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HETEROGENEOUS BELIEFS, RISK, AND LEARNING IN A SIMPLE ASSET-PRICING MODEL WITH A MARKET MAKER

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This paper studies the dynamics of a simple discounted present-value asset-pricing model where agents have different risk attitudes and follow different expectation formation schemes for the price distribution. A market-maker scenario is used as the market-clearing mechanism, in contrast to the more usual Walrasian scenario. In particular, the paper concentrates on models of fundamentalists and trend followers who follow recursive geometric-decay (learning) processes (GDP) with both finite and infinite memory. The analysis depicts how the dynamics are affected by various key elements (or parameters) of the model, such as the adjustment speed of the market maker, the extrapolation rate of the trend followers, the decay rate of the GDP, the lag length used in the learning GDP, and external random factors.

Keywords: Heterogeneous Beliefs, Market Maker, Asset Pricing, Geometric-Decay Learning Process

1. INTRODUCTION

Research into the dynamics of financial asset prices resulting from the interaction of heterogeneous agents having different attitudes about risk and having different expectations about the future evolution of prices has flourished in recent years [e.g., Day and Huang (1990), Bullard (1994), Lux (1995, 1997, 1998), Sethi (1996), Brock and Hommes (1997a), Franke and Sethi (1998), Bullard and Duffy (1999), Franke and Nesemann (1999), and Chiarella and He (2002c)]. This literature has developed as a result of a growing dissatisfaction with (i) models of asset price dynamics based on the representative-agent paradigm, as expressed, for example, by Kirman (1992); and (ii) the extreme informational assumptions of rational expectations.

Recently, Brock and Hommes (1997b, 1998) have introduced the concept of an *adaptively rational equilibrium*, where agents base decisions upon predictions of

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future values of endogenous variables whose actual values are determined by equilibrium equations. A key aspect of these models is that they exhibit expectations feedback. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance as measured by realized profits. The resulting dynamical system is nonlinear and, as Brock and Hommes (1998) show, capable of generating the entire "zoo" of complex behavior from local stability to high-order cycles and chaos as various key parameters of the model change.

The paper builds on the work of Chiarella and He (2002c), who extend the model of Brock and Hommes (1998) by allowing the agents to have different risk attitudes and different expectation formation schemes for both first and second moments of the price distribution. Both papers, in common with much of the earlier cited literature, use the Walrasian auctioneer scenario as the market-clearing mechanism. However, this scenario is an unsatisfactory explanation as to how market-clearing prices are arrived at in financial markets; see in this regard O'Hara (1995), who cites only one market in which market-clearing prices are arrived at via the Walrasian auctioneer scenario, and who also highlights the inadequacy of assuming this type of market-clearing mechanism. The paper presented here takes the model of Chiarella and He (2002c), but uses a market-maker scenario as the mechanism generating the market-clearing price.

In particular, the present paper investigates a simple model for the price dynamics involving only a few trading strategies.[1] It is assumed that there are three types of participants in the asset market: two groups of traders—fundamentalists and trend followers—and a market maker. The market maker at the beginning of each trading period announces a price and then receives all of the buy and sell orders for the risky asset in that time period formed by agents on the basis of the announced price. The market maker hence determines the excess demand and then takes an offsetting long or short position on the risky asset so as to clear the market. The market maker announces the price for the next trading interval as a function of the excess demand in the current period. Although this scenario is still highly stylized, it does bring the analysis closer to the functioning of real markets than does the Walrasian scenario.

There is a related literature on the behavior of the market maker in securities markets, starting with Garman (1976) and continuing with Stoll (1978), Beja and Goldman (1980), Ho and Stoll (1981), Day and Huang (1990), Peck (1990), Chiarella (1992), Sethi (1996), and Farmer and Joshi (2002). Broadly speaking, these papers can be categorized into two types. The first type illustrates the trade-offs faced by the market maker. In Ho and Stoll (1981), holding costs arise from the return risk faced by risk-averse market makers. Peck (1990) considers the opposite extreme in which a market maker faces no return risk but a significant risk of being forced to honor his or her promises as a result of adverse price changes. The second type belongs to the class of behavioral models for the price dynamics under heterogeneous expectations. Assuming linear trading rules for each type of trader, Beja and Goldman (1980) show that equilibrium is unstable when the fraction of trend

followers is sufficiently high. Using nonlinear investment rules, Day and Huang (1990) show that this could result in chaotic price series. The Beja and Goldman model was extended by Chiarella (1992), who made the trend-following rule nonlinear. It is found that, when the fraction of trend followers is sufficiently low, the equilibrium is stable, but when it exceeds a critical value, it becomes unstable and is replaced by a limit cycle. The excess demand of each trader type oscillates as the cycle is traversed, causing sustained deviations from the equilibrium price. This model was further enhanced by Sethi (1996), who studied inventory accumulation, cash flow, and the cost of information acquisition. Farmer and Joshi (2002) use a market-maker-based method of price formation to study the price dynamics induced by several commonly used financial trading strategies to show how they amplify noise, induce structure in prices, and cause phenomena such as excess and clustered volatility.

The market maker in this paper still remains highly stylized in that he or she does not change behavior, irrespective of the size of his or her long or short position. We assume that the market maker is risk neutral, setting the price in response to excess demands from the traders, without worrying about accumulated inventory. The market framework and price formation mechanism are similar to those of Kyle (1985) and Farmer and Joshi (2002). Farmer and Joshi (2002) regard a trading strategy as a signal-processing element that uses external information and past prices as inputs and incorporates them into future prices. They show numerically how trading strategies can amplify noise, induce structure in prices, and cause phenomena such as excess and clustered volatility.

Our focus in this paper is on both the theoretical and numerical behavior of asset prices resulting from the interaction of heterogeneous investors under a (admittedly stylized) market-maker price-setting mechanism. The analysis below seeks to determine how the results of Brock and Hommes (1998) and Chiarella and He (2002c) are affected by this alternative market-clearing mechanism and how the price dynamics change under boundedly rational geometric-decay learning processes with finite and infinite memory. The focus is on a number of aspects of the modeling framework. First, as in Chiarella and He (2002c), traders are allowed to have differing attitudes toward risk.[2] Second, the trend followers are assumed to use a geometric-decay (learning) process, based on past observations, to form their forecast rule on the price of the risky asset. With homogeneous beliefs and traders following moving-average learning processes, Balasko and Royer (1996) show that the intuition, that the larger the lag length of past information used the more stable is the Walrasian equilibrium, is essentially correct. However, with heterogeneous beliefs, the influence of the lag length becomes more complicated. For a general discussion on the dynamics of homogeneous and heterogeneous beliefs under various recursive (learning) processes, the reader is referred to Chiarella and He (2002a,b). In some learning analysis, such as that of Grandmont (1985), Balasko and Royer (1996) and Honkapohja and Mitra (2000), memory is bounded because the agents forget data that has occurred far enough in the past. In other analyses of learning, such as that of Evans and Honkapohja (1994, 1999) and Barucci (2001),

memory is unbounded or, alternatively, past data are discounted geometrically. This paper first introduces the geometric decay processes (GDP) with bounded or unbounded memory into the asset-pricing model with heterogeneous agents and then considers the learning dynamics when the trend followers use GDP. A feature of the analysis is that adaptive learning is considered in a framework that allows the fractions of the agents to vary, as in Brock and Hommes (1998) and Chiarella and He (2002c).

The plan of the paper is as follows. Section 2 outlines the model of interacting heterogeneous agents with the market-clearing price set by a market maker, introduces the expectations function and learning mechanisms of the heterogeneous groups, and sets up the adaptive beliefs system (ABS). Section 3 considers the existence of steady states of the deterministic ABS. In Section 4, the local stability and bifurcation of the fundamental steady state are analyzed when the trend followers use the GDP with finite memory. In Section 5, various numerical tools are employed to obtain global information about the dynamics when the fundamental steady state is locally unstable, focusing in particular on how the dynamics are affected by various key elements (or parameters) of the model, such as the adjustment speed of the market maker, the extrapolation rate of the trend followers, the decay rate of the GDP, the lag length used in the learning GDP, and external random factors. A discussion on the price dynamics when the memory is infinite is included in Section 6. Section 7 concludes, and all proofs are included in the Appendix.

2. A HETEROGENEOUS-AGENTS MODEL UNDER A MARKET MAKER

This section sets up a standard discounted-value asset-pricing model with heterogeneous agents, which is closely related to the frameworks of Brock and Hommes (1997b, 1998) and Chiarella and He (2002c). However, as stated in the Introduction, the market-clearing price is arrived at via a market-maker scenario rather than the Walrasian scenario used in the studies just cited. We focus on the simple case in which there are three classes of participants in the asset market: two groups of traders—fundamentalists and trend chasers—and a market maker.

2.1. Market Equilibrium Price Under a Market Maker

Following the framework of Brock and Hommes (1998), consider an asset-pricing model with one risky asset and one risk-free asset. It is assumed that the risk-free asset is perfectly elastically supplied at gross return $R = 1 + r > 1$ (with a fixed risk-free rate r) for each period. Let p_t be the price (ex dividend) per share of the risky asset at time t and $\{y_t\}$ be the stochastic dividend process of the risky asset. Then, the wealth of a typical investor at $t + 1$ is given by

$$W_{t+1} = RW_t + [p_{t+1} + y_{t+1} - Rp_t]z_t, \tag{1}$$

where W_t is wealth at time t and z_t is the number of shares of the risky asset purchased by the investor at t.

Denote by $F_t = \{p_t, p_{t-1}, \ldots; y_t, y_{t-1}, \ldots\}$ the information set at time t. Let E_t, V_t be the conditional expectation and variance, respectively, based on F_t, and $E_{h,t}$ and $V_{h,t}$ be the "beliefs" of investor type h about the conditional expectation and variance of quantities at $t+1$. Denote by R_{t+1} the excess return at $t+1$; that is,

$$R_{t+1} = p_{t+1} + y_{t+1} - Rp_t. \tag{2}$$

Then, it follows from (1) and (2) that

$$\begin{aligned} E_{h,t}(W_{t+1}) &= RW_t + E_{h,t}(p_{t+1} + y_{t+1} - Rp_t)z_{h,t} \\ &= RW_t + E_{h,t}(R_{t+1})z_{h,t}, \\ V_{h,t}(W_{t+1}) &= z_{h,t}^2 V_{h,t}(p_{t+1} + y_{t+1} - Rp_t) \\ &= z_{h,t}^2 V_{h,t}(R_{t+1}), \end{aligned} \tag{3}$$

where $z_{h,t}$ is the demand by agent h for the risky asset.

Assume each investor type is an expected-utility maximizer, but different investors (e.g., type h) have different attitudes toward risk, characterized by the risk-aversion coefficient, a_h. Then, for type h, the demand $z_{h,t}$ of the risky asset is given by[3]

$$z_{h,t} = \frac{E_{h,t}(R_{t+1})}{a_h V_{h,t}(R_{t+1})}. \tag{4}$$

Let $n_{h,t}$ be the fraction of investors of type h at t (so that $\sum_h n_{h,t} = 1$). The fraction $n_{h,t}$ is updated on the basis of the fitness function, which is specified in the following discussion. Assume zero supply of outside shares. Then, the excess demand $z_{e,t}$ is given by

$$z_{e,t} \equiv \sum_h n_{h,t} z_{h,t}, \tag{5}$$

or [using (4)]

$$z_{e,t} = \sum_h n_{h,t} \frac{E_{h,t}[R_{t+1}]}{a_h V_{h,t}[R_{t+1}]}. \tag{6}$$

To complete the model the price changes must be made explicit. The role of the market maker is to take a long (when $z_{e,t} < 0$) or short (when $z_{e,t} > 0$) position so as to clear the market. At the end of period t, after the market maker has carried out all transactions, he or she adjusts the price for the next period in the direction of the observed excess demand. Using μ to denote the corresponding speed of adjustment for each period, then, with a simple linear price adjustment process, the price would be given by

$$p_{t+1} = p_t + \mu z_{e,t},$$

which by use of (5) becomes

$$p_{t+1} = p_t + \mu \sum_h n_{h,t} z_{h,t}. \tag{7}$$

Note that the market-maker behavior in this model is highly stylized. For instance, the inventory of the market maker built up as a result of the accumulation of various long and short positions is not considered. This could affect his or her behavior; for example, the market-maker price-setting role in (7) could be a function of the inventory. Allowing μ to be a function of inventory would be one way to model such behavior. Such considerations are left to future research. Future research should also seek to explore the microfoundations of the coefficient μ. In the present paper, it is best thought of as a market friction, and an aim of our analysis is to understand how this friction affects the market dynamics.

2.2. Heterogeneous Beliefs

To get a benchmark notion of the rational-expectation *fundamental solution* p_t^*, consider the equation

$$Rp_t^* = E_t[p_{t+1}^* + y_{t+1}],$$

where E_t is expectation conditional on the information set F_t. In the case in which the dividend process $\{y_t\}$ is i.i.d., $E_t(y_{t+1}) = \bar{y}$ (assumed constant), the only solution satisfying the "no bubbles" condition[4] ($\lim_{t\to\infty} Ep_t/R^t = 0$) is the constant solution $\bar{p} = \bar{y}/r$.

Regarding the heterogeneous beliefs of fundamentalists (type 1) and trend chasers (type 2) about the deviations from the fundamental solution, we assume that expectations are formed according to

$$E_{1t}(p_{t+1} + y_{t+1}) = E_t(p_{t+1}^* + y_{t+1}), \tag{8}$$

$$E_{2t}(p_{t+1} + y_{t+1}) = E_t(p_{t+1}^* + y_{t+1}) + df_t, \tag{9}$$

with expected variance[5]

$$V_{1t}(p_{t+1} + y_{t+1}) = V_{2t}(p_{t+1} + y_{t+1}) = \sigma^2. \tag{10}$$

Here, $d > 0$, σ^2 are constant, and

$$f_t = \sum_{i=0}^{L-1} b_i p_{t-i} - p_t^*, \qquad b_i \geq 0, \qquad \sum_{i=0}^{L-1} b_i = 1 \tag{11}$$

is a weighted moving-average price process with a window length $L \geq 1$. This type of process, known as a general a_L process is considered by Chiarella and He (2002a,b), where the dynamics of (Walrasian) steady states with both homogeneous and heterogeneous a_L processes are studied in detail. It is found that such recursive (learning) processes can generate very rich and complicated

dynamics. In the following discussion, three simple weighted averaging processes are considered:

(i) Moving-average process (MAP),

$$f_t = \frac{1}{L}\sum_{i=0}^{L-1} p_{t-i} - p_t^*; \tag{12}$$

(ii) Geometric decay process (GDP),

$$f_t = b\left[p_t + \omega p_{t-1} + \omega^2 p_{t-2} + \cdots + \omega^{L-1} p_{t-(L-1)}\right] - p_t^*, \tag{13}$$

where $b = 1/[1 + \omega + \omega^2 + \cdots + \omega^{L-1}]$, $\omega \in [0, 1]$ measures the decay rate of the memory;

(iii) Limiting geometric decay process (LGDP),

$$f_t = \omega f_{t-1} + (1 - \omega)[p_t - p_t^*], \qquad \omega \in [0, 1). \tag{14}$$

One could interpret GDP as one of bounded memory. As $L \to \infty$, the limiting process of GDP becomes LGDP, in which memory is unbounded. For $\omega = 1$, GDP leads to MAP and hence MAP can be treated as a degenerate case of GDP. Therefore, the following discussion concentrates on the geometric decay processes with both finite and infinite memory.

Under assumptions (8) and (9) with $d = 0$, the fundamentalists believe that the prices will return to their fundamental values. Under the same assumption but with $d > 0$, the trend chasers chase the trend of the prices by extrapolating from the weighted moving-average prices f_t at an extrapolation rate of d. Here, a_i $(i = 1, 2)$ are allowed to be different to characterize the different risk attitudes of the two types of investors. Typically, as pointed by Campbell and Kyle (1993), one would expect the fundamentalists to be more risk averse than the trend chasers (that is, $a = a_2/a_1 < 1$).

2.3. Adaptive Belief System

Let the benchmark fundamental price be the no-bubble constant price $\bar{p}$ and let $x_t = p_t - \bar{p}$ denote the deviation of p_t from the no-bubble fundamental price $\bar{p}$. Rewrite the excess return at period t in the form

$$R_t = x_t - Rx_{t-1} + \delta_t, \tag{15}$$

where

$$\delta_t = p_t^* + y_t - E_t[p_{t+1}^* + y_{t+1}] \tag{16}$$

is a martingale difference sequence with respect to F_t; that is, $E[\delta_t | F_t] = 0$ for all t. Then, under (8)–(11), one obtains

$$E_{1t}(R_{t+1}) = -Rx_t, \qquad E_{2t}(R_{t+1}) = d\, g_t - Rx_t, \tag{17}$$

where

$$g_t = \sum_{i=0}^{L-1} b_i x_{t-i}, \qquad b_i \geq 0, \qquad \sum_{i=0}^{L-1} b_i = 1. \tag{18}$$

In particular, $g_t = (1/L)\sum_{i=0}^{L-1} x_{t-i}$ for MAP, $g_t = b[x_t + \omega x_{t-1} + \cdots + \omega^{L-1} x_{t-(L-1)}$ for GDP and $g_t = \omega g_{t-1} + (1-\omega)x_t$ for LGDP.

Let $\pi_{h,t}$ be the "fitness function," which is defined by the realized profits of trader type h at period t so that

$$\pi_{h,t} = R_t z_{h,t-1} = [x_t - Rx_{t-1} + \delta_t] z_{h,t-1},$$

with

$$z_{1,t} = \frac{-Rx_t}{a_1\sigma^2}, \qquad z_{2,t} = \frac{dg_t - Rx_t}{a_2\sigma^2}.$$

It is assumed that the fractions are updated on the basis of discrete-choice probability, as in Manski and McFadden (1981), Anderson et al. (1993), and Brock and Hommes (1997b, 1998); that is,

$$n_{h,t} = \exp[\beta(\pi_{h,t} - C_h)]/Z_t \quad (h = 1, 2), \qquad Z_t = \sum_h \exp[\beta(\pi_{h,t} - C_h)], \tag{19}$$

where $C_h \geq 0$ is the constant cost incurred by the investors of type h, and $\beta(>0)$ is the *intensity of choice* measuring how fast agents switch between different prediction strategies. In particular, $\beta = +\infty$ means the entire mass of traders uses the strategy that has highest fitness; $\beta = 0$ means that the mass of traders distributes itself evenly across the set of available strategies. Typically, it is assumed that, as informed traders, the fundamentalists incur greater costs than the chartists in the gathering of information in the formation of expectations. Here, set $C_1 = C$ and $C_2 = 0$.

To sum up, the dynamics of the time evolution of the market are described by the ABS,

$$\begin{cases} x_{t+1} = x_t + \mu \sum_h n_{h,t} z_{h,t}, \\ n_{h,t} = \exp[\beta(\pi_{h,t} - C_h)]/Z_t \quad (h = 1, 2), \end{cases} \tag{20}$$

where

$$\begin{cases} Z_t = \sum_h \exp[\beta(\pi_{h,t} - C_h)], \\ \pi_{h,t} = [x_t - Rx_{t-1} + \delta_t] z_{h,t-1}, \\ z_{1,t} = \dfrac{-Rx_t}{a_1\sigma^2}, \\ z_{2,t} = \dfrac{d\,g_t - Rx_t}{a_2\sigma^2}. \end{cases} \tag{21}$$

Let $m_t = n_{1,t} - n_{2,t}$. Then, $n_{1,t} = (1+m_t)/2$, $n_{2,t} = (1-m_t)/2$ and hence the ABS (20) can be expressed as

$$\begin{cases} x_{t+1} = x_t + \dfrac{\mu}{2}[(1+m_t)z_{1,t} + (1-m_t)z_{2,t}], \\ m_t = \tanh\left(\dfrac{\beta}{2}[x_t - Rx_{t-1} + \delta_t][z_{1,t-1} - z_{2,t-1}] - \dfrac{\beta C}{2}\right). \end{cases} \tag{22}$$

Because of the random term δ_t, (22) is a nonlinear stochastic difference system. When $\delta_t = 0$, the ABS (22) becomes a deterministic difference system of order $L+1$. An analysis of this deterministic system is some guide in understanding the underlying dynamics of the corresponding stochastic system.

The following sections consider the dynamic behavior of the ABS (22) under the geometric decay processes with both finite and infinite memory. As pointed out earlier, the moving-average process can be considered as a degenerate case of GDP when $\omega = 1$. The existence of fundamental and nonfundamental steady states is considered first. The local stability of the fundamental steady state is analyzed. Numerical simulations are then used to verify the local stability and to demonstrate the global dynamics of the nonlinear model. In particular, the analysis depicts how the dynamics are affected by various key elements (or parameters) of the model, such as the adjustment speed of the market maker μ, the extrapolation rate of the trend followers d, the decay rate of the GDP ω, the lag length used in the learning GDP L, and external random factors. Some comparison with the results of Brock and Hommes (1998) and Chiarella and He (2002c) under the Walrasian scenario are also made.

3. EXISTENCE OF STEADY STATE OF THE DETERMINISTIC ABS

Consider the existence of the steady state of the deterministic ABS (22) when g_t follows the general $\boldsymbol{a}_L$ process (18). Let

$$a = \frac{a_2}{a_1}, \qquad m^{eq} = \tanh\left(-\frac{\beta C}{2}\right), \qquad m^* = \frac{(1+a)R - d}{(1-a)R - d}, \qquad \gamma = \frac{\mu}{4a_2\sigma^2}, \tag{23}$$

and

$$d^* = R\left[1 + \frac{1+m^{eq}}{1-m^{eq}}a\right], \qquad \gamma^* = \frac{1}{(1-m^{eq})d^*}. \tag{24}$$

PROPOSITION 1. *Let $\delta = 0$ and x^* be the positive solution (if it exists) of*

$$\tanh\left[\frac{\beta}{2a_2\sigma^2}(R-1)[d-(1-a)R](x^*)^2 - \frac{\beta C}{2}\right] = m^*. \tag{25}$$

Then, the (deterministic) ABS (22) of (x_t, m_t) with g_t defined by (18) always has a (fundamental) steady state $E(0, m^{eq})$. In addition, the ABS (22) has another two nonfundamental steady states $E_\pm(x^*, m^*)$ if $d > d^*$.

(All proofs, unless otherwise stated, are given in the Appendix.)

Proposition 1 corresponds to the existence results obtained by Brock and Hommes (1998) and Chiarella and He (2002c) under the Walrasian scenario. Therefore, in terms of the existence of fixed equilibria, both the Walrasian and market-maker scenarios lead to the same outcomes. Proposition 1 indicates that the existence of the nonfundamental steady states depends on how the trend chasers extrapolate the trend and are independent of the adjustment of the market maker. When the trend chasers extrapolate weakly (i.e., $d < d^*$), the fundamental steady state is the only steady state. However, when the trend chasers extrapolate strongly (i.e., $d > d^*$), the system generates two other nonfundamental steady states. Intuitively, the fundamental steady state is stable when the trend traders extrapolate weakly; otherwise, it becomes unstable and the nonfundamental steady states may become stable or the prices may become more volatile. The discussion in the following sections shows that this is indeed the case in general. Moreover, a broad and precise characterization of the roles of the traders and the market maker is drawn through stability and bifurcation analysis and numerical simulations.

4. DYNAMICS OF ABS UNDER THE GEOMETRIC DECAY PROCESS WITH FINITE MEMORY

This section considers the dynamic behavior of the ABS (22) under the geometric decay process with finite memory.

Assume that the expectations of the trend chasers follow the geometric decay process

$$g_t = b\left[x_t + \omega x_{t-1} + \omega^2 x_{t-2} + \cdots + \omega^{L-1} x_{t-(L-1)}\right], \tag{26}$$

where $b = 1/[1 + \omega + \omega^2 + \cdots + \omega^{L-1}]$, $\omega \in [0, 1]$. Then, the deterministic ABS (22) assumes the form

$$\begin{cases} x_{t+1} = x_t + \dfrac{\mu}{2}[(1 + m_t)z_{1,t} + (1 - m_t)z_{2,t}] \\ m_{t+1} = \tanh\left[\dfrac{\beta}{2}(x_{t+1} - Rx_t + \delta_{t+1})(z_{1,t} - z_{2,t}) - \dfrac{\beta C}{2}\right], \end{cases} \tag{27}$$

where

$$z_{1,t} = \frac{-R}{a_1\sigma^2}x_t, \qquad z_{2,t} = \frac{dg_t - Rx_t}{a_2\sigma^2}.$$

4.1. Stability Analysis

It is generally difficult to derive sufficient conditions for the local stability of the fundamental steady state E for general lag length. Some insights into the stability when the trend chasers follow the GDP may be gained by considering the simple cases when $L = 2$ and 3. For $L \geq 4$, Rouche's theorem (see Appendix A.3) can be used to obtain some stability results.

The stability analysis of the fundamental steady state can be conducted by the eigenvalues analysis of the corresponding characteristic equations. The following Proposition 2 gives the necessary and sufficient conditions for all the eigenvalues (e.g., λ_i) to satisfy $|\lambda_i| < 1$.

PROPOSITION 2. *Let $\delta_t \equiv 0$. Then, the fundamental steady state E of* (27) *is locally asymptotically stable* (*LAS*) *if* $(d, \gamma) \in D_L$ *for* $L = 1, 2, 3$, *respectively, where*

$$D_L = \{(d, \gamma), 0 < d < d^*, 0 < \gamma < \gamma_L^*\}$$

and

$$\begin{cases} \gamma_1^* = \dfrac{1}{(1 - m^{eq})(d^* - d)} \\ \gamma_2^* = \dfrac{1}{(1 - m^{eq})\left(d^* - \dfrac{1-\omega}{1+\omega} d\right)} \\ \gamma_3^* = \dfrac{1}{(1 - m^{eq})\left(d^* - \dfrac{1-\omega+\omega^2}{1+\omega+\omega^2} d\right)}. \end{cases}$$

In addition, along $\gamma = \gamma_i^*(i = 1, 2, 3)$, *flip bifurcations* (*with* -1 *as one of the eigenvalues*) *occur and, along* $d = d^*$, *saddle-node bifurcations* (*with* 1 *as one of the eigenvalues*) *occur. Furthermore,*

$$D_2 \subset D_3 \subset D_1.$$

With the Walrasian market-clearing scenario in Brock and Hommes (1998) and Chiarella and He (2002c), the local stability of the fundamental steady state is completely characterized by the extrapolation rate of the trend followers and the relative risk-aversion ratio between the fundamentalists and the trend followers, not the lag length used in the moving-average learning process for the trend followers. Also, complicated price dynamics can only be generated through saddle-node-type bifurcations. However, Proposition 2 tells us that, with the market-maker scenario, the stability of the fundamental steady state is maintained only when the speed of the adjustment of the market maker is low and is balanced with the extrapolation rate of the trend followers. In addition, complicated price dynamics can be generated through either saddle-node or flip-type bifurcations.

In general, for any lag length, the following result on the local stability of the fundamental steady state can be derived.

PROPOSITION 3. *Let* $\delta_t \equiv 0$ *and* $L > 1$.

(i) *The fundamental steady state E of the ABS* (22) *is LAS if* $0 < d < d^*$ *and*

$$0 < \gamma < \gamma_4^* \equiv \frac{1}{(1 - m^{eq})(d + d^* - 2bd)}. \tag{28}$$

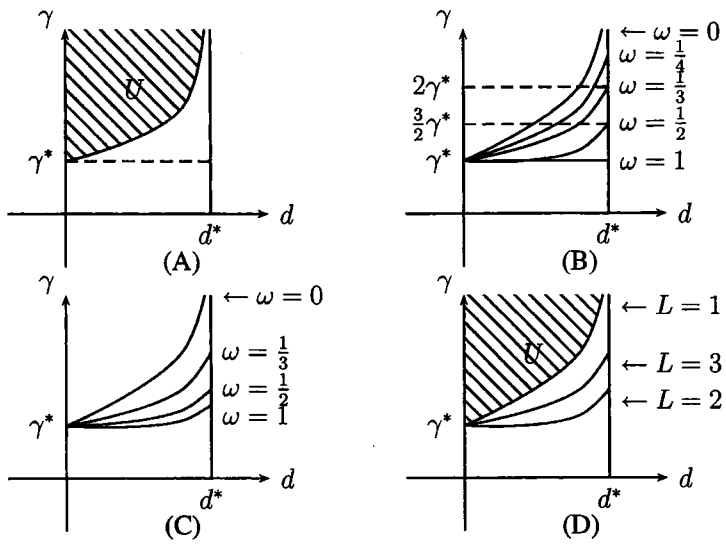

Figure 1. Unstable region (as indicated by U) and local stability regions (whose upper bounds are given by the various curves) of the fundamental steady state: (A) $L = 1$; (B) $L = 2$ with different $\omega \in [0, 1]$; (C) $L = 3$ with different values of ω; (D) Comparison of the local stability regions for $L = 1, 2, 3$ with fixed $\omega \in (0, 1)$.

(ii) *The fundamental steady state E of the ABS* (22) *is unstable if* $0 < d < d^*$ *and* $\gamma > \gamma_1^*$.

Based on Propositions 2 and 3, the local unstable and stable regions of the fundamental steady state are plotted in Figure 1 for different lag length and different values of the decay rate. One can see the following:

- In terms of the stability of the fundamental steady state, it follows from $D_2 \subset D_3 \subset D_1$ (as indicated by Figure 1D) that an increase in the lag length of the GDP does not necessarily enlarge the stability region for the extrapolation rates and the adjustment speed of the market maker. In fact, D_1 is the largest stability region when $d < d^*$ for all the lag lengths $L \geq 2$ (as indicated by Figure 1B for $L = 2$ and Figure 1C for $L = 3$). This contradicts a common belief that the stability regions are enlarged when agents include more historical data in forecasting rules.
- For fixed lag length $L = 2, 3$, a decrease in the memory decay rate ω enlarges the local stability region in terms of d and γ, as indicated by Figures 1B and 1C.
- When $\omega = 0$, $\gamma_L^* = \gamma_1^*$ for $L = 1, 2, 3$, implying that the stability regions D_L for $L = 2, 3$ become D_1 as the decay rate of the GDP tends to zero. At $\omega = 1$, $\gamma_2^* = \gamma^* = 1/[(1 - m^{eq})\, d^*]$ and $\gamma_3^* = \gamma_3 \equiv 1/[(1 - m^{eq})(d^* - d/3)]$, which leads to the same stability regions when the trend chasers follow MAP with $L = 2$ and 3, respectively. Therefore, the MAP can be treated as a degenerate case of the GDP. The following discussion illustrates those two special cases.

4.2. The Case of $L = 1$

Consider first the simplest case when either $L = 1$ or $\omega = 0$; that is, the trend chasers follow the forecasting rule

$$E_{2,t}(p_{t+1}) = p_t^* + d[p_t - p_t^*].$$

The trend chasers predict that prices will rise (or fall) by a constant rate d. Correspondingly, $g_t = x_t$ and $L = 1$ for MAP or $\omega = 0$ for GDP. In this case, the deterministic ABS (22) has the form

$$\begin{aligned} x_{t+1} &= F_1(x_t, m_t) \\ m_{t+1} &= \tanh\left[-\frac{\beta}{2a_2\sigma^2}(F_1(x_t, m_t) - Rx_t + \delta_{t+1})(d - (1-a)R)x_t - \frac{\beta C}{2}\right], \end{aligned} \tag{29}$$

where

$$F_1(x, m) = x\left[1 - \frac{\mu}{2a_2\sigma^2}(aR(1+m) + (R-d)(1-m))\right].$$

COROLLARY 1. *The fundamental steady state E of the (deterministic) ABS (29) is locally asymptotically stable (LAS) if*

$$0 < d < d^*, \qquad 0 < \gamma < \gamma_1^*. \tag{30}$$

In addition, along $\gamma = \gamma_1^*$, flip bifurcations occur and, along $d = d^*$, saddle-node bifurcations occur. Furthermore, if $0 < d < R$, then E is globally asymptotically stable under any one of the following conditions:

$$\begin{aligned} &a \geq 1, 0 < \gamma < 1/[2aR]; \\ &a < 1, d > (1-a)R, 0 < \gamma < 1/[2aR]; \\ &a < 1, d < (1-a)R, 0 < \gamma < 1/[2(R-d)]. \end{aligned}$$

Corollary 1 indicates that, with the market-maker scenario, the stability of the fundamental steady state E depends on the speed of the adjustment of the market maker, the extrapolation rate of the trend chasers, and the relative risk-aversion ratio between the fundamentalists and trend chasers. Following from Corollary 1, E is LAS in a region of (d, γ), defined by $0 < \gamma < \gamma_1^*$ and $0 < d < d^*$. Stability regions (both local and global) of the fundamental equilibrium E are shown in Figure 2 in which

$$\gamma_{1,0} = 1/[2aR], \qquad \gamma_{2,0} = 1/[2R], \qquad \gamma_{3,0} = 1/[(1 - m^{eq})\, d^*].$$

One can verify that both d^* and γ_1^* are increasing functions of a and hence, as a increases, the local stability region becomes larger. The condition (30) can be

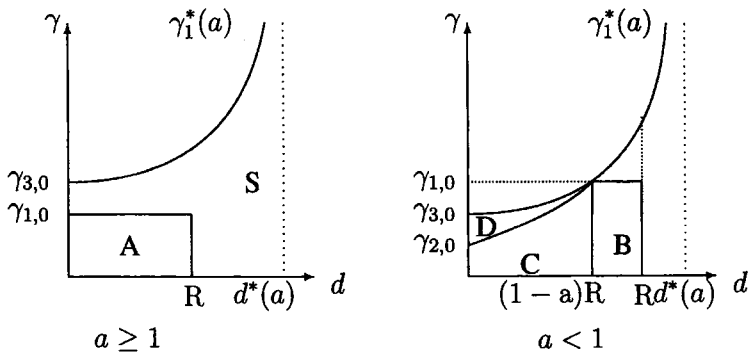

FIGURE 2. Stability regions of the fundamental steady state S; $a \geq 1$, global stability region A and local stability region $A \cup S$; $a < 1$, global stability region $B \cup C$ and local stability region $B \cup C \cup D$.

rewritten as $0 < d < d^*$ and $0 < \mu < \mu^*$, with

$$\mu^* = \frac{4a_1 a\sigma^2}{aR(1+m^{eq}) + (R-d)(1-m^{eq})}.$$

One can verify that both d^* and μ^* are increasing functions of a. This shows that, to maintain the (local) stability of the fundamental price, the speed of adjustment of the market maker must be proportional to the relative risk ratio a. Recall that a high risk ratio a indicates that the trend followers are more risk averse than the fundamentalists. When the trend followers become less risk averse (relative to the fundamentalists), a lower speed of the adjustment is needed for the market maker to stabilize the fundamental price. The intuition behind this result is that an overreaction of the trend followers accompanied by a quick price adjustment (toward the fundamental price) by the market maker can in fact cause the price to be pushed away from the fundamental price.

When the trend chasers follow naive expectations, that is, $L = 1$ or $\omega = 0$, the global dynamics can be characterized by bifurcation diagrams and Lyapunov exponent plots. Let the risk-free rate $r = 10\%$ (i.e., $R = 1.1$) and the cost coefficient C for the fundamentalists be standardized at $C = 1$. We also standardize the variance σ^2 and the risk-aversion coefficients a_1 and a_2 so that $a_1\sigma^2 = 1$ and $a = a_2/a_1$. Consider first the situation in which both fundamentalists and trend followers have the same risk-aversion coefficients $a = 1$. Choose the intensity-of-choice coefficient $\beta = 3.5$ and the extrapolation rate of the trend followers $d = 1.2$. Assume that the speed of the adjustment μ of the market maker varies. One can verify that the fundamental steady state is $E(0, -0.9414)$ and $d^* = 1.13322$, as defined in (24). Since $d = 1.2 > d^*$, it follows from Proposition 1 that there exist two other nonfundamental steady states $E_\pm(\pm x^*, m^*) = E_\pm(\pm 1.62, -0.83)$. For the parameter μ, Figure 3A indicates that there exist $\mu_1 \in (0.82, 0.83)$ and $\mu_2 \in (1.06, 1.07)$ such that (i) for $\mu < \mu_1$, all the solutions converge to either one of the two nonfundamental equilibria $E_\pm$, depending on the initial values; (ii) for $\mu_1 < \mu < \mu_2$, the model has either two attracting invariant circles around

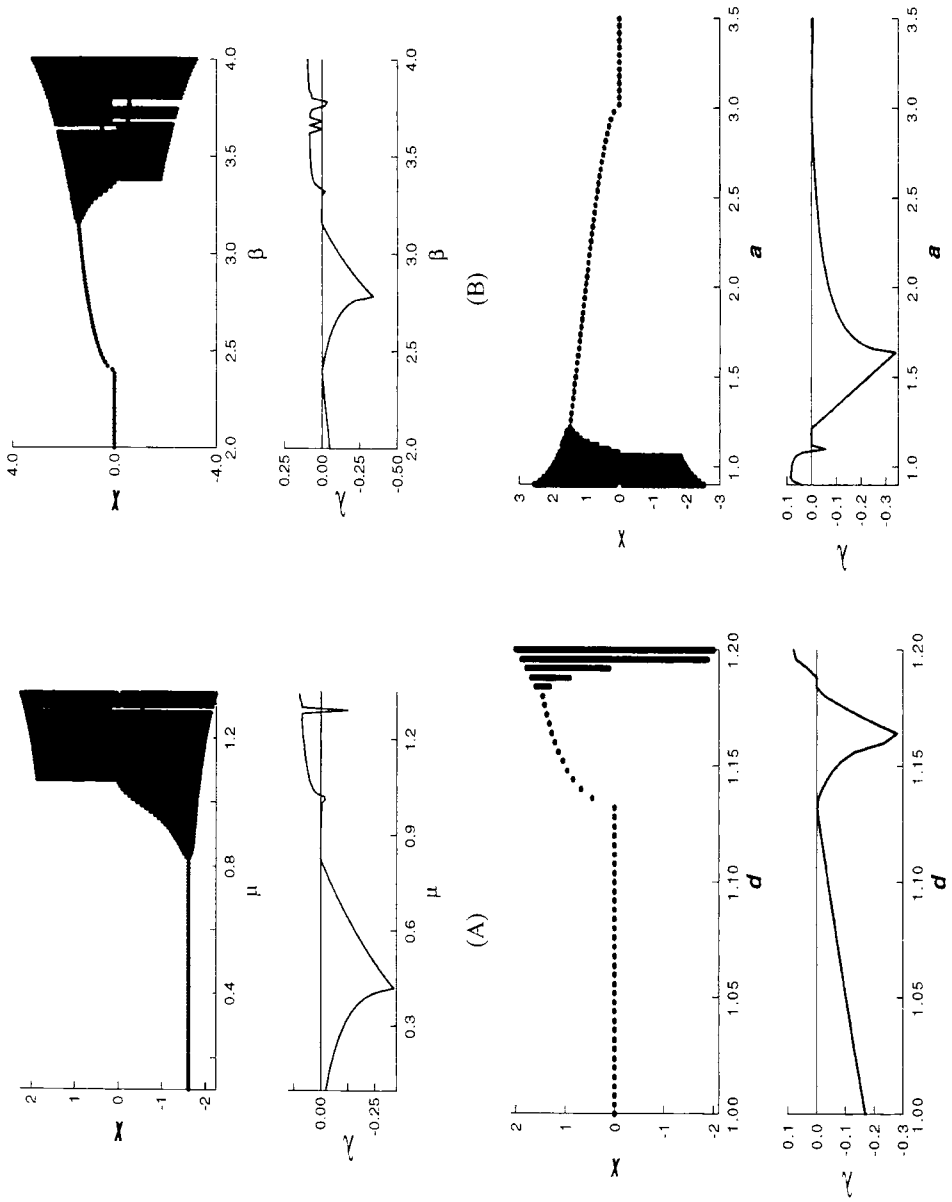

Figure 3. Bifurcation diagrams and Lyapunov plots for parameters (A) μ, (B) β, (C) d, and (D) a for the model of fundamentalists versus trend chasers with $L = 1$.

$E_{\pm}$ or strange attractors (which is indicated by the positive Lyapunov exponent in the plot); (iii) for $\mu > \mu_2$, the invariant circles and strange attractors cover both sides of the x-axis. Thus, the dynamics of the market-maker model are similar to Brock and Hommes' Walrasian model when μ is small, in that prices converge to or fluctuate around one or another of the nonfundamental equilibria. However, differences between the two market-clearing scenarios appear for large μ when fluctuations between both nonfundamental equilibria occur.

In Figure 3B, the intensity-of-choice parameter β is selected as the bifurcation parameter. For fixed $a = 1$, $\mu = 1.2$, and $d = 1.2$, the figure indicates that all the solutions converge to the fundamental steady state when β is small ($\beta < 2.4$). As β increases further (up to 3.14), solutions converge to either one of the nonfundamental equilibria and, for large β, the solutions display chaotic behavior, which is indicated by the positive Lyapunov exponent. Under the Walrasian scenario, the bifurcation diagrams have similar features, but the attractors for the solutions can only stay either nonpositive or nonnegative. With the market-maker scenario, the attractors can extend into both positive and negative phases. In Figure 3C, $a = 1$, $\beta = 3.5$, $\mu = 1.2$, and d is selected as the bifurcation parameter. It illustrates that all the solutions converge to the fundamental equilibrium for $d < 1.132$ and to one of the nonfundamental equilibria for $1.132 < d < 1.18$. As d increases further, the solutions either converge to some strange attractor or explode. In Figure 3D, $\beta = 3.5$, $\mu = 1.2$, $d = 1.2$, and $a = a_2$ is selected as the bifurcation parameter. The figure indicates that the fundamental value is attracting when the fundamentalists are more risk averse than the trend followers [i.e., when $a(\geq 3.0)$ is large], and it becomes unstable, whereas the nonfundamental equilibria are stable as a decreases. As a decreases further, periodic cycles, quasi-periodic orbits, and even chaotic behavior are observed.

4.3. The Case of the GDP with $\omega = 1$ and $L = 2, 3$

Recall that with $\omega = 1$ the GDP degenerates to the MAP.

COROLLARY 2. *The fundamental steady state E of the (deterministic) ABS (22) is LAS if $d < d^*$ and*

$$0 < \gamma < \gamma_2 \equiv 1/[(1 - m^{eq})\, d^*] \tag{31}$$

for $L = 2$, and

$$0 < \gamma < \gamma_3 \equiv 1/[(1 - m^{eq})(d^* - d/3)] \tag{32}$$

for $L = 3$.

Furthermore, $\gamma = \gamma_i$ corresponds to a flip-type bifurcation and $d = d^*$ $(i = 1, 2)$ corresponds to a saddle-node-type bifurcation.

Note that condition (31) for $L = 2$ implies the conditions (30) and (32) for $L = 1$ and $L = 3$, respectively. Therefore, the local stability of E_1 for $L = 2$ implies that for $L = 1$ and $L = 3$. It is difficult to obtain explicit conditions on local stability for

general lag length. However, Appendix A.6 gives a sufficient condition on local stability if μ is chosen from a specific set of values.

Overall, these results suggest that, all other things being equal, the fundamental steady state of the ABS with the GDP is stabilized as the trend chasers put more weight on the most recent prices. Also, increasing the extrapolation rate d up to d^* enlarges the local stability region. However, there does not seem to be any connection between the lag length and the size of the stability region.

5. NUMERICAL ANALYSIS OF THE DYNAMICS OF ABS UNDER THE GEOMETRIC DECAY PROCESS WITH FINITE MEMORY

To verify the stability result established in Proposition 2 and, more importantly, to gain some insights into the price dynamics when either the market maker over-adjusts the price or the trend chasers extrapolate strongly, the nonlinear adaptive beliefs system is simulated numerically.

The effect of the intensity of the choice and the relative risk-aversion (between the two traders) on the price dynamics has been studied by Brock and Hommes (1998) and Chiarella and He (2002c). The following discussion concentrates on the effect on the nonlinear dynamics of various key elements (or parameters) of the model, such as the adjustment speed of the market maker μ, the extrapolation rate of the trend followers d, the decay rate of the GDP ω, the lag length used in the GDP L, and external random factors (e.g., a noisy dividend process). Let

$$R = 1.1, \qquad C = 1.0, \qquad a_1\sigma^2 = 1, \qquad a_2\sigma^2 = 1, \qquad \beta = 3.5.$$

In this case, both the fundamentalists and the trend chasers have the same risk-aversion coefficient. Then, the fundamental steady state is $E(0, -0.9414)$ and $d^* = 1.13322$, as defined in (24). When Proposition 3 is applied, the fundamental steady state $E(0, -0.9414)$ is LAS for $d < d^* = 1.13322$ and $\gamma < \gamma_L^*$ and unstable for either $d > d^*$ or $\gamma > \gamma_L^*$ ($L = 1, 2, 3$). This result is confirmed numerically. The solutions become explosive when γ crosses the flip boundary $\gamma = \gamma_L^*$, implying that weak extrapolation from the trend chasers and overadjustment from the market maker can lead to unbounded prices.

In the following discussion,[6] we consider situations when the trend chasers extrapolate strongly $d = 1.2 > d^*$. In such cases, it follows from Proposition 1 that there exist two other nonfundamental steady states $E_\pm(\pm x^*, m^*) = E_\pm(\pm 1.62, -0.83)$.

5.1. Effect of the Speed of Adjustment of the Market Maker μ

For $L = 1$, Figure 4 illustrates the corresponding time series for $\mu = 0.6, 0.9$, and 1.2 without noise (left panel). It indicates that, when the trend chasers extrapolate strongly, a small adjustment from the market maker (e.g., $\mu = 0.6$) leads the prices to converge to one of the nonfundamental steady states. However, as the market

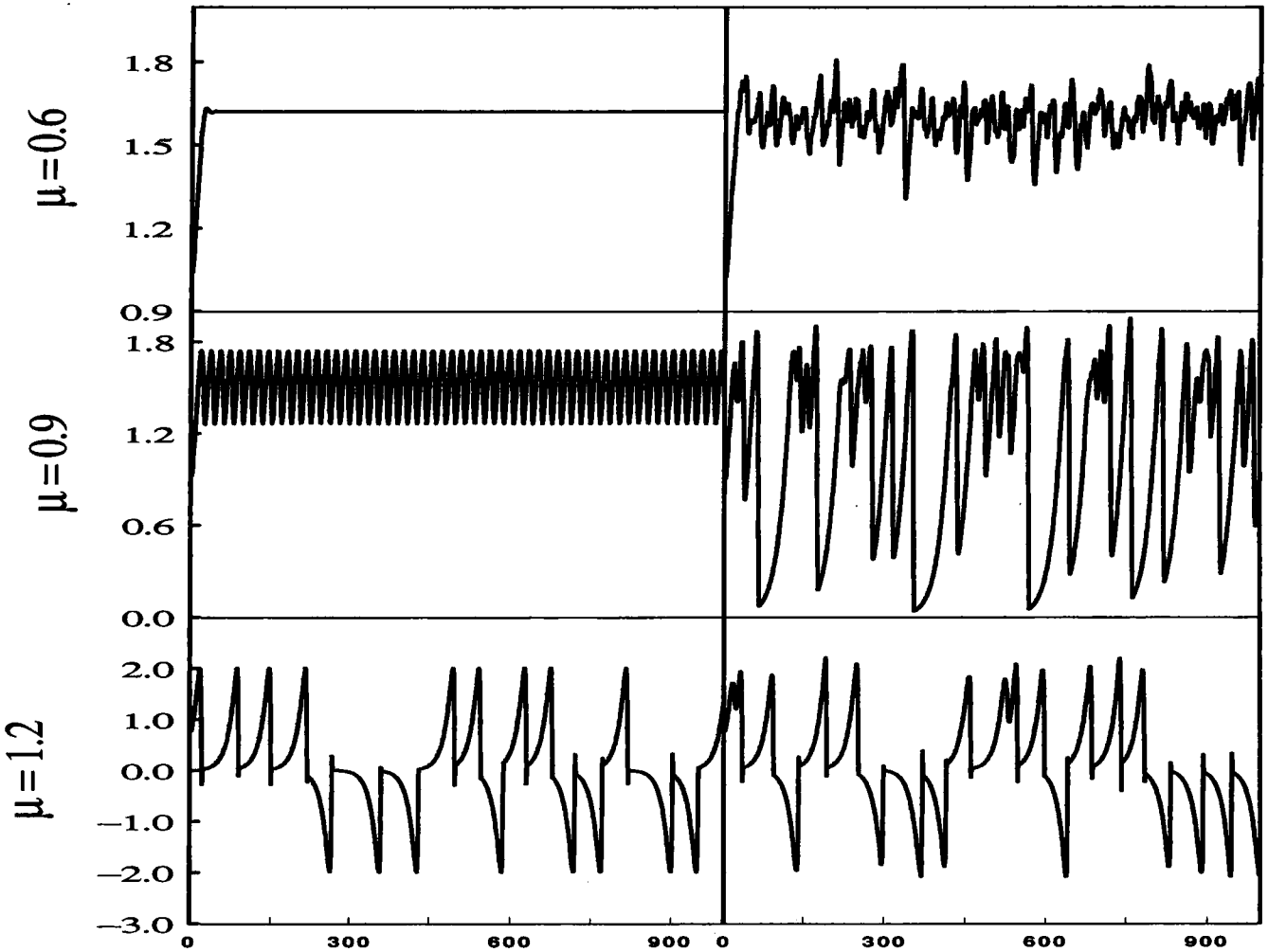

FIGURE 4. Time series of the GDP for $L = 1$ with different values of $\mu = 0.6, 0.9, 1.2$ without noise (left panel) and with noise (right panel).

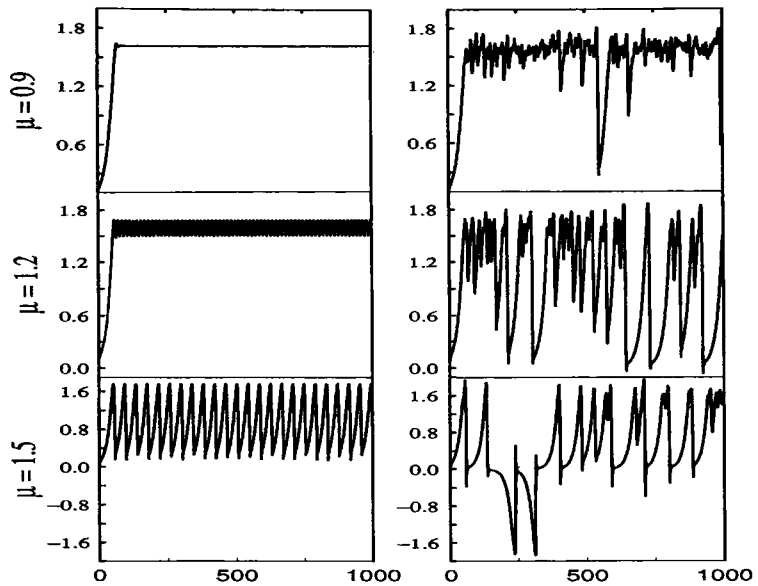

FIGURE 5. Time series of the GDP for $L = 2$ with different values of $\mu = 0.9$, 1.2, and 1.5 without noise (left panel) and with noise (right panel) for $\omega = 0.5$.

maker adjusts more strongly in a way such that both the forces from extrapolation of the trend chasers and the adjustment of the market maker are balanced, the prices fluctuate periodically or quasi-periodically around one of the nonfundamental steady states, as indicated when $\mu = 0.9$. Large adjustment from the market maker results in the breaking of such balance, leading the prices to fluctuate among the three steady states, as indicated when $\mu = 1.2$.

For $L = 2$, let $\omega = 0.5$ be fixed. The price dynamics are similar to the case of $L = 1$, as shown in Figure 5. However, for $d = 1.2 > d^*$, an increase in L (from 1 to 2) can dampen the price fluctuations in terms of the adjustment speed μ of the market maker, which is indicated by comparing the time series of $L = 1$ and $L = 2$ in Figures 4 and 5, respectively, for $\mu = 0.9$ and $\mu = 1.2$.

5.2. Effect of the Decay Rate ω in the GDP

To demonstrate the effect of the decay rate ω on the price dynamics when the trend chasers extrapolate strongly, for fixed $d = 1.2$ and $\mu = 0.95$, we selected $\omega = 0.01, 0.3$, and 0.95, successively. Without noise, the price series converge to one of the nonfundamental steady states for $\omega = 0.95$, as shown in Figure 6 (top plot on left panel). However, for $\omega = 0.3$ and 0.01, the price series display quasi-periodic cycles or even chaotic behavior, which can be characterized by the phase plots in the (x_{t-1}, x_t) plane. In these cases, trajectories converge to either closed orbits or strange attractors. We have generally observed that large decay rates ω stabilize the price dynamics, whereas small decay rates ω destabilize the price dynamics, leading to quasi-periodic cycles and chaotic behavior.

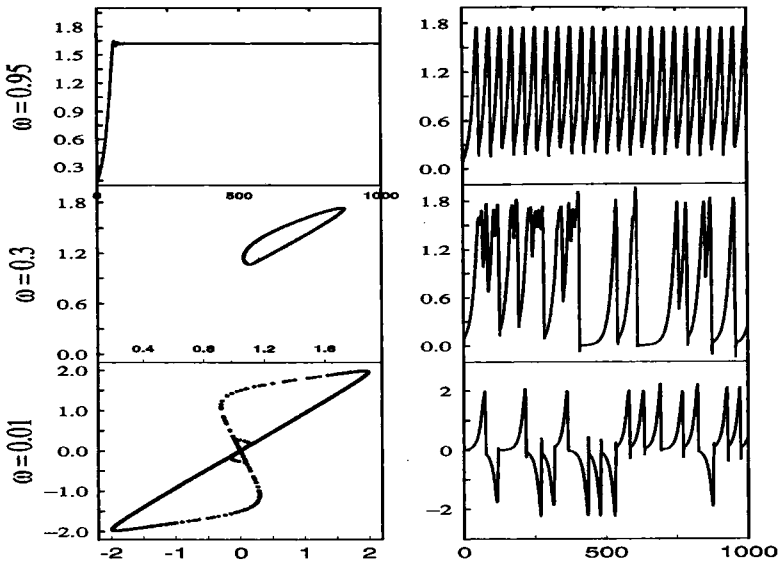

FIGURE 6. Time series and phase plots of the GDP for $L=2$ with different values of $\omega=0.01, 0.3$, and 0.95 without noise (left panel) and the corresponding time series with noise (right panel) for $\mu=0.95$ and $d=1.2$.

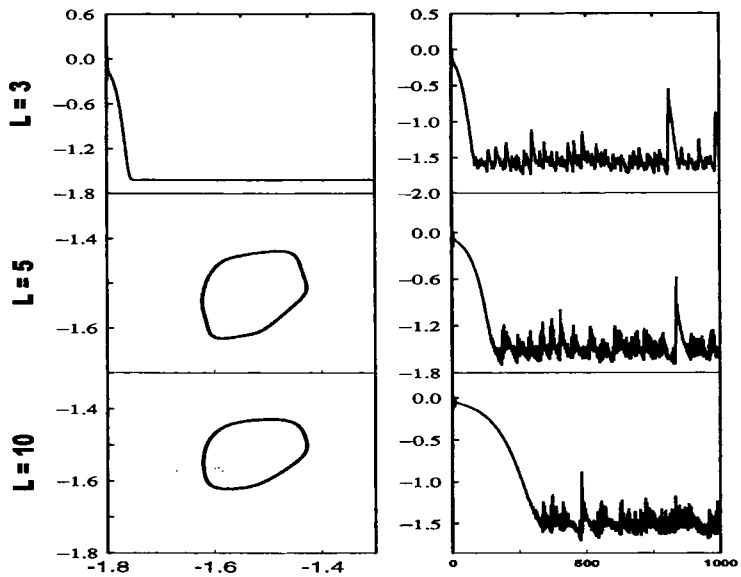

FIGURE 7. Time series and phase plots of the GDP for $L=3$, 5, and 10 without noise (left panel) and with noise (right panel).

5.3. Effect of the Lag Length L

To demonstrate the effect of the leg length L on the price dynamics, we select $d=1.2$, $\mu=1.2$, $\omega=0.95$, and $L=3$, 5, and 10, respectively. The corresponding time series and phase plots are shown in Figure 7. Without noise, the price series converge to one of the nonfundamental equilibria for $L=3$. However, for $L=5$ and 10, the phase plots of the price series tend to closed orbits encircling the nonfundamental steady states.

For the MAP, unlike the asset price model under the Walrasian scenario in Chiarella and He (2002c), the stability conditions are much more related to the speed of the adjustment of the market maker for different lag lengths and, in general, increasing the lag length does not necessary increase the stability of the fundamental steady state. It may or may not be the case that the stability of the nonfundamental equilibria are improved as lag length increases, as indicated by the time-series plots in Figure 8, where the time series for $L = 1$ is stabilized to one of the nonfundamental equilibria for $L = 2$ and 3, but increasing L further to 5 and 10 leads to some periodic cycles, which can be regarded as bifurcations from the nonfundamental equilibria.[7] Figure 8 has been generated using the parameter set

$$R = 1.1, \quad C = 1.0, \quad d = 1.2, \quad a_1\sigma^2 = a_2\sigma^2 = 1, \quad \beta = 3.5, \quad \mu = 1.2$$

and $L = 1, 2, 3, 5$, and 10.

5.4. Effect of the Noise

The analysis of this paper (and indeed that of most of the cited literature) has focused on the dynamics of ABS purely as deterministic difference equation systems. Of course, such systems will be impacted by external random factors (e.g., a noisy dividend process). The way in which these highly nonlinear ABS difference equations process external noise is an important topic for future research. Here, a very preliminary attempt is made to gauge the impact of a noisy dividend process. To this end, Figures 4–8 show the corresponding time-series plots with noise. The noise comes from a stochastic dividend process $y_t = \bar{y} + \epsilon_t$ with i.i.d. noise ϵ_t, uniformly distributed on the interval $[-0.05, 0.05]$, added to the constant dividend process $\bar{y}$. For $L = 1$ and 2, prices are characterized by switching between a phase with prices close to the fundamental price and phases of upward and downward trends near the nonfundamental steady states. This result is different from the Brock–Hommes model under the Walrasian scenario, where the switching is between a phase with prices close to the fundamental price and a phase of upward (or downward) trend. In both the noise-free and noisy cases, the switching among the fundamental and nonfundamental steady states seems to be irregular. The prices fluctuate around one of the nonfundamental steady states when the market maker adjusts the prices weakly (e.g., $\mu = 0.6$ for $L = 1$ in Figure 4 and $\mu = 0.9$ for $L = 2$ in Figure 5, respectively). When such a price adjustment of the market maker is balanced by the extrapolation from the trend followers, the addition of a noisy process can cause the price to fluctuate around the nonfundamental equilibrium with occasional excursions to a neighborhood of the fundamental equilibrium. When such balanced forces disappear, the noisy process causes the prices to fluctuate among the three equilibria ($\mu = 1.2$ for $L = 1$ in Figure 4 and $\mu = 1.5$ for $L = 2$ in Figure 5). Similar price dynamics are observed as the decay rate of the GDP ω decreases (as shown in Figure 6) and the lag length of the GDP L increases. More importantly, addition of the noise process causes prices to take a longer time to come to the level near the

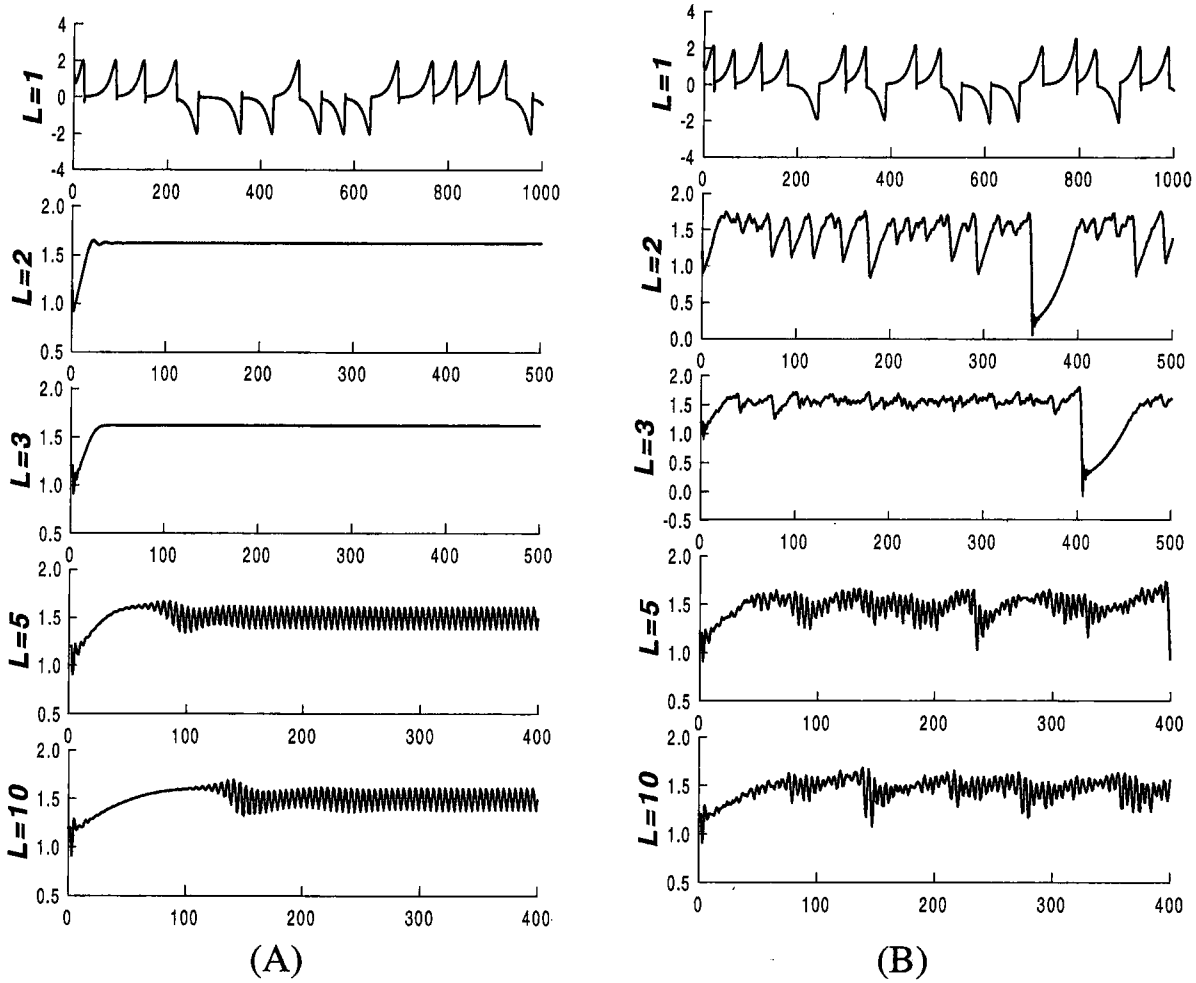

FIGURE 8. Time series of the MAP for different lag lengths $L = 1, 2, 3, 5$, and 10 without noise (A) and with noise (B).

nonfundamental steady state for long lags and, more interestingly, increases the intensity of the apparent volatility clustering (see Figure 7). As might be expected, the addition of noise makes the deterministic dynamic patterns irregular. What is of interest are the occasional excursions to the other nonfundamental steady state; see, for example, the $L = 3$ case in Figure 8, which suggests bimodality of the price distribution.

All of these very preliminary simulations of the noisy nonlinear adaptive belief systems are, of course, very tentative. A worthwhile future research agenda would be to conduct a Monte Carlo–type analysis of the noisy nonlinear ABS with a view to studying its statistical characteristics in terms of price distribution under the noisy dividend process.

6. DYNAMICS OF ABS UNDER THE GEOMETRIC DECAY PROCESS WITH INFINITE MEMORY

Considering the limit of the geometric decay process (when $L \to \infty$), it follows from (14) that

$$g_{t+1} = \omega g_t + (1 - \omega)x_{t+1}. \tag{33}$$

The ABS (22) then becomes

$$\begin{cases} x_{t+1} = x_t + \dfrac{\mu}{2}[(1 + m_t)z_{1,t} + (1 - m_t)z_{2,t}], \\ m_{t+1} = \tanh\left(\dfrac{\beta}{2}[x_{t+1} - Rx_t + \delta_{t+1}][z_{1,t} - z_{2,t}] - \dfrac{\beta C}{2}\right), \\ g_{t+1} = \omega g_t + (1 - \omega)x_{t+1}, \end{cases} \tag{34}$$

where

$$z_{1,t} = \frac{-Rx_t}{a_1\sigma^2}, \qquad z_{2,t} = \frac{d\,g_t - Rx_t}{a_2\sigma^2},$$

which is a third-order nonlinear difference equation system. With regard to the existence of steady state $E(x^*, m^*, \bar{g}^*)$, it can be seen that the result of Proposition 1 still holds with $g^* = x^*$.

6.1. Stability Analysis

The eigenvalue analysis of the fundamental steady state in this case shows that the stability region is given by $D_2(d, \gamma)$, corresponding to the case of $L = 2$ for the GDP.

PROPOSITION 4. *Let* $\delta_t = 0$, $\omega \in (0, 1)$. *Then the fundamental steady state* E_1 *of the ABS* (34) *is LAS if*

$$(d, \gamma) \in D(d, \gamma) = D_2(d, \gamma) = \{(d, \gamma) : 0 < d < d^*, 0 < \gamma < \gamma_2^*\}. \tag{35}$$

It is perhaps surprising that, when the memory is infinite, the stability region of the fundamental steady state is the same as the GDP with lag

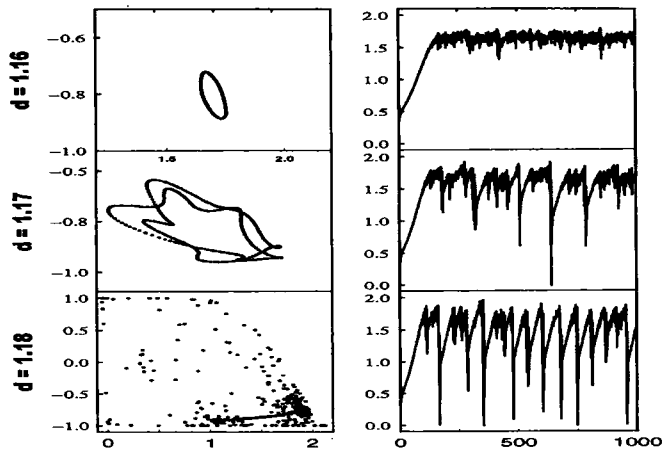

FIGURE 9. Limiting phase plot in (x_t, m_t) plane without noise (left panel) and price time series with noise (right panel) for $\omega = 0.75$, $a_1 = 1$, $a_2 = 0.5$, $\mu = 1$, and $d = 1.16$, 1.17, and 1.18.

Comparison of the stability regions of GDP with $L = 1, 2$, and 3, when memory is infinite leads to the smallest stability region for the fundamental steady state. Furthermore, when the trend followers extrapolate weakly, a decrease in the decay rate enlarges the stability region.

6.2. Numerical Analysis

Numerical simulations indicate that the prices converge to the fundamental price when both the extrapolation rate and the adjustment speed are weak (so that they are located in the local stability regions) and prices become unbounded when the prices are strongly adjusted by the market maker. Also, when the trend chasers extrapolate strongly (so that $d > d^*$), the effect of the decay rate ω on the price dynamics is similar to the case for GDP with $L \geq 2$. The following discussion considers a situation in which the trend chasers are less risk averse than the fundamentalists; that is, $a < 1$. It is found that, in this case, the decay rate has a different influence on the price dynamics when the trend followers extrapolate strongly. In the following simulations, let

$$R = 1.1, \quad C = 1.0, \quad a_1 = 1, \quad a_2 = 0.5, \quad \beta = 3.5, \quad \sigma^2 = 1, \quad \mu = 1.0.$$

6.2.1. Effect of the extrapolation rate d. Let $\omega = 0.75$ be fixed. For different values of d, phase plots in the (x_t, m_t) plane without noise and price time series with noise are plotted in Figure 9. For $d = 1.16$, the (x_t, m_t) trajectories converge to a closed orbit encircling one of the nonfundamental steady states. As d increases, the attractors become more complicated. Addition of the noise process leads prices to fluctuate around the nonfundamental steady-state level for $d = 1.16$ and, as d increases further, the volatility pattern becomes more complicated, no doubt reflecting the interaction of the underlying complex deterministic dynamics and the noise. Also of interest are the not infrequent excursions to a neighborhood

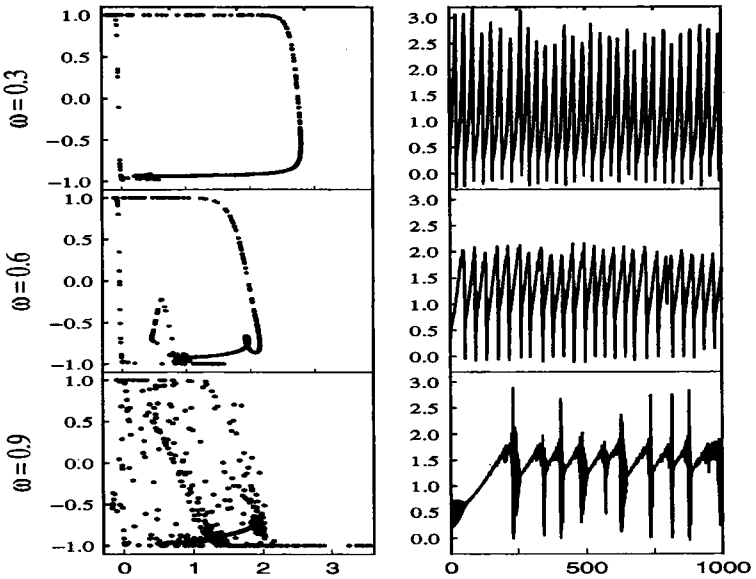

FIGURE 10. Limiting phase plot in (x_t, m_t) without noise (left panel) and price time series with noise (right panel) for $d = 1.2$, $a_1 = 1$, $a_2 = 0.5$, $\mu = 1$, and $\omega = 0.3$, 0.6, and 0.9.

fundamental steady state. Here again, one might conjecture that the noisy nonlinear ABS is characterized by a bimodal distribution for prices.

6.2.2. Effect of the decay rate ω. Let $d = 1.2$ be fixed. For different values of $\omega = 0.3$, 0.6, and 0.9, phase plots in the (x_t, m_t) plane without noise and price time series with noise are plotted in Figure 10. For $\omega = 0.3$, the time series (x_t, m_t) converge to some strange attractor encircling one of the nonfundamental steady states. As ω increases, the attractors become more complicated. The addition of the noisy dividend process in this case seems to break the price pattern of the deterministic case. Now, prices fluctuate between the fundamental and one nonfundamental equilibria ($\omega = 0.3$ and $\omega = 0.6$). As ω increases further ($\omega = 0.9$), one also observes high apparent volatility clustering of the prices.

7. CONCLUSION

Using a market-maker scenario, the present paper has incorporated risk and learning schemes into an asset-pricing model with heterogeneous beliefs. Fundamentalists and trend chasers are the main trading groups that drive the various dynamics of price changes. The importance of different learning schemes for price dynamics has been highlighted in earlier literature (and in this paper). However, the precise way in which the market clearing mechanism affects the dynamics of asset prices in models involving heterogeneity and learning has not received a great deal of attention in the literature. The current paper has sought to address this deficiency.

Under a market-maker scenario, this paper studies how the dynamics of asset prices are affected by different risk attitudes and different learning schemes of different types of investors. In particular, the paper concentrates on models of

fundamentalists and trend followers who follow recursive geometric decay (learning) processes with both finite and infinite memory. The analysis depicts how the dynamics are affected by various key elements (or parameters) of the model, such as the adjustment speed μ of the market maker, the extrapolation rate d of the trend followers, the decay rate ω of the GDP, the lag length L used in the learning GDP, and external random factors. The results of the paper can be summarized as follows:

- With the Walrasian market-clearing scenario in Brock and Hommes (1998) and Chiarella and He (2002c), the local stability of the fundamental steady state is completely characterized by the extrapolation rate of the trend followers and the relative risk-aversion ratio between the fundamentalists and the trend followers, not the lag length used in the moving-average learning process for the trend followers, and complicated price dynamics can only be generated through saddle-node-type bifurcation. In contrast, this paper has found that, with the market-maker scenario, the stability of the fundamental steady state is maintained only when the speed of the adjustment of the market maker is low and balanced with the extrapolation rate of the trend followers. Furthermore, complicated price dynamics can be generated through either saddle-node or flip-type bifurcations.
- Similar to the Walrasian scenario, in general, different lag lengths can complicate the price dynamics, but an increase in lag length may not necessarily enlarge the local stability region.
- The decay rate of the geometric-decay learning process for the trend followers has more complicated effects on the price dynamics. When the trend followers extrapolate weakly, a decrease in the decay rate for both finite and infinite memory processes enlarges the local stability region of the fundamental steady state. However, when the trend followers extrapolate strongly, the price dynamics can be stabilized by an increase in the decay rate when both the fundamentalists and trend followers have the same risk-aversion coefficients, but can be destabilized when the trend followers are less risk averse than the fundamentalists.
- When external random factors (e.g., a noisy dividend process) impinge on the dynamics, some interesting additional features are observed, in particular, excursions to other equilibria and volatility clustering. The volatility appears to be much greater than that of the external i.i.d. dividend process. However, the paper has only been able to give some brief and preliminary insights into this issue that should be topics for future research.

In summary, the study finds that the resulting dynamical behavior of the asset pricing model under the market-maker scenario is considerably enriched and has some significant differences from the Walrasian scenario. However, the interaction of external noise with the nonlinear dynamics of the model is a topic that requires more extensive research. The techniques discussed by Arnold (1998) may be useful in this regard.

The model considered here has concentrated on the interaction of two types of investors. Of course, to mimic the price behavior observed in real financial markets, it would be preferable to allow for a larger number of investor types and/or a richer set of learning schemes from which the various investor types update their beliefs. A number of authors [see Chen and Yeh (1997, 1999), Lux and Marchesi (1999)] have developed models (similar in spirit to the one of this paper) containing many different trader types choosing from a rich set of learning schemes. Much of the analysis by those authors is necessarily based on numerical simulations. However, their models do mimic the types of price patterns observed in real financial markets, such as ARCH effects, volatility clustering, and fat tails. This paper should be seen as complementing these more numerically oriented studies by obtaining analytical results concerning local stability analysis and thus gaining some insight into the maps determining the nonlinear dynamics of this class of model.

NOTES

1. A more extensive study involving more types of agents appears in Chiarella and He (2000).
2. When different types of investors, for example, "smart-money" investors and "noise traders," are involved in the market, it is believed [see Miller (1977), Black (1986), Summers (1986), Fama and French (1988), Poterba and Summers (1988), DeLong et al. (1990), Campbell and Kyle (1993)] that the *smart-money* investors are more risk averse than *noise traders*. One message from this strand of literature is that the differing attitudes to risk of the various types of investors have an influence on the price dynamics observed.
3. Equation (4) may be derived using a mean-variance framework or by assuming that agents have a utility of wealth function $-\exp(-a_h W)$.
4. See Hommes (2001) for further discussion of fundamental price, speculative bubbles, and rational bubbles.
5. Gaunersdorfer (2000) investigates the Brock–Hommes evolutionary asset-pricing model with time-varying variances, determined by a weighted average (with exponentially decreasing weights) of past squared returns. She shows that the results are quite similar to the Brock–Hommes results with constant variance, which is also confirmed by the analysis by Chiarella and He (2002c). For the market-maker model, it would also be feasible to introduce a changing variance. However, numerical simulations (not reported here) also indicate that the results are quite similar to those obtained here with constant variance. Therefore, to simplify the following analysis, constant variance is assumed.
6. The generation of the simulations with noise is discussed in Section 5.4.
7. The task of conducting a stability analysis of the nonfundamental equilibria is not included in this paper due to space limitations.

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APPENDIX A

A.1. PROOF OF PROPOSITION 1

Let $\bar{x}$, $\bar{m}$ be the steady state of (22) and $A_1 = R/a_1$, $A_2 = (R-d)/a_2$. Then,

$$\begin{aligned} \bar{x} &= \bar{x} + \frac{\mu}{2}[(1+\bar{m})\bar{z}_1 + (1-\bar{m})\bar{z}_2] \\ \bar{m} &= \tanh\left(\frac{\beta(1-R)}{2}\bar{x}[\bar{z}_1 - \bar{z}_2] - \frac{\beta C}{2}\right) \end{aligned} \tag{A.1}$$

with $\bar{z}_1 = -R\bar{x}/(a_1\sigma^2) = -A_1\bar{x}/\sigma^2$ and $\bar{z}_2 = (d-R)\bar{x}/(a_2\sigma^2) = -A_2\bar{x}/\sigma^2$. Hence the solution of (A.1) satisfies

$$\begin{aligned} &[A_1(1+\bar{m}) + A_2(1-\bar{m})]\bar{x} = 0 \\ &\bar{m} = \tanh\left[\frac{\beta}{2\sigma^2}(R-1)(A_1 - A_2)\bar{x}^2 - \frac{\beta}{2}C\right]. \end{aligned} \tag{A.2}$$

The first equation implies $\bar{x} = 0$, or $\bar{m} = (A_2 + A_1)/(A_2 - A_1) \equiv m^* \cdot \bar{x} = 0$ implies $\bar{m} = m^{eq}$ which leads to the existence of E. Assume $\bar{x} \neq 0$; then, $\bar{m} = m^*$.

Note that $A_1 > 0$. If $d < R$, then $A_2 > 0$ and hence $m^* > 1$ for $A_2 > A_1$ and $m^* < -1$ for $A_2 < A_1$. Therefore, there is no other solution when $d < R$. When $d > R$, then $A_2 < 0$ and one can verify that $\bar{m} \in (-1, 1)$. If $A_2 < -A_1$, that is, $d > (1+a)R$, then $0 < m^* < 1$ and hence equation (A.2) always has a solution for some $\pm\bar{x}$. This indicates the existence of $E_\pm$. Now, if $A_2 > -A_1$—that is, $d < (1+a)R$—then $-1 < m^* < 0$. Then, equation (A.2) has a solution for some $\pm\bar{x}$ if and only if $m^{eq} < m^*$, which is equivalent to $d > d^*$. Note that $d^* < (1+a)R$. Hence, in both cases, equation (A.2) has a solution for some $\pm\bar{x}$. This completes the proof.

A.2. PROOF OF PROPOSITION 2

The system (27) is equivalent to an $L+1$ order difference equation

$$x_{t+1} = F(x_t, x_{t-1}, \ldots, x_{t-(L-1)}, x_{t-L}),$$

where

$$\begin{cases} F(x_t, x_{t-1}, \ldots, x_{t-L}) = x_t + \dfrac{\mu}{2}\left[(1+m_t)\dfrac{(-Rx_t)}{a_1\sigma^2} + (1-m_t)\dfrac{dg_t - Rx_t}{a_2\sigma^2}\right] \\ g_t = b_1 x_t + b_2 x_{t-1} + \cdots + b_L x_{t-(L-1)} \end{cases}$$

and

$$m_t = \tanh\left[\frac{\beta}{2}(x_t - Ry_t)\left(\frac{-Ry_t}{a_1\sigma^2} - \frac{dg_t - Ry_t}{a_2\sigma^2}\right) - \frac{\beta C}{2}\right].$$

At the steady state E, $x^* = 0$. Evaluate the partial derivatives of F at $(x_t, x_{t-1}, \ldots, x_{t-L}) = (x^*, x^*, \ldots, x^*) = (0, 0, \ldots, 0)$:

$$\begin{aligned} \frac{\partial F}{\partial x_t} &= 1 + \frac{\mu}{2a_2\sigma^2}(1 - m^{eq})\left[b_1 d - R\left(1 + a\frac{1+m^{eq}}{1-m^{eq}}\right)\right] \\ &= 1 + 2\gamma(1 - m^{eq})[b_1 d - d^*] \equiv A, \\ \frac{\partial F}{\partial x_{t-i}} &= \frac{\mu}{2a_2\sigma^2}(1 - m^{eq})\, db_{i+1} = 2\gamma(1 - m^{eq})\, db_{i+1} \equiv B_{i+1}, \\ &\qquad\qquad (i = 1, 2, \ldots, L-1), \\ \frac{\partial F}{\partial x_{t-L}} &= 0. \end{aligned}$$

The characteristic equation of the fundamental steady state E is given by

$$\lambda\left[\lambda^L - A\lambda^{L-1} - B_2\lambda^{L-2} - \cdots - B_L\right] = 0$$

Thus, one eigenvalue is zero and the remaining L eigenvalues are given by the solutions to

$$\Gamma_L(\lambda) \equiv \lambda^L - A\lambda^{L-1} - B_2\lambda^{L-2} - \cdots - B_L = 0.$$

- For $L = 1$, $b_1 = 1$, and
$$\Gamma_1(\lambda) \equiv \lambda - A = 0,$$
one obtains $|\lambda| < 1$ iff $|A| < 1$; that is,
$$-1 < 1 + 2\gamma(1 - m^{eq})[d - d^*] < 1.$$
That is,
$$0 < \gamma(1 - m^{eq})(d^* - d) < 1,$$
which is equivalent to
$$0 < d < d^*, \qquad 0 < \gamma < \frac{1}{(1 - m^{eq})(d^* - d)} = \gamma_1^*.$$
Also, $\lambda = 1$ along $d = d^*$ and $\lambda = -1$ along $\gamma = \gamma_1^*$.
- For $L = 2$, $b_1 = 1/(1+\omega)$, $b_2 = \omega/(1+\omega)$. Hence, $B = 2\gamma(1 - m^{eq})\, d\omega/(1+\omega)$ and
$$\Gamma_2(\lambda) \equiv \lambda^2 - A\lambda - B = 0.$$
Using Jury's test, $|\lambda_i| < 1$ iff $\Gamma_2(1) > 0$ and $(-1)^2\Gamma_2(-1) > 0$; that is,
$$1 - A - B > 0,$$
$$1 + A - B > 0.$$
One can verify that these conditions are equivalent to
$$0 < d < d^*, \qquad 0 < \gamma < \gamma_2^*.$$
Also, $\lambda = 1$ is one of the eigenvalues along $d = d^*$ and $\lambda = -1$ is one of the eigenvalues along $\gamma = \gamma_2^*$.
- For $L = 3$, $b_1 = 1/(1+\omega+\omega^2)$, $b_2 = \omega/(1+\omega+\omega^2)$, $b_3 = \omega^2/(1+\omega+\omega^2)$, and
$$\begin{cases} B_2 = 2\gamma(1 - m^{eq})\dfrac{\omega d}{1+\omega+\omega^2} \\ B_3 = 2\gamma(1 - m^{eq})\dfrac{\omega^2 d}{1+\omega+\omega^2}. \end{cases}$$
$$\Gamma_3(\lambda) \equiv \lambda^3 - A\lambda^2 - B_2\lambda - B_3 = 0.$$
Using Jury's test, $|\lambda_i| < 1$ iff $-B_2 < 3$ and
$$\pi_1 \equiv 1 - A - B_2 - B_3 > 0,$$
$$\pi_2 \equiv 1 + A - B_2 + B_3 > 0,$$
$$\pi_3 \equiv 1 + B_2 + AB_3 - B_3^2 > 0.$$
Furthermore, $\lambda = 1(-1)$ is one of the eigenvalues along $\pi_1 = 0$ ($\pi_2 = 0$). One can verify that $\pi_1 > 0$ if $d < d^*$, $\pi_2 > 0$ if $0 < \gamma < \gamma_3^*$ and $\pi_1 > 0$, $\pi_2 > 0$ imply $\pi_3 > 0$. Note that
$$\frac{1-\omega}{1+\omega} < \frac{1-\omega+\omega^2}{1+\omega+\omega^2} < 1,$$

which implies that

$$\gamma_2^* < \gamma_3^* < \gamma_1^*.$$

Therefore

$$D_2 \subset D_3 \subset D_1.$$

A.3. ROUCHE'S PROPOSITION

If the complex functions $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve γ and if $|g(z)| < |f(z)|$ on γ, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside γ.

A.4. PROOF OF PROPOSITION 3

(i) To apply Rouche's Proposition, take γ to be the unit circle $|z|=1$, the function $f(z)=z^L$, while $g(z)=-Az^{L-1}-B_2z^{L-2}-\cdots-B_2$. Then, on γ: $|z|=1$, $|f(z)|=1$ and $|g(z)| \le |A| + \sum_{i=2}^{L} |B_i|$. Note that

$$\sum_{i=2}^{L} |B_i| = 2\gamma(1-m^{eq})d(1-b_1), \qquad b_1 = b.$$

It can be verified that

$$|A| + \sum_{i=2}^{L} |B_i| < 1 \tag{A.3}$$

is equivalent to the conditions $d < d^*$ and (28) of Proposition 3, leading to $|g(z)| < |f(z)|$ on γ. By Rouche's Proposition, it follows that $p(z)=f(z)+g(z)$ and $f(z)=z^L$ have the same number of zeros inside the unit circle. Since $f(z)=z^L=0$ has L repeated zeros $z=0$ inside the unit circle, all the zeros of $p(\lambda)$ lie inside the unit circle and, consequently, E is locally asymptotically stable.

(ii) To apply Rouche's Proposition, take γ to be the unit circle $|z|=1$. The function $f(z)=(B_1-A)z^{L-1}$, while $g(z)=z^L-B_1z^{L-1}-B_2z^{L-2}-\cdots-B_2$ with $B_1=b$. Then, on γ: $|z|=1$, $|f(z)|=|b-A|$ and $|g(z)| \le 1+\sum_{i=1}^{L}|B_i| = 1+2\gamma(1-m^{eq})d$. Under the condition $0<d<d^*$ and $\gamma > \gamma_1^*$, one can verify that $|g(z)| \le 1+\sum_{i=1}^{L}|B_i| \le |b-A| = |f(z)|$ on γ. It follows from Rouche's Proposition that $p(z)=f(z)+g(z)$ and $f(z)=(b-A)z^{L-1}$ have the same number of zeros inside the unit circle. Since $f(z)=(b-A)z^{L-1}=0$ has $L-1$ repeated zeros $z=0$ inside the unit circle, there are only $L-1$ zeros of $p(\lambda)$ lying inside the unit circle, which implies that there is one zero satisfying $|z| \ge 1$. Consequently, E is unstable.

A.5. PROOF OF COROLLARY 1

In the case of $L=1$, the local stability result follows from the Proposition 2.

Now assume $0<d<R$ and let

$$h(m) = 1 - \frac{\mu}{2a_2\sigma^2}[aR(1+m)+(R-d)(1-m)].$$

Then it follows from $-1 < m < 1$ and

$$h'(m) = -\frac{\mu}{2a_2\sigma^2}[(a-1)R + d]$$

that if $d > (1-a)R$, then $h(1) < h(m) < h(-1)$, and if $d < (1-a)R$, then $h(-1) < h(m) < h(1)$. Note that $h(1) = 1 - 4\gamma aR$ and $h(-1) = 1 - 4\gamma(R-d)$. From the first equation of (29), one can see that E is globally asymptotically stable iff $-1 < h(1), h(-1) < 1$ for $d > (1-a)R$ and $-1 < h(-1), h(1) < 1$ for $d < (1-a)R$, which can be verified under either one of the conditions in Corollary 1.

A.6. A STABILITY RESULT FOR THE MAP WITH ANY LAG LENGTH

It is generally difficult to obtain explicit sufficient conditions on the local stability of the GDP for any lag length. However, when $\omega = 1$, the GDP leads to the MAP and $B_i = B \equiv 1 + 2\gamma(1 - m^{eq})[d/L - d^*]$. In this case, the following Lemma from Chiarella and He (2002c) can be used to obtain a stability result.

LEMMA A.1. *Let*

$$p(\lambda) = \lambda^L + \gamma\lambda^{L-1} + \gamma\lambda^{L-2} + \cdots + \gamma\lambda + \gamma.$$

Then, zeros of $p(\lambda)$ lie inside the unit circle if and only if

$$-\frac{1}{L} < \gamma < 1.$$

If μ is selected such that $A = B$, then Lemma A.1 leads to the following stability result for $L \geq 1$.

COROLLARY A.1. *Assume that*

$$\gamma = 1/[2(1 - m^{eq})\, d^*]. \tag{A.4}$$

Then, E of system (22) is LAS if $0 < d < d^*$.

Under the Walrasian scenario, the same sufficient condition $d < d^*$ on the local stability of the Walrasian fixed equilibrium E_1 is obtained by Chiarella and He (2002c). Note that the condition $d < d^*$ is independent of the lag length L. Under the market-maker scenario, Corollary A.1 indicates that, when the speed of the adjustment of the market maker is selected as in (A.4), the stability condition is independent of the lag length L. Let

$$a^* = \left[\frac{d^*}{R} - 1\right]\frac{1 - m^{eq}}{1 + m^{eq}}. \tag{A.5}$$

Hence, under condition (A.4), E_1 is LAS for $a > a^*$. In the special case that the information costs and the risk-aversion coefficients are the same (i.e., $C = 0$ and $a_1 = a_2$), (A.4) and $d < d^*$ lead to

$$\gamma = \frac{\mu}{4a_2\sigma^2} = \frac{4}{R}, \qquad d < 2R.$$

This indicates that, in this case, the fundamental steady state is LAS if the trend chasers do not extrapolate strongly ($0 < d < 2R$) and the speed of the adjustment of the market maker

is proportional to the risk-aversion coefficient (a_2) with the product of the variance (σ^2) and the discount rate of return ($1/R$) as the coefficient.

A.7. PROOF OF PROPOSITION 4

The ABS has the following form:

$$\begin{cases} x_{t+1} = F(x_t, y_t, g_t) \\ y_{t+1} = x_t \\ g_{t+1} = G(x_t, y_t, g_t), \end{cases}$$

where

$$\begin{cases} F(x_t, y_t, g_t) = x_t + \frac{\mu}{2}\left[(1+m_t)\dfrac{-Rx_t}{a_1\sigma^2} + (1-m_t)\dfrac{dg_t - Rx_t}{a_2\sigma^2}\right] \\ G(x_t, y_t, g_t) = \omega g_t + (1-\omega)F(x_t, y_t, g_t) \end{cases}$$

and

$$m_t = \tanh\left[\frac{\beta}{2}(x_t - Ry_t)\left(\frac{-Ry_t}{a_1\sigma^2} - \frac{dg_t - Ry_t}{a_2\sigma^2}\right) - \frac{\beta C}{2}\right].$$

One can verify that, at the steady state $(x^*, y^*, g^*) = (0, 0, 0)$,

$$\begin{cases} \dfrac{\partial F}{\partial x_t} = A = 1 - 2\gamma(1 - m^{eq})\,d^*, & \dfrac{\partial G}{\partial x_t} = (1-\omega)A \\ \dfrac{\partial F}{\partial y_t} = 0, & \dfrac{\partial G}{\partial y_t} = 0 \\ \dfrac{\partial F}{\partial g_t} = B = 2\gamma d(1 - m^{eq}), & \dfrac{\partial G}{\partial g_t} \equiv C = \omega + (1-\omega)B. \end{cases}$$

Hence, the characteristic equation is

$$\Gamma_3(\lambda) \equiv \lambda[(\lambda - A)(\lambda - C) - AB(1-\omega)] = 0.$$

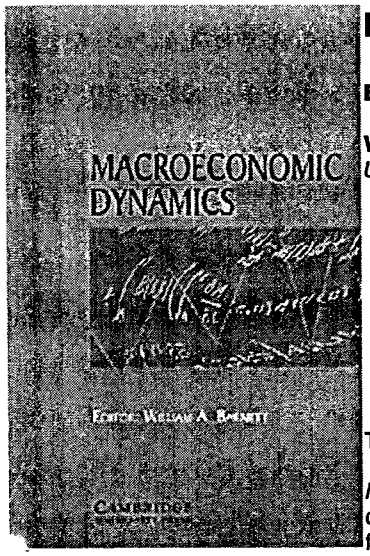
Note that

$$D \equiv AC - AB(1-\omega) = \omega A.$$

Then,

$$\Gamma_3(\lambda) = \lambda[\lambda^2 - (A+C)\lambda + \omega A] = 0$$

and $|\lambda_i| < 1$ iff $0 < d < d^*$, $0 < \gamma < \gamma_2^*$. Furthermore, $\lambda = 1 (\lambda = -1)$ is one of the eigenvalues along $d = d^* (\gamma = \gamma_2^*)$.



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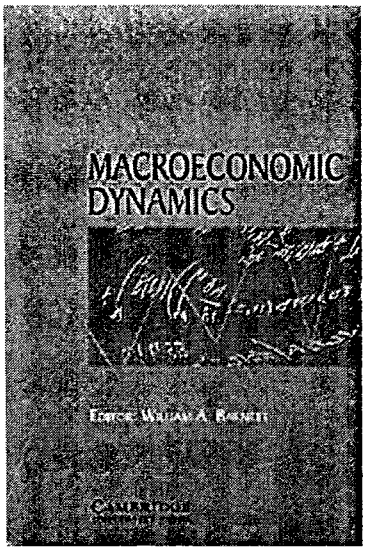

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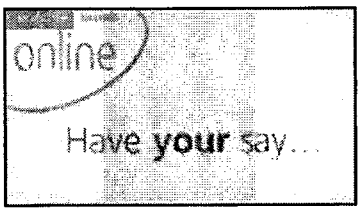

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