Minimal multirealisation of MIMO linear systems

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Abstract

This paper explores the minimal multirealisation problem, which is the determination of a minimal degree, parameter-dependent, state variable description to express a finite set of linear multivariable systems. The form of the parameter-dependent state variable description is selected as a feedback form to implement “state sharing” and “bumpless transfer”, which are possible ways to improve poor transient responses for switching control. The problem is solved by finding a special kind of minimal multiplier for a finite set of polynomial matrices.

Index Terms – System multirealisation, linear multivariable systems, switching systems, Multiple Model Adaptive Control.

I. INTRODUCTION

As an extension of the concept of state variable realisation of a single transfer function, the multirealisation of linear systems deals with the task of finding a parameter-dependent state variable description to realise a finite set of linear systems. The minimal multirealisation problem is that of ensuring that the parameter-dependent state variable realisation is of minimal degree.

The motivation for investigating minimal multirealisation problems partially originates from multiple model adaptive control (MMAC) algorithms [1] [2] [3] [4] [5] [6] [7]. Multirealisation

This research was supported by the US Office of Naval Research, grant number N00014-97-1-0946 and by the Australian Research Council and Australian Department of Communication, Information Technology and the Arts under the Australian Government Centre of Excellence Program.

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is relevant in efficiently realising the multicontroller structure of MMAC. It is observed [5] that a MMAC system only needs to generate one control signal at any time, because only one of the constituent controllers is applied to the plant at any one instant of time. It may be possible to efficiently generate all control signals by just using a single system with adjustable parameters, an observation giving rise to the multirealisation problem. In this paper, the multicontroller structure is implemented by using a single stable linear system with adjustable parameters. The state of the stable linear system is shared by the family of controllers, and the bumpless switching of controllers is implemented by adjusting the parameter dependent feedback (see Definition 1). This implementation is termed a state sharing multirealisation using parameter dependent feedback. State sharing additionally has the potential to ameliorate the poor transient response problem that can arise due to controller switching, and also is efficient in avoiding use of unnecessarily many parameters. Our previous work [8] presented an algorithm for multirealisation that did not necessarily secure minimal degree. Minimal degree multirealisation is the focus of this paper.

The realisation of a **single** linear system involves more tools than just state-variable realisation themselves, especially in the MIMO case, with matrix fraction descriptions being valuable in connecting to canonical state variable realisations, see eg [9] [10] [11] [12] [13] [14] [15]. The same turns out to be true for the multirealisation problem, which has been much less studied: in the MIMO case, introducing matrix fraction descriptions is of great utility. Past work on the implementation of multicontroller structure includes that of Morse [12], in the context of examining MMAC for scalar plants, and forthcoming work [8] by the authors on MIMO multirealisation obtaining realisations which are minimal only generically so that the algorithm on occasions does not lead to a minimal multirealisation.

For different purposes, the form of realisation could be different. For studying uncertain systems, linear fractional transformation (LFT) form representations of systems with parametric [16] or structured [17] uncertainties have been developed for the establishment of a comprehensive theory of system analysis and synthesis [18] [19]. To efficiently realise multicontroller structure of switching control systems, this paper investigate two multirealisation forms \( \{A_0 + B_0 K_i, B_0, C_i\} \) and \( \{A_0 + F_i C_0, B_i, C_0\} \) (which are dual). The multirealisation form \( \{A_0 + F_i C_0, B_i, C_0\} \) is prefered for the implementation of multicontroller structures, because it can ensure that the output of the switched system remains continuous across switching instants, provided its input is reasonably well behaved, e.g. is piecewise continuous, i.e. “bumpless” transfer [5] is achieved.
However, it is slightly more convenient to investigate the dual form \{A_0 + B_0K_i, B_0, C_i\} because for this multirealisation form we can directly lift known results on the invariant description of linear multivariable systems [11] [13]. Corresponding results for the multirealisation form \{A_0 + F_iC_0, B_i, C_0\} can be easily achieved by using the duality relationship (e.g. see Method 2).

The definition of the concept of minimal stably based multirealisation is given as below:

**Definition 1:** Assume that there is given a number \(N\) of \(m\)-input \(p\)-output strictly proper real rational transfer function matrices \(P_i (i \in \{1, \cdots, N\})\). A multirealisation of the set of systems \(P_i\) is a set of state variable realisations \{\(A_0 + B_0K_i, B_0, C_i\)\} (with the pair \((A_0, B_0)\) being controllable and adjustable parameter matrices \(C_i\) and \(K_i\)) realising all the systems \(P_i (i \in \{1, \cdots, N\})\). If all eigenvalues of \(A_0\) are in the left half plane, \{\(A_0 + B_0K_i, B_0, C_i\)\} is termed a stably based multirealisation of the set of systems \(P_i (i \in \{1, \cdots, N\})\). Furthermore, if the dimension of \(A_0\) is the smallest of all such stably based multirealisations, then we call \{\(A_0 + B_0K_i, B_0, C_i\)\} a minimal stably based multirealisation of the set of systems \(P_i (i \in \{1, \cdots, N\})\).

Because of the assumption of controllability of the pair \((A_0, B_0)\), it is evident that the requirement that the multirealisation be stably based poses no extra theoretical challenge (If \(A_0\) is not stable, find \(\bar{K}\) so that \(A_0 + B_0\bar{K}\) is stable, and replace \(K_i\) by \(K_i - \bar{K}\)). It is important in implementation for a multirealisation to be stably based [4].

Standard concepts and notations, such as column reduced polynomial matrices, are defined as in [11]. A new operator \((\mathcal{D}_{hc}\{\}\})\) is introduced as below:

**Definition 2:** Given a polynomial matrix \(D(s)\), it is always possible to write \(D(s) = D_{hc}S(s) + D_{lc}\Psi(s)\). Where, \(S(s) \overset{\Delta}{=} \text{diag}\{s^{k_1}, s^{k_2}, \cdots, s^{k_m}\}\), \(k_i\) is the degree of the \(i\)-th column \(^1\) of \(D(s)\), \(D_{hc}\) is a matrix formed from the coefficients of the highest degree polynomials in the columns of \(D(s)\) (highest-degree-coefficient matrix), \(\Psi^T(s) \overset{\Delta}{=} \text{block diag}\{[s^{k_1-1}, \cdots, s, 1], [s^{k_m-1}, \cdots, s, 1]\}\), and \(D_{lc}\) is a matrix formed from the remaining coefficients of polynomials in the columns of \(D(s)\) (lower-degree-coefficient matrix).

Define the operator \(\mathcal{D}_{hc}(\cdot)\) as \(\mathcal{D}_{hc}(D(s)) = D_{hc}S(s)\).

In the next section, necessary and sufficient conditions for the multirealisation of multivari-

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\(^1\)The \(i\)-th column degree of a polynomial matrix is the highest degree of all entries in the \(i\)-th column.
able systems are presented first. Then, the solution of the minimal multirealisation problem is provided.

II. MINIMAL STABLY BASED MULTIREALISATION FOR MULTIVARIABLE SYSTEMS

A. Conditions for multirealisations

Necessary and sufficient conditions for any given set of linear systems with compatible input and output dimensions are presented in the following theorem.

Theorem 1: Consider a set of $m$-input $p$-output strictly proper systems $H_i(s)$ ($i \in \{1, 2, \ldots, N\}$).

The following two statements 1 and 2 are equivalent.

1. There exists a controllable pair $(A_0, B_0)$ ($\text{dim}\{A_0\} = n$), and appropriately dimensioned real matrices $C_i$ and $K_i$ (for $i \in \{1, 2, \ldots, N\}$) such that $A_0$ is stable, and \{\(A_0 + B_0K_i\), $B_0$, $C_i$\} is a controllable realisation of system $H_i(s)$, (for $i \in \{1, 2, \ldots, N\}$).

And

2. There exists a right polynomial MFD for each system $H_i(s)$ described by $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$ (where $D_{Ei}(s)$ is a Popov polynomial matrix [11] [13] with degree $n$, i.e. $\text{deg}\{D_{Ei}(s)\} = n, \forall i \in \{1, 2, \ldots, N\}$) such that

i) $k_{il} = k_{jl}$ for $i, j \in \{1, 2, \ldots, N\}$ and $l \in \{1, 2, \ldots, m\}$, where $k_{ij}$ is the $j$-th column degree of the matrix $D_{Ei}(s)$, and

ii) the matrix $D_{hcEi}$, which is the highest-degree-coefficient matrix of the $D_{Ei}(s)$, is identical for $i \in \{1, 2, \ldots, N\}$.

Proof: The theorem can be proved by using the relationship between invariant Popov parameters of a controllable pair $(A, B)$ and the coefficients in a Popov form matrix $D_{E}(s)$ [11] [13]. The detailed proof can be seen in [8] or [20].

From Theorem 1, we can derive that the minimal degree of the multirealisation of SISO systems is equal to the maximum McMillan degree of any of the $H_i(s)$. For the MIMO systems, Theorem 1 gives no guidance as to the minimal dimension, it may turn out that it is not possible to obtain a dimension as low as the maximum McMillan degree of any of the $H_i(s)$, due to varying possible column degrees. In the next subsection, we provide a method to find the minimal degree of the multirealisation of MIMO systems.
B. Minimal multirealisation

In order to simplify our discussion, we present a problem that is equivalent to the minimal stably based multirealisation problem. We call it the “minimal common hc- (highest column degree) multiplier problem” for a set of polynomial matrices.

Problem 1: Given a finite set of square \((m \times m)\) column-reduced polynomial matrices \(D_i(s)\), find nonsingular stable polynomial matrices \(X_i(s)\) (that is, the zeros of \(\det(X_i(s))\) lie in the left half plane \(Re(s) < 0\)) such that there exists a column-reduced polynomial matrix \(D_{min}(s)\) with the property that

\[
D_{hc}[D_i(s)X_i(s)] = D_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\},
\]

and \(D_{min}(s)\) has the lowest possible degree.

Although the minimal common hc- (highest column degree) multiplier problem is actually equivalent to the minimal stably based multirealisation problem, here we are only particularly interested in whether it is possible to construct the minimal stably based multirealisation from the solution of the minimal common hc- (highest column degree) multiplier problem.

Theorem 2: Consider a set of \(m\)-input \(p\)-output strictly proper systems \(H_i(s) (i \in \{1, 2, \cdots, N\})\) described by right polynomial MFDs, i.e. \(H_i(s) = N_i(s)D_i^{-1}(s)\), and \((N_i(s), D_i(s))\) are right coprime polynomial matrices. If for the set of polynomial matrices \(D_i(s)\), one can find a minimal common hc-multiplier (as stated in Problem 1) \(D_{min}(s)\), i.e. the column reduced polynomial matrix \(D_{min}(s)\) satisfies equation (1) with the lowest possible degree, then, a minimal stably-based multirealisation \(\{A_0 + B_0K_{qi}, B_0, C_{qi}\}\) with \(\dim\{A_0\} = \deg\{D_{min}\}\) for the set of systems \(H_i(s)\) can be constructed.

Proof: The proof is based on the following two steps. The first step is to construct a stably based multirealisation with \(\{A_0 + B_0K_{qi}, B_0, C_{qi}\}\) with \(\dim\{A_0\} = \deg\{D_{min}\}\). The second step is to prove by contradiction based on the results of Theorem 1 that this multirealisation is minimal.

In order to solve Problem 1, we introduce a new concept, hc-\((highest\ column\ degree)\) dependence on a set of polynomial vectors.

Definition 3: A polynomial vector \(d_i(s)_{n \times 1}\) is hc-(highest column degree) dependent on a collection of polynomial vectors \(d_i(s)_{n \times 1}, i = 1, 2, \cdots, m\) if there exists a set of scalar polynomials
\( r_i(s) \) such that

\[
\mathcal{D}_{hc}\{d_e(s)\} = \mathcal{D}_{hc}\{\sum_{i=1}^{m} r_i(s)d_i(s)\}.
\]

In Problem 1, it can be seen that each column of the minimal polynomial matrix \( D_{min}(s) \) must be \( hc \)-dependent on the columns of \( D_i(s) \) for each \( i \in \{1, 2, \cdots, N\} \). The following theorem provides necessary and sufficient condition for \( hc-(\text{highest column degree}) \) dependence.

**Theorem 3:** Assume there is given a collection of polynomial vectors \( d_i(s)_{n \times 1}, i = 1, 2, \cdots, m \), such that their column degrees, \( k_i \), are ordered as

\[
k_1 \leq k_2 \leq \cdots k_m.
\]

Assume further that the matrix \([d_1(s) d_2(s) \cdots d_m(s)]\) is such that \( D_{hc} = [d_{hc}^1 d_{hc}^2 \cdots d_{hc}^m] \) has full column rank. Then a given polynomial vector \( d_e(s)_{n \times 1} \) (with column degree \( k_e \)) is \( hc \)-dependent on the collection of polynomial vectors \( d_i(s), i = 1, 2, \cdots, m \) if and only if the real vector \( d_{hc}^e \) (the highest-(column)degree-coefficient vector of \( d_e(s) \)) is a linear combination of real vectors \( d_{hc}^1, d_{hc}^2, \cdots, d_{hc}^l \) where \( l = \max_i\{\arg\{k_i \leq k_e\}\} \).

**Proof:** (Forward Implication) If \( \mathcal{D}_{hc}\{d_e(s)\} = \mathcal{D}_{hc}\{\sum_{i=1}^{m} r_i(s)d_i(s)\} \), for some polynomial \( r_i(s) \), then \( d_e(s) + g(s) = \sum_{i=1}^{m} r_i(s)d_i(s) \), where \( g(s) \) is a polynomial vector with column degree less than \( k_e \). According to Theorem 6.3-13 in pp387 of [11], if \( k_i > k_e \), we must have \( r_i(s) = 0 \), and the ordering of \( k_i \) and the definition of \( l \) imply that

\[
d_e(s) + g(s) = \sum_{i=1}^{l} r_i(s)d_i(s).
\]

If \( d_{hc}^e \) is not a linear combination of real vectors \( d_{hc}^1, d_{hc}^2, \cdots, d_{hc}^l \), then \( d_e(s), d_1(s), \cdots, d_l(s) \) are linearly independent. Considering that the column degree of \( g(s) \) is less than \( k_e \), equation (2) is impossible. Then, the necessity is proved.

(Reverse Implication) If the real vector \( d_{hc}^e \) is a linear combination of real vectors \( d_{hc}^1, d_{hc}^2, \cdots, d_{hc}^l \), then

\[
d_{hc}^e = \sum_{i=1}^{l} r_i d_{hc}^i,
\]

where \( r_i, i \in \{1, \cdots, l\} \) are real numbers.

It follows that

\[
d_{hc}^e s^{k_e} = \sum_{i=1}^{l} r_i s^{k_e-k_i} d_{hc}^i s^{k_i} = \mathcal{D}_{hc}\{\sum_{i=1}^{l} r_i s^{k_e-k_i} d_{hc}^i s^{k_i}\}.
\]
Therefore, setting $r_i(s) = r_is^{k_e-k_i}$, we have
\[
D_{hc}\{d_e(s)\} = d_{hc}^{ke}s^{ke} = D_{hc}\{\sum_{i=1}^{m} r_i(s)d_i(s)\}.
\]

Let us first indicate two simplifications to Problem 1. If in the problem statement any $D_i(s)$ is replaced by $\tilde{D}_i(s) = D_i(s)U_i(s)$ where $U_i(s)$ is unimodular, but otherwise arbitrary, then the problem is effectively unchanged. In fact, the solution $X_i(s)$ is just replaced by $U_i^{-1}(s)X_i(s)$. Second, if $D_{min}(s)$ is a minimal common $hc$-multiplier for a set $D_i(s)$ for $i = 1, 2, \ldots, N$, so is $D_{min}(s)V$, where $V$ is a permutation matrix. Effectively, $X_i(s)$ is replaced by $X_i(s)V$.

In particular then, without loss of generality, we can assume $D_i(s)$ is a Popov form matrix $D_{Ei}(s)$, and seek a column degree ordered (see Definition 4 below) $D_{min}(s)$.

**Definition 4:** A column reduced polynomial matrix $D(s)$ is said to be “column degree ordered” [21] if the columns of the matrix $D(s)$ are ordered according to increasing column degrees $k_1 \leq k_2 \leq \cdots \leq k_m$. Suppose
\[
k_1 = \cdots = k_{r_1} < k_{r_1+1} = \cdots = k_{r_1+r_2} < k_{r_1+r_2+1} \cdots \leq k_m, \tag{3}
\]
i.e., the columns are arranged in groups of $r_j$ columns with the same column degree.

Let the number of groups of columns with equal degree be $q$, so that $k_{r_1+r_2+\cdots+r_q} = k_m$ and note that each column group has the same column degree $k_{j}^{\text{group}} (j \in \{1, 2, \ldots, q\})$. Further define
\[
\sigma_j = \sum_{l=1}^{j} r_l, j \in \{1, 2, \ldots, q\},
\]
and also define $D_j(s)$ as the sub-matrix of $D(s)$ obtained by deleting the columns whose column degree are greater than $k_{j}^{\text{group}} (j \in \{1, 2, \ldots, q\})$, and $D_{hc}^{j}$ to be the highest-(column)degree-coefficient matrix of the polynomial matrix $D_j(s)$.

**Definition 5:** A column reduced polynomial matrix $D_{mu}(s)_{m \times m}$ is termed $hc$-dependent on another square polynomial matrix $D(s)_{m \times m}$ if there exists a polynomial matrix $X(s)$ such that
\[
\mathcal{D}_{hc}\{D(s)X(s)\} = \mathcal{D}_{hc}\{D_{mu}(s)\}.
\]

In the method below, we will consider simultaneous $hc$-dependence of $D_{min}(s)$ on a number of Popov matrices $D_{Ei}(s)$, $i \in \{1, 2, \cdots, N\}$. The following theorem considers dependence on just one of these matrices.
Theorem 4: Assume that a column reduced polynomial matrix \( D_m(s)_{m \times m} \) is "column degree ordered", and \( k^\text{group}_j \) is defined corresponding to Definition 4. Consider also a particular Popov polynomial matrix \( D_{Ei}(s)_{m \times m} \). Let \( D^\text{hc}_{Ei} \) denote the highest-(column)degree-coefficient matrix of the polynomial matrix \( D_{Ei}(s) \), let \( D_{Ei}(s) \) denote the sub-matrix derived from \( D_{Ei}(s) \) by deleting the columns whose column degree are greater than \( k^\text{group}_j \) \( (j \in \{1, 2, \ldots, q\}) \), and let \( D^\text{hc}_{Ei} \) denote the highest-(column)degree-coefficient matrix of the polynomial matrix \( D_{Ei}(s) \).

Then the polynomial matrix \( D_m(s) \) is \( hc \)-dependent on the Popov polynomial matrix \( D_{Ei}(s) \) if and only if there exists a set of real matrices \( X_{ij} \) with \( j \in \{1, 2, \ldots, q\} \) such that

\[
D^\text{hc}_{Ei} X_{ij} = D^\text{hc}_m, \forall j \in \{1, 2, \ldots, q\}.
\]

(4)

Proof: By considering the necessary and sufficient conditions for \( hc \)-dependence on one polynomial vector given in Theorem 3 and noting that \( D_m(s) \) and \( D_{Ei}(s) \) are "column degree ordered", the conclusion is straightforward.

Theorem 4 presents a condition (see equation (4)) for \( hc \)-dependence of a polynomial matrix. Next, we will consider the minimal common \( hc \)-multiplier for a set of polynomial matrices based on this theorem. Specifically, we present a method which uses elementary column operations and multiplication of columns by powers of \( (s + a) \) to achieve a common \( hc \)-multiplier of a set of Popov polynomial matrices \( D_{Ei}(s) \) \( (i \in \{1, 2, \ldots, N\}) \). This method consists of searching for a set \( k^\text{group}_j \) and \( \sigma_j \) \( (j \in \{1, 2, \ldots, q\}) \) in order to construct a common \( hc \)-multiplier \( D_m(s) \). Later, we will prove the method provides a minimal common \( hc \)-multiplier.

Method 1: \(^2\) Step 1. Consider the matrices \( D_{Ei}(s) \), define \( k^\text{max}_1 \) as the highest degree of the first column in all \( D_{Ei}(s) \), i.e. \( k^\text{max}_1 = \max\{k_{ij}\} \). By multiplication by \( (s + a)^{k^\text{max}_1 - k_{ij}} \) of any column whose column degree \( k_{ij} \) is less than \( k^\text{max}_1 \), one can make each \( D_{Ei}(s) \) to have the lowest column degree \( k^\text{max}_1 \). Here \( k_{ij} \) is the \( j \)-th column degree of the matrix \( D_{Ei}(s) \). Denote each transformed matrix as \( D^0_{Ei}(s) \).

Step 2. We search for a value of \( k^\text{group}_1 \) starting from \( k^\text{max}_1 \), and trying in turn \( k^\text{max}_1, k^\text{max}_1 + 1, \ldots \) until a certain condition (given by equation (5) below) is satisfied.

In more detail, try \( k^\text{group}_1 = k^\text{max}_1 \) first. For each \( D_{Ei}(s) \), denote \( D_{Ei}(s) \) as the sub-matrix derived from \( D_{Ei}(s) \) by deleting the columns whose column degree are greater than \( k^\text{group}_1 \)

\(^2\)The reader may find it helpful to review Example 1 below partway through the description of the method.
(\(i \in \{1, 2, \ldots, N\}\)), and \(D_{Ei}^{hc}\) as the highest-(column)degree-coefficient matrix of the polynomial matrix \(D_{Ei}(s) (i \in \{1, 2, \ldots, N\})\).

a) If for the set of real matrices \(D_{Ei}^{hc}\), there exist constant real matrices \(X_{i1}\) and a real matrix \(D_{m1}^{hc}\), such that

\[
D_{Ei}^{hc} X_{i1} = D_{m1}^{hc}, \forall i,
\]

where \(D_{m1}^{hc}\) has full column rank and the largest possible number of columns (\(\sigma_1 > 0\)), then it is possible to post-multiply each \(D_{Ei}^{hc}(s)\) (generated in Step 1) by a real constant matrix to make them have the same \(D_{m1}^{hc} \in \mathbb{R}^{m \times \sigma_1}\) for the first \(\sigma_1\) columns. Denote each transformed matrix as \(D_1^{hc}(s)\).

b) If for \(k_{i1}^{group} = k_{i1}^{max}\), equation (5) has no solution (i.e. \(\sigma_1 = 0\)), then we increase \(k_{i1}^{group}\) by one (that is \(k_{i1}^{group} = k_{i1}^{max} + 1\) and repeat the process above, seeking a nontrivial solution to (5)). Keep on searching until a minimal value of \(k_{i1}^{group}\) is achieved such that equation (5) has a solution \(X_{i1}\) for \(i = 1, \cdots, N\). By multiplication by \((s + a)^{k_{i1}^{group} - k_{ij}}\) of any column whose column degree \(k_{ij}\) is less than \(k_{i1}^{group}\), one can make each \(D_{Ei}(s)\) have the lowest column degree \(k_{i1}^{group}\). If we denote each transformed matrix as \(D_1^{hc}(s)\), then it is possible to post-multiply it by a real constant matrix to make each \(D_1^{hc}(s)\) have the same corresponding \(D_{m1}^{hc} \in \mathbb{R}^{m \times \sigma_1}\) for the first \(\sigma_1\) columns. Denote each transformed matrix as \(D_1^{hc}(s)\).

There always exists a value of \(k_{i1}^{group} \leq k_{i1}^{max}\) (where \(k_{i1}^{max}\) is the highest column degree of all \(D_{Ei}(s)\), i.e. \(k_{i1}^{max} = \max_{i,j} \{k_{ij}\}\), for \(i \in \{1, \cdots, N\}, j \in \{1, \cdots, m\}\) such that equation (5) has a solution. This is because, for the case \(k_{i1}^{group} = k_{i1}^{max}\), a common \(hc\)-multiplier will be \(sk_{i1}^{max} I_m\).

Step 3. Search for a minimal integer \(k_{i2}^{group}\) (searching from \(k_{i1}^{group} + 1\)) and a set of real matrix \(X_{i2}\) for the set of polynomial matrices \(D_{Ei}(s)\) such that

\[
D_{Ei2}^{hc} X_{i2} = D_{m2}^{hc}, \forall i,
\]

with \(D_{m2}^{hc} \in \mathbb{R}^{m \times \sigma_2}\) having full column rank and the largest possible number (\(\sigma_2 > \sigma_1\)) of columns. Recall that \(D_{Ei2}^{hc}\) is the highest-(column)degree-coefficient matrix of each \(D_{Ei2}(s)\) which is a sub-matrix derived from \(D_{Ei}(s)\) by deleting the columns whose column degrees are greater than \(k_{i2}^{group}\). Based on equations (5) and (6), we can find a nonsingular real matrix \(R_2\) such that

\[
D_{m2}^{hc} R_2 = [D_{m1}^{hc}; D_{m2}^{hc}]\Delta_2.
\]
For each $D_{Ei}(s)$, define $l_{Ei2}$ as the number of columns of $D_{Ei}(s)$ whose degree is no more than $k_2^{\text{group}}$ ($i \in \{1,2,\ldots,N\}$), i.e. $l_{Ei2} = \max_j \{\arg_j \{k_{ij} \leq k_2^{\text{group}}\} \}$. Multiply the polynomial matrices $D^l_{Ei}(s)$ (achieved in Step 3) by $(s+a)^{k_2^{\text{group}}-k_{ij}}$ ($\sigma_1 < j \leq l_{Ei2}$) from the $(\sigma_1+1)$-th column to the $l_{Ei2}$-th column, and denote the new matrices so obtained as $D^U_{Ei}(s)$. According to equation (7), it is possible to post multiply by a corresponding unimodular polynomial matrix to transform each matrix $D^U_{Ei}(s)$ to have the same $D_{hc}^{m_1} \in \mathbb{R}^{m \times \sigma_1}$ for the first $\sigma_1$ columns (with column degree all equal to $k_1^{\text{group}}$), and the same $D_{hc}^{m_\Delta} \in \mathbb{R}^{m \times (\sigma_2-\sigma_1)}$ for the columns from $(\sigma_1+1)$-th columns to $\sigma_2$-th columns (with column degree all equal to $k_2^{\text{group}}$). Denote each transformed matrix as $D^2_{Ei}(s)$.

Repeat Step 3. This will eventually derive the common $hc$-multiplier

$$D_m(s) = D_{hc}\{D^q_{Ei}(s)\}, \forall i \in \{1,2,\ldots,N\}$$

for all Popov polynomial matrices $D_{Ei}(s)$.

In this process, the values of $k_j^{\text{group}}$ and $\sigma_j$ are determined for $j > 2$ in an identical manner to the determination of $k_1^{\text{group}}$ in Step 2-3. If we define $D_{Eij}(s)$ ($i \in \{1,2,\ldots,N\}$ and $j \in \{1,2,\ldots,q\}$) as a sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degrees are greater than $k_j^{\text{group}}$, and $D_{Eij}^{hc}$ is the highest-(column)degree-coefficient matrix of $D_{Eij}(s)$, then, there exist a set of real matrices $X_{ij}$ ($i \in \{1,2,\ldots,N\}$ and $j \in \{1,2,\ldots,q\}$) such that

$$D_{Eij}^{hc} X_{ij} = D_{m_j}^{hc}, \forall i \in \{1,2,\ldots,N\}, \forall j \in \{1,2,\ldots,q\}.$$  \hspace{1cm} (8)

The real matrix $D_{m_j}^{hc}$ has full column rank, and $\sigma_j$ is equal to the number of columns in the matrix $D_{m_j}^{hc}$.

To derive a solution for equations (5) and (6) or to identify that no such solution exists is not difficult, because each column of each $D_{Ei}^{hc}$, the highest-(column)degree-coefficient matrix of $D_{Ei}(s)$, has a unique pivot index. Method 1 presents a way to achieve a common $hc$-multiplier for a set of polynomial matrices. The following theorem confirms that it is also a minimal common $hc$-multiplier.
Similarly, we assume \( \tilde{J} \in \{ D(4) \) for fixed \( \tilde{\sigma} \).
Also \( \sigma \) matrix \( D^{hc}_{\tilde{J}} \) associated values \( k_{\tilde{J}}^{group} \) and \( \sigma_{\tilde{J}} \). The next step of the proof is just to confirm that the common \( hc \)-multiplier form defined by \( k_{\tilde{J}}^{group} \) and \( \sigma_{\tilde{J}} \) is also a minimal common \( hc \)-multiplier. Denote \( D_{Ei}(s) \) as the Popov polynomial matrix of matrix \( D_i(s) (i \in \{ 1, 2, \ldots, N \}) \).

Suppose that a minimal common \( hc \)-multiplier is given by \( \tilde{D}_{min}(s) \), without loss of generality with column degree ordered. We now prove the desired result by contradiction. To this end, assume that the common \( hc \)-multiplier \( D_m(s) \) (with parameters \( k_{\tilde{J}}^{group} \) and \( \sigma_{\tilde{J}} \)) achieved by using Method 1 is not a minimal common \( hc \)-multiplier. Then, there should exist an integer \( l (l \in \{ 1, 2, \ldots, m \}) \) such that the \( l \)-th column degree \( k_i \) of the multirealisation polynomial matrix \( D_m(s) \) is bigger than the \( l \)-th column degree \( \tilde{k}_i \) of the minimal common \( hc \)-multiplier \( \tilde{D}_{min}(s) \), i.e. \( \tilde{k}_i < k_i \).
Without loss of generality, we assume \( k_i = k_{\tilde{J}}^{group} \) (here, \( \tilde{J} \) is fixed and \( J \in \{ 1, 2, \ldots, q \} \)), so that \( \sigma_{\tilde{J} - 1} < l \leq \sigma_{\tilde{J}} \) and

\[
\tilde{k}_i < k_i = k_{\tilde{J}}^{group}.
\]
Similarly, we assume \( \tilde{k}_i = \tilde{k}_{\tilde{J}}^{group} \), then \( \tilde{\sigma}_{\tilde{J} - 1} < l \leq \tilde{\sigma}_{\tilde{J}} \) (here, \( \tilde{J} \) is fixed and \( \tilde{J} \in \{ 1, 2, \ldots, \tilde{q} \} \)). Also \( \sigma_0 = 0 \) and \( \tilde{\sigma}_0 = 0 \).

Because \( \tilde{D}_{min}(s) \) is a \( hc \)-multiplier of each \( D_{Ei}(s) \), according to Theorem 4 (See equation (4) ), for fixed \( \tilde{j} = \tilde{J} \), there exists a real matrices \( \tilde{X}_{i,\tilde{J}} \) for each \( D_{Ei}(s) \) such that

\[
D^{hc}_{Ei,\tilde{J}} \tilde{X}_{i,\tilde{J}} = \tilde{D}^{hc}_{min,\tilde{J}}, \forall i \in \{1, 2, \ldots, N\},
\]
where \( \tilde{D}^{hc}_{min,\tilde{J}} \) has full column rank \( \tilde{\sigma}_{\tilde{J}} \geq l \), and \( D^{hc}_{Ei,\tilde{J}} \) is the highest-(column)degree-coefficient matrix of \( D_{Ei,\tilde{J}}(s) \) which is the sub-matrix derived from \( D_{Ei}(s) \) by deleting the columns whose column degree is greater than \( \tilde{k}_i \).

From equation (8) (substituting \( (J - 1) \) in place of \( j \)), we have

\[
D^{hc}_{Ei,J - 1} X_{i,J - 1} = D^{hc}_{m,J - 1}, \forall i \in \{1, 2, \ldots, N\}.
\]
Comparing equation (11) with equation (10) (and noting that the matrix \( D^{hc}_{m,J - 1} \) has full column rank \( \sigma_{J - 1} \), while the matrix \( \tilde{D}^{hc}_{min,\tilde{J}} \) has full column rank \( \tilde{\sigma}_{\tilde{J}} \geq l \), and also noting the fact
that $\sigma_{J-1} < l$, we conclude that the column rank of the matrix $\tilde{D}_{hc}^{m,J}$ is greater than that of the matrix $\tilde{D}_{hc}^{m,J-1}$. Therefore, we have

$$\tilde{k}_l > k_{J-1}^{group}. \quad (12)$$

Assume that we are seeking the common $hc$-multiplier of $D_m(s)$ following the steps of Method 1, and $k_{J-1}^{group}$ and $\sigma_{J-1}$ (for $j = J - 1$) are already achieved (see equation (11)). We are in the step of searching $k_{J+1}^{group}$ and $\sigma_{J+1}$. Obviously, $k_{J}^{group}$ and $\sigma_{J}$ is one possible choice for $k_{J+1}^{group}$ and $\sigma_{J+1}$ because they are actually assumed to be achieved by using Method 1. On the other hand, $\tilde{k}_l$ and $\tilde{\sigma}_J$ are also possible choice of $k_{J+1}^{group}$ and $\sigma_{J+1}$ (see equation (10) and considering that $\tilde{k}_l > k_{J-1}^{group}$ (see equation(12)) and $\tilde{\sigma}_J \geq l > \sigma_{J-1}$). On consideration of Step 3 of Method 1, we recall that each $k_{J}^{group}$ is the minimal integer (searching from $\sigma_{J-1} + 1$ ) such that equation (8) can be satisfied, and hence we conclude that $k_{J}^{group} \leq \tilde{k}_l$. However, this contradicts our earlier statement that $\tilde{k}_l < k_{J}^{group}$ (see equation(9)).

Hence, the assumption is incorrect, and the conclusion of Theorem 5 holds.

We will present a simple example to explain how to use Method 1 to achieve a minimal common $hc$-multiplier for two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$.

Example 1: Using Method 1 to achieve a minimal common $hc$-multiplier for two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$, where

$$D_{E1}(s) = \begin{bmatrix}
0 & 2s^2 & s^3 & 0 & 0 \\
0 & s^2 + 5s & 0 & 0 & 0 \\
s + 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s^5 & 0 \\
0 & 0 & 0 & s^4 & 0
\end{bmatrix},$$

$$D_{E2}(s) = \begin{bmatrix}
0 & 2s^2 + 1 & 0 & 0 & s^5 \\
0 & s^2 & 0 & 0 & 0 \\
s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s^4 & 0 \\
0 & 0 & s^3 + s^2 & 0 & 0
\end{bmatrix}. $$

1. The highest degree of the first column in the two Popov polynomial matrices is equal to 1, i.e. $k_{1}^{max} = 1$. So, we begin searching from $k_{1}^{max} = 1$, and we achieve that $k_{1}^{group} = 1$, $\sigma_1 = 1$, and

$$D_{E1}^{hc} \cdot X_{11} = D_{E2}^{hc} \cdot X_{21} = \begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
0
\end{bmatrix} \cdot 1 = D_{m1}^{hc}.$$
2. $k_2^{\text{group}} = 2$, $\sigma_2 = 2$, and

$$D_{E12}^{hc} \cdot X_{12} = D_{E22}^{hc} \cdot X_{22} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = I_2 = D_{m2}^{hc}.$$ 

3. $k_3^{\text{group}} = 4$, $\sigma_3 = 3$ (If we try $k_3^{\text{group}} = 3$, the maximum value of $\sigma_3$ ensuring satisfaction of (8) is equal to $\sigma_2 = 2$, which is not acceptable since the algorithm requires $\sigma_3 > \sigma_2$.)

$$D_{E13}^{hc} \cdot X_{13} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D_{m3}^{hc},$$

and

$$D_{E23}^{hc} \cdot X_{23} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = D_{m3}^{hc}.$$ 

4. $k_4^{\text{group}} = 5$, $\sigma_4 = 5$,

$$D_{E14}^{hc} \cdot X_{14} = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = D_{m4}^{hc},$$

and

$$D_{E24}^{hc} \cdot X_{24} = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = D_{m4}^{hc}.$$
5. Then, we achieve the minimal common $hc$-multiplier $D_m(s)$ for two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$:

$$D_m(s) = \begin{bmatrix} 0 & 2s^2 & 0 & 0 & s^5 \\ 0 & s^2 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ 0 & 0 & s^5 & 0 \\ 0 & 0 & s^4 & 0 & 0 \end{bmatrix}.$$

Based on former results and the dual relationship between multirealisation forms $\{A_0 + B_0K_i, B_0, C_0\}$ and $\{A_0 + F_iC_0, B_i, C_0\}$, a minimal multirealisation $\{A_0 + F_iC_0, B_i, C_0\}$ which ensure bumpless transfer can then be constructed according to the following method.

Method 2: 1. Find a right irreducible MFD for each $H_i^T(s) i \in \{1, \cdots , N\}$, and transfer them to Popov MFDs. That is $H_i^T(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$.

2. According to Method 1, a minimal common $hc$-multiplier $D_m(s) = \text{diag}\{s^\gamma_1, \cdots , s^\gamma_m\}$ can be constructed for the set of Popov polynomial matrices $D_{Ei}(s) i \in \{1, \cdots , N\}$. Each $H_i^T(s)$ can be rewritten as $N_{Ei}(s)D_{Ei}^{-1}(s) = \tilde{N}_{Ei}(s)\tilde{D}_{Ei}^{-1}(s) = \tilde{N}_{Ei}(s)\Lambda_i(s) [\tilde{D}_{Ei}(s)\Lambda_i(s)]^{-1} = N_{Ei}(s)D_{Ei}^{-1}(s)$ (See Step 2 of Method 1).

3. Construct a stable polynomial matrix $D_{ms}(s)$ such that $D_{hc}[D_{ms}(s)] = D_m(s)$. By using the method in [11] pp403-407, a controller form realisation $\{A_{c0}, B_{c0}, C_{c0}\}$ of $D_{ms}^{-1}(s)$ can be found with the pair $(A_{c0}, B_{c0})$ controllable and $A_{c0}$ stable. Let $C_{ci} = N_{Ei,c}$ and $K_i = D_{ms,c} - D_{Ei,c}$. A generic minimal multirealisation for the set of linear multivariable systems $H_i^T(s) i \in \{1, \cdots , N\}$ is $\{A_{c0} + K_iB_{c0}, B_{c0}, C_{ci}\}$.

4. Denote $A_0 = A_{c0}^T$, $B_i = C_{ci}^T$, $C_0 = B_{c0}^T$ and $F_i = K_i^T$. Then, $\{A_0 + F_iC_0, B_i, C_0\}$ is a generic minimal stably based multirealisation for the set of linear multivariable systems $H_i(s) i \in \{1, \cdots , N\}$.

III. CONCLUSION

This paper deals with the minimal multirealisation problem for linear multivariable systems, one motivation for which comes from Multiple Model Adaptive Control. This problem is simplified to a minimal common $hc$-multiplier problem, which is then solved. The results provides an efficient and practical way to implement the multi-controllers for the MMAC approach.

November 6, 2004 DRAFT
REFERENCES


