_{min}(s)$ is a minimal common hc-multiplier for a set $D_i(s)$ for $i = 1, 2, \cdots, N$, so is $D_{min}(s)V$, where V is a permutation matrix. Effectively, $X_i(s)$ is replaced by $X_i(s)V$.

In particular then, without loss of generality, we can assume $D_i(s)$ is a Popov form matrix $D_{Ei}(s)$, and seek a column degree ordered (see Definition 4 below) $D_{min}(s)$.

Definition 4: A column reduced polynomial matrix $D(s)$ is said to be "column degree ordered" [21] if the columns of the matrix $D(s)$ are ordered according to increasing column degrees $k_1 \leq k_2 \leq \cdots \leq k_m$. Suppose

$$k_1 = \cdots = k_{r_1} < k_{r_1+1} = \cdots = k_{r_1+r_2} < k_{r_1+r_2+1} \cdots \leq k_m, \tag{3}$$

i.e., the columns are arranged in groups of r_j columns with the same column degree.

Let the number of groups of columns with equal degree be q, so that $k_{r_1+r_2+\cdots+r_q} = k_m$ and note that each column group has the same column degree k_j^{group} ($j \in \{1, 2, \ldots, q\}$). Further define

$$\sigma_j = \sum_{l=1}^{j} r_l, j \in \{1, 2, \ldots, q\},$$

and also define $D_j(s)$ as the sub-matrix of $D(s)$ obtained by deleting the columns whose column degree are greater than k_j^{group} ($j \in \{1, 2, \ldots, q\}$), and D_j^{hc} to be the highest-(column)degree-coefficient matrix of the polynomial matrix $D_j(s)$.

Definition 5: A column reduced polynomial matrix $D_{mu}(s)_{m\times m}$ is termed hc-dependent on another square polynomial matrix $D(s)_{m\times m}$ if there exists a polynomial matrix $X(s)$ such that

$$\mathcal{D}_{hc}\{D(s)X(s)\} = \mathcal{D}_{hc}\{D_{mu}(s)\}.$$

In the method below, we will consider simultaneous hc-dependence of $D_{min}(s)$ on a number of Popov matrices $D_{Ei}(s)$, $i \in \{1, 2, \cdots, N\}$. The following theorem considers dependence on just one of these matrices.

Theorem 4: Assume that a column reduced polynomial matrix $D_m(s)_{m\times m}$ is "column degree ordered", and k_j^{group} is defined corresponding to Definition 4. Consider also a particular Popov polynomial matrix $D_{Ei}(s)_{m\times m}$. Let D_{Ei}^{hc} denote the highest-(column)degree-coefficient matrix of the polynomial matrix $D_{Ei}(s)$, let $D_{Ei_j}(s)$ denote the sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degree are greater than k_j^{group} ($j \in \{1, 2, \ldots, q\}$), and let $D_{Ei_j}^{hc}$ denote the highest-(column)degree-coefficient matrix of the polynomial matrix $D_{Ei_j}(s)$.

Then the polynomial matrix $D_m(s)$ is hc-dependent on the Popov polynomial matrix $D_{Ei}(s)$ if and only if there exists a set of real matrices X_{ij} with $j \in \{1, 2, \cdots, q\}$ such that

$$D_{Ei_j}^{hc} X_{ij} = D_{m_j}^{hc}, \forall\, j \in \{1, 2, \cdots, q\}. \tag{4}$$

Proof: By considering the necessary and sufficient conditions for hc-dependence on one polynomial vector given in Theorem 3 and noting that $D_m(s)$ and $D_{Ei}(s)$ are "column degree ordered", the conclusion is straightforward. ■

Theorem 4 presents a condition (see equation (4)) for hc-dependence of a polynomial matrix. Next, we will consider the minimal common hc-multiplier for a set of polynomial matrices based on this theorem. Specifically, we present a method which uses elementary column operations and multiplication of columns by powers of $(s + a)$ to achieve a common hc-multiplier of a set of Popov polynomial matrices $D_{Ei}(s)$ ($i \in \{1, 2, \cdots, N\}$). This method consists of searching for a set k_j^{group} and σ_j ($j \in \{1, 2, \cdots, q\}$) in order to construct a common hc-multiplier $D_m(s)$. Later, we will prove the method provides a minimal common hc-multiplier.

Method 1: [2] Step 1. Consider the matrices $D_{Ei}(s)$, define k_1^{max} as the highest degree of the first column in all $D_{Ei}(s)$, i.e. $k_1^{max} = \max_i\{k_{i1}\}$. By multiplication by $(s + a)^{k_1^{max} - k_{ij}}$ of any column whose column degree k_{ij} is less than k_1^{max}, one can make each $D_{Ei}(s)$ to have the lowest column degree k_1^{max}. Here k_{ij} is the j-th column degree of the matrix $D_{Ei}(s)$. Denote each transformed matrix as $D_{Ei}^0(s)$.

Step 2. We search for a value of k_1^{group} starting from k_1^{max}, and trying in turn k_1^{max}, $k_1^{max}+1, \cdots$ until a certain condition (given by equation (5) below) is satisfied.

In more detail, try $k_1^{group} = k_1^{max}$ first. For each $D_{Ei}(s)$, denote $D_{Ei_1}(s)$ as the sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degree are greater than k_1^{group}

[2] The reader may find it helpful to review Example 1 below *partway* through the description of the method.

($i \in \{1, 2, \ldots, N\}$), and $D^{hc}_{Ei_1}$ as the highest-(column)degree-coefficient matrix of the polynomial matrix $D_{Ei_1}(s)$ ($i \in \{1, 2, \ldots, N\}$).

a) If for the set of real matrices $D^{hc}_{Ei_1}$, there exist constant real matrices X_{i1} and a real matrix $D^{hc}_{m_1}$, such that

$$D^{hc}_{Ei_1} X_{i1} = D^{hc}_{m_1}, \forall i, \tag{5}$$

where $D^{hc}_{m_1}$ has full column rank and the **largest** possible number of columns ($\sigma_1 > 0$), then it is possible to post-multiply each $D^0_{Ei}(s)$ (generated in Step 1) by a real constant matrix to make them have the same $D^{hc}_{m_1} \in R^{m \times \sigma_1}$ for the first σ_1 columns. Denote each transformed matrix as $D^1_{Ei}(s)$.

b) If for $k^{group}_1 = k^{max}_1$, equation (5) has no solution (i.e. $\sigma_1 = 0$), then we increase k^{group}_1 by one (that is $k^{group}_1 = k^{max}_1 + 1$ and repeat the process above, seeking a nontrivial solution to (5)). Keep on searching until a **minimal** value of k^{group}_1 is achieved such that equation (5) has a solution X_{i1} for $i = 1, \cdots, N$. By multiplication by $(s + a)^{k^{group}_1 - k_{ij}}$ of any column whose column degree k_{ij} is less than k^{group}_1, one can make each $D_{Ei}(s)$ have the lowest column degree k^{group}_1. If we denote each transformed matrix as $D^{0'}_{Ei}(s)$, then it is possible to post-multiply it by a real constant matrix to make each $D^{0'}_{Ei}(s)$ have the same corresponding $D^{hc}_{m_1} \in R^{m \times \sigma_1}$ for the first σ_1 columns. Denote each transformed matrix as $D^1_{Ei}(s)$.

There always exists a value of $k^{group}_1 \le k^{max}$ (where k^{max} is the highest column degree of all $D_{Ei}(s)$, i.e. $k^{max} = max_{i,j}\{k_{ij}\}$, for $i \in \{1, \cdots, N\}$, $j \in \{1, \cdots, m\}$) such that equation (5) has a solution. This is because, for the case $k^{group}_1 = k^{max}$, a common hc-multiplier will be $s^{k^{max}} I_m$.

Step 3. Search for a **minimal** integer k^{group}_2 (searching from $k^{group}_1 + 1$) and a set of real matrix X_{i2} for the set of polynomial matrices $D_{Ei}(s)$ such that

$$D^{hc}_{Ei_2} X_{i2} = D^{hc}_{m_2}, \forall i, \tag{6}$$

with $D^{hc}_{m_2} \in \mathcal{R}^{m \times \sigma_2}$ having full column rank and the **largest** possible number ($\sigma_2 > \sigma_1$) of columns. Recall that $D^{hc}_{Ei_2}$ is the highest-(column)degree-coefficient matrix of each $D_{Ei_2}(s)$ which is a sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degrees are greater than k^{group}_2. Based on equations (5) and (6), we can find a nonsingular real matrix R_2 such that

$$D^{hc}_{m_2} R_2 = [D^{hc}_{m_1} \vdots D^{hc}_{m\Delta 2}]. \tag{7}$$

For each $D_{E_i}(s)$, define l_{Ei_2} as the number of columns of $D_{Ei}(s)$ whose degree is no more than k_2^{group} ($i \in \{1, 2, \ldots, N\}$), i.e. $l_{Ei_2} = \max_j\{\arg_j\{k_{ij} \leq k_2^{group}\}\}$. Multiply the polynomial matrices $D^1_{Ei}(s)$ (achieved in Step 3) by $(s+a)^{k_2^{group}-k_{ij}}$ ($\sigma_1 < j \leq l_{Ei_2}$) from the (σ_1+1)-th column to the l_{Ei_2}-th column, and denote the new matrices so obtained as $D^{1'}_{Ei}(s)$. According to equation (7), it is possible to post multiply by a corresponding unimodular polynomial matrix to transform each matrix $D^{1'}_{Ei}(s)$ to have the same $D^{hc}_{m_1} \in \mathcal{R}^{m\times\sigma_1}$ for the first σ_1 columns (with column degree all equal to k_1^{group}), and the same $D^{hc}_{m\Delta 2} \in \mathcal{R}^{m\times(\sigma_2-\sigma_1)}$ for the columns from (σ_1+1)-th columns to σ_2-th columns (with column degree all equal to k_2^{group}). Denote each transformed matrix as $D^2_{Ei}(s)$.

Repeat Step 3. This will eventually derive the common hc-multiplier

$$D_m(s) = \mathcal{D}_{hc}\{D^q_{Ei}(s)\}, \forall i \in \{1, 2, \ldots, N\}$$

for all Popov polynomial matrices $D_{Ei}(s)$.

In this process, the values of k_j^{group} and σ_j are determined for $j > 2$ in an identical manner to the determination of k^{group_1} in Step 2-3. If we define $D_{Ei_j}(s)$ ($i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, q\}$) as a sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degrees are greater than k_j^{group}, and $D^{hc}_{Ei_j}$ is the highest-(column)degree-coefficient matrix of $D_{Ei_j}(s)$, then, there exist a set of real matrices X_{ij} ($i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, q\}$) such that

$$D^{hc}_{Ei_j} X_{ij} = D^{hc}_{m_j}, \forall i \in \{1, 2, \ldots, N\}, \forall j \in \{1, 2, \ldots, q\}. \tag{8}$$

The real matrix $D^{hc}_{m_j}$ has full column rank, and σ_j is equal to the number of columns in the matrix $D^{hc}_{m_j}$.

To derive a solution for equations (5) and (6) or to identify that no such solution exists is not difficult, because each column of each D^{hc}_{Ei}, the highest-(column)degree-coefficient matrix of $D_{Ei}(s)$, has a unique pivot index. Method 1 presents a way to achieve a common hc-multiplier for a set of polynomial matrices. The following theorem confirms that it is also a minimal common hc-multiplier.

Theorem 5: (Main result) The common hc-multiplier $D_m(s)$ for a set of square column reduced polynomial matrices $D_i(s)$ ($i \in \{1, 2, \ldots, N\}$) achieved by using Method 1 is also a **minimal** common hc-multiplier for this set of polynomial matrices $D_i(s)$ (see Problem 1).

Proof: Method 1 has already confirmed that a common hc-multiplier can be achieved with associated values k_j^{group} and σ_j. The next step of the proof is just to confirm that the common hc-multiplier form defined by k_j^{group} and σ_j is also a **minimal** common hc-multiplier. Denote $D_{Ei}(s)$ as the Popov polynomial matrix of matrix $D_i(s)$ ($i \in \{1, 2, \cdots, N\}$).

Suppose that a minimal common hc-multiplier is given by $\tilde{D}_{min}(s)$, without loss of generality with column degree ordered. We now prove the desired result by contradiction. To this end, **assume** that the common hc-multiplier $D_m(s)$ (with parameters k_j^{group} and σ_j) achieved by using Method 1 is not a **minimal** common hc-multiplier. Then, there should exist an integer l ($l \in \{1, 2, \cdots, m\}$) such that the l-th column degree k_l of the multirealisation polynomial matrix $D_m(s)$ is bigger than the l-th column degree $\tilde{k}_l$ of the minimal common hc-multiplier $\tilde{D}_{min}(s)$, i.e. $\tilde{k}_l < k_l$. Without loss of generality, we assume $k_l = k_{\mathcal{J}}^{group}$ (here, $\mathcal{J}$ is fixed and $\mathcal{J} \in \{1, 2, \cdots, q\}$), so that $\sigma_{\mathcal{J}-1} < l \leq \sigma_{\mathcal{J}}$ and

$$\tilde{k}_l < k_l = k_{\mathcal{J}}^{group}. \tag{9}$$

Similarly, we assume $\tilde{k}_l = \tilde{k}_{\tilde{\mathcal{J}}}^{group}$, then $\tilde{\sigma}_{\tilde{\mathcal{J}}-1} < l \leq \tilde{\sigma}_{\tilde{\mathcal{J}}}$ (here, $\tilde{\mathcal{J}}$ is fixed and $\tilde{\mathcal{J}} \in \{1, 2, \cdots, \tilde{q}\}$). Also $\sigma_0 = 0$ and $\tilde{\sigma}_0 = 0$.

Because $\tilde{D}_{min}(s)$ is a hc-multiplier of each $D_{Ei}(s)$, according to Theorem 4 (See equation (4)), for fixed $\tilde{j} = \tilde{\mathcal{J}}$, there exists a real matrices $\tilde{X}_{i\tilde{\mathcal{J}}}$ for each $D_{Ei}(s)$ such that

$$D_{Ei_{\tilde{\mathcal{J}}}}^{hc} \tilde{X}_{i\tilde{\mathcal{J}}} = \tilde{D}_{min_{\tilde{\mathcal{J}}}}^{hc}, \forall i \in \{1, 2, \ldots, N\}, \tag{10}$$

where $\tilde{D}_{min_{\tilde{\mathcal{J}}}}^{hc}$ has full column rank $\tilde{\sigma}_{\tilde{\mathcal{J}}} \geq l$, and $D_{Ei_{\tilde{\mathcal{J}}}}^{hc}$ is the highest-(column)degree-coefficient matrix of $D_{Ei_{\tilde{\mathcal{J}}}}(s)$ which is the sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degree is greater than $\tilde{k}_l$.

From equation (8) (substituting $(\mathcal{J} - 1)$ in place of j), we have

$$D_{Ei_{\mathcal{J}-1}}^{hc} X_{i,\mathcal{J}-1} = D_{m_{\mathcal{J}-1}}^{hc}, \forall i \in \{1, 2, \ldots, N\}. \tag{11}$$

Comparing equation (11) with equation (10) (and noting that the matrix $D_{m_{\mathcal{J}-1}}^{hc}$ has full column rank $\sigma_{\mathcal{J}-1}$, while the matrix $\tilde{D}_{min_{\tilde{\mathcal{J}}}}^{hc}$ has full column rank $\tilde{\sigma}_{\tilde{\mathcal{J}}} \geq l$,), and also noting the fact that $\sigma_{\mathcal{J}-1} < l$, we conclude that the column rank of the matrix $\tilde{D}^{hc}_{min_{\tilde{\mathcal{J}}}}$ is greater than that of the matrix $D^{hc}_{m_{\mathcal{J}-1}}$. Therefore, we have

$$\tilde{k}_l > k^{group}_{\mathcal{J}-1}. \tag{12}$$

Assume that we are seeking the common hc-multiplier of $D_m(s)$ following the steps of Method 1, and $k^{group}_{\mathcal{J}-1}$ and $\sigma_{\mathcal{J}-1}$ (for $j = \mathcal{J} - 1$) are already achieved (see equation (11)). We are in the step of searching k^{group}_{j+1} and σ_{j+1}. Obviously, $k^{group}_{\mathcal{J}}$ and $\sigma_{\mathcal{J}}$ is one possible choice for k^{group}_{j+1} and σ_{j+1} because they are actually assumed to be achieved by using Method 1. On the other hand, $\tilde{k}_l$ and $\tilde{\sigma}_{\tilde{\mathcal{J}}}$ are also possible choice of k^{group}_{j+1} and σ_{j+1} (see equation (10) and considering that $\tilde{k}_l > k^{group}_{\mathcal{J}-1}$ (see equation(12)) and $\tilde{\sigma}_{\tilde{\mathcal{J}}} \geq l > \sigma_{\mathcal{J}-1}$). On consideration of Step 3 of Method 1, we recall that each $k^{group}_{\mathcal{J}}$ is the **minimal** integer (searching from $\sigma_{\mathcal{J}-1} + 1$) such that equation (8) can be satisfied, and hence we conclude that $k^{group}_{\mathcal{J}} \leq \tilde{k}_l$. However, this **contradicts** our earlier statement that $\tilde{k}_l < k^{group}_{\mathcal{J}}$ (see equation(9)).

Hence, the assumption is incorrect, and the conclusion of Theorem 5 holds. ■

We will present a simple example to explain how to use Method 1 to achieve a minimal common hc-multiplier for two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$.

Example 1: Using Method 1 to achieve a minimal common hc-multiplier for two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$, where

$$D_{E1}(s) = \begin{bmatrix} 0 & 2s^2 & s^3 & 0 & 0 \\ 0 & s^2+5s & 0 & 0 & 0 \\ s+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s^5 \\ 0 & 0 & 0 & s^4 & 0 \end{bmatrix}, D_{E2}(s) = \begin{bmatrix} 0 & 2s^2+1 & 0 & 0 & s^5 \\ 0 & s^2 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^4 & 0 \\ 0 & 0 & s^3+s^2 & 0 & 0 \end{bmatrix}.$$

1. The highest degree of the first column in the two Popov polynomial matrices is equal to 1, i.e. $k^{max}_1 = 1$. So, we begin searching from $k^{max}_1 = 1$, and we achieve that $k^{group}_1 = 1$, $\sigma_1 = 1$, and

$$D^{hc}_{E1_1} \cdot X_{11} = D^{hc}_{E2_1} \cdot X_{21} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot 1 = D^{hc}_{m_1}.$$

2. $k_2^{group} = 2, \sigma_2 = 2$, and

$$D_{E1_2}^{hc} \cdot X_{12} = D_{E2_2}^{hc} \cdot X_{22} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot I_2 = D_{m_2}^{hc}.$$

3. $k_3^{group} = 4$, $\sigma_3 = 3$ (If we try $k_3^{group} = 3$, the maximum value of σ_3 ensuring satisfaction of (8) is equal to $\sigma_2 = 2$, which is not acceptable since the algorithm requires $\sigma_3 > \sigma_2$.)

$$D_{E1_3}^{hc} \cdot X_{13} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D_{m_3}^{hc},$$

and

$$D_{E2_3}^{hc} \cdot X_{23} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = D_{m_3}^{hc}.$$

4. $k_4^{group} = 5$, $\sigma_4 = 5$,

$$D_{E1_4}^{hc} \cdot X_{14} = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = D_{m_4}^{hc},$$

and

$$D_{E2_4}^{hc} \cdot X_{24} = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = D_{m_4}^{hc}.$$

5. Then, we achieve the minmal common hc-multiplier $D_m(s)$ for two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$:

$$D_m(s) = \begin{bmatrix} 0 & 2s^2 & 0 & 0 & s^5 \\ 0 & s^2 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^5 & 0 \\ 0 & 0 & s^4 & 0 & 0 \end{bmatrix}.$$

Based on former results and the dual relationship between multirealisation forms $\{A_0+B_0K_i, B_0, C_i\}$ and $\{A_0+F_iC_0, B_i, C_0\}$, a minimal multirealisation $\{A_0+F_iC_0, B_i, C_0\}$ which ensure **bumpless** transfer can then be constructed according to the following method.

Method 2: 1.Find a right irreducible MFD for each $H_i^T(s)$ $i \in \{1, \cdots, N\}$, and transfer them to Popov MFDs. That is $H_i^T(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$.

2.According to Method 1, a **minimal** common hc-multiplier $D_m(s) = diag\{s^{\gamma_1}, \cdots, s^{\gamma_m}\}$ can be constructed for the set of Popov polynomial matrices $D_{Ei}(s)$ $i \in \{1, \cdots, N\}$. Each $H_i^T(s)$ can be rewritten as $N_{Ei}(s)D_{Ei}^{-1}(s) = \tilde{N}_{Ei}(s)\tilde{D}_{Ei}^{-1}(s) = \tilde{N}_{Ei}(s)\Lambda_i(s)\ [\tilde{D}_{Ei}(s)\Lambda_i(s)]^{-1}$ $= N_{Ei}(s)D_{Ei}^{-1}(s)$ (See Step 2 of Method 1).

3.Construct a **stable** polynomial matrix $D_{ms}(s)$ such that $\mathcal{D}_{hc}[D_{ms}(s)] = D_m(s)$. By using the method in [11] pp403-407, a controller form realisation $\{A_{c0}, B_{c0}, C_{c0}\}$ of $D_{ms}^{-1}(s)$ can be found with the pair (A_{c0}, B_{c0}) controllable and A_{c0} stable. Let $C_{ci} = N_{Eilc}$ and $K_i = D_{mslc} - D_{Eilc}$. A generic minimal multirealisation for the set of linear multivariable systems $H_i^T(s)$ $i \in \{1, \cdots, N\}$ is $\{A_{c0} + K_iB_{c0}, B_{c0}, C_{ci}\}$.

4.Denote $A_0 = A_{c0}^T$, $B_i = C_{ci}^T$, $C_0 = B_{c0}^T$ and $F_i = K_i^T$. Then, $\{A_0 + F_iC_0, B_i, C_0\}$ is a generic minimal stably based multirealisation for the set of linear multivariable systems $H_i(s)$ $i \in \{1, \cdots, N\}$.

III. Conclusion

This paper deals with the minimal multirealisation problem for linear multivariable systems, one motivation for which comes from Multiple Model Adaptive Control. This problem is simplified to a minimal common hc-multiplier problem, which is then solved. The results provides an efficient and practical way to implement the multi-controllers for the MMAC approach.

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