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THE GLOBAL SOLUTION OF AN INITIAL VALUE PROBLEM FOR A GENERALIZED BOUSSINESQ EQUATION

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Abstract: This paper deals with the well-posedness of the global solution of a small initial value problem for a generalized Boussinesq equation. The conditions for the existence and uniqueness of the solution to the problem are established in a Sobolev space. It is also proved that the global solution decays exponentially in time.

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1. Introduction

In recent years, many mathematicians have focused on studying various nonlinear evolution equations. One of the equations attracting attention is the following Boussinesq equation

$$u_{tt} = \alpha_1 u_{xxxx} + u_{xx} - \beta(u^2)_{xx}, \quad t > 0, \quad x \in R^1, \tag{1}$$

where $u(x,t)$ is the elevation of the free surface of fluid, the subscripts denote partial derivatives, and the constant coefficients α_1 and β depend on the depth of fluid and the characteristic speed of the long waves.

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Equation (1) governs the motion of long waves on shallow water and the propagation of nonlinear oscillations of an elastic string. This equation possesses traveling waves solution called solitary waves and a scientific explanation of the existence of solitary waves has been given by Boussinesq [1].

Milewsky and Keller [3] have deduced an isotropic pseudo-differential equation for the motion of waves in fluids with a free surface and an arbitrary uniform depth. Various versions of the Boussinesq equations discussed in the literature [2, 4-9] can be obtained in the way similar to that proposed in [3]. They all possess one common characteristics, namely, they are perturbations of the linear wave equation that take into considerations of the effects of weak nonlinearty and dispersion.

In references [10-13], equation (1) has been researched from various aspects. Clarkson [10] proposed a general method for obtaining similarity reductions and constructing exact solutions. Hirota [11] deduced conservation laws and then examined the numerical solutions. Yazhima [13] investigated the nonlinear evolution of a linearly stable solution. The exponentially decaying solutions of the spherical Boussinesq equation have been established by Nakamura [14]. Galkin et al [15] have devoleped rational solutions of the one dimensional Boussinesq equation with both zero and nonzero boundary conditions at the infinity in space. The structure of the solution in [15] extends a family of rational solutions of the Korteweq-de Vries equation to the case of two-wave processes. Various types of generalization of the classical Boussinesq equation have been studied in various aspects [8, 16], in particular the well-posedness of the Cauchy problem for the following equation

$$u_{tt} = -u_{xxxx} + u_{xx} - (f(u))_{xx}, t > 0, x \in R^1. \tag{2}$$

In [8], it has been established that the special solitary wave solutions of (2) are nonlinearly stable for a range of wave speeds. The local and global well-posedness of the problem has been proved by transforming equation (2) into a system of nonlinear Schrödinger equations. Also the authors conclude that initial data lying relatively close to a stable solitary wave evolves into a global solution of equation (2) under some assumptions restricting the data and nonlinear term $f(u)$. Sufficient conditions for nonexistence of global solutions in time to various initial boundary value problems of a Boussinesq equation were obtained in [17, 18].

Varlamov [19] considered the initial-boundary value problem for the following damped Boussinesq equation

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} - \beta(u^2)_{xx}, t > 0, x \in R^1, \tag{3}$$

where the second term on the left hand side is responsible for dissipation, $\beta \in R^1, \alpha$ and b are positive constants and satisfy the condition $\alpha > b^2$. The initial boundary value problem for equation (3) with small initial value is established for the case of one space dimension. The classical solution is constructed and its uniqueness is proved. Also the long-time asymptotics is obtained in explicit form. The existence of global asymptotic solutions of the initial value problem for equation (3) has been studied by Varlamov [20, 21]. However, proof for the uniqueness of the solution has not yet been established.

The aim of this paper is to study the initial value problem for the following generalized Boussinesq equation

$$u_{tt} - au_{ttxx} - 2bu_{txx} + mu_t = -nu_{xxxx} + u_{xx} + \beta(f(u))_{xx},$$
$$t > 0, \quad x \in R^1, \tag{4}$$

where $a \geq 0, b > 0, m > 0, n > 0$ are constants and $\beta \in R^1$ and the function f is a polynomial with order p and $f(0) = 0$. The existence and uniqueness of the global solution in Sobolev space $C([0, +\infty), H^s(R^1)) \cap C^1([0, +\infty), H^{s-1}(R^1))$ will be established for equation (4) with the following initial conditions

$$\begin{aligned} u(x,0) &= \varepsilon\varphi(x), \quad x \in R^1, \\ u_t(x,0) &= \varepsilon\psi(x), \quad x \in R^1, \end{aligned} \tag{5}$$

where $\varepsilon > 0$ is a small parameter.

2. Existence and Uniqueness Theorem

Throughout this paper, we let $C\left([0,T], H^{s+1}\left(R^1\right)\right) \cap C^1\left([0,T], H^s\left(R^1\right)\right)$ denote the Sobolev space with norm defined by

$$|||u||| = \sup_{t\in[0,T]} \left(\|u\|_{s+1} + \|u_t\|_s\right),$$

where

$$\|u\|_s = \left(\int_{-\infty}^{+\infty} < \xi >^{2s} |\widehat{u}(\xi, t)|^2 d\xi \right)^{\frac{1}{2}},$$

$$\widehat{u}(\xi, t) = F[u(x,t)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ix\xi} u(x,t) dx,$$

$$< \xi > = \left(1 + |\xi|^2\right)^{\frac{1}{2}}.$$

We also denote $X_s(T)$ by

$$X_s(T) = C\left([0,T], H^{s+1}\left(R^1\right)\right) \cap C^1\left([0,T], H^s\left(R^1\right)\right),$$

with

$$X_s(\infty) = C\left([0,+\infty), H^{s+1}\left(R^1\right)\right) \cap C^1\left([0,+\infty), H^s\left(R^1\right)\right).$$

In addition, we assume that $\varphi(x) \in H^{s+1}(R^1)\psi \in H^s(R^1)$.

Theorem 1. *Suppose $\varphi(x) \in H^{s+1}(R^1)$ and $\psi(x) \in H^s(R^1)$ with $s > \frac{1}{2}$. If $a \geq 0$, $b > 0$, $m > 0$, $n > 0$, $p \geq 0$, $a+n > b^2$, and the nonlinear term $f(u)$ is a polynomial with order p and $f(0) = 0$, then there exists a unique solution $u(x,t) \in X_s(\infty)$ to the problem defined by (4)-(5). Furthermore, the solution $u(x,t)$ satisfies the following inequality*

$$\|u\|_{s+1} < e^{-kt}(2\varepsilon A),$$

where $k = \min\{m, \frac{b}{a}\}$ and A is a constant independent of ε.

Proof. We divide the proof of the theorem into two parts. Firstly, we prove that there exists a solution $u(x,t) \in X_s(\infty)$ to the problem defined by(4)-(5). Then, we prove that the solution is unique.

(a) *Existence.* Taking the Fourier transform of equations (4)-(5), we get

$$\begin{aligned} &(1 + a\xi^2)\widehat{u}''(\xi,t) + 2(b\xi^2 + m)\widehat{u}'(\xi,t) + (n\xi^4 + \xi^2 + p^2)\widehat{u}(\xi,t) \\ &\qquad = -\beta\xi^2\widehat{f(u)}(\xi,t), \\ &\widehat{u}(\xi,0) \ = \varepsilon\widehat{\varphi}(\xi), \\ &\widehat{u}_t(\xi,0) = \varepsilon\widehat{\psi}(\xi), \end{aligned} \tag{6}$$

which give the following solution

$$\widehat{u}(\xi,t) = \varepsilon e^{-\frac{b\xi^2+m}{1+a\xi^2}t}\left\{\left[\cos(\sigma_\xi t) + \frac{b\xi^2+m}{1+a\xi^2}\cdot\frac{\sin(\sigma_\xi t)}{\sigma_\xi}\right]\widehat{\varphi}(\xi) + \frac{\sin(\sigma_\xi t)}{\sigma_\xi}\widehat{\psi}(\xi)\right\}$$
$$-\frac{\beta\xi^2}{(1+a\xi^2)\sigma_\xi}\int_0^t \exp\left[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)\right]\sin[\sigma_\xi(t-\tau)]\widehat{f(u)}(\xi,\tau)d\tau, \tag{7}$$

where

$$\sigma_\xi = \frac{\sqrt{an\xi^6 + (a+c-b^2)\xi^4 + (pa+2mb+1)\xi^2 + p^2 + m^2}}{1+a\xi^2},$$
$$a + n - b^2 > 0.$$

Denoting u_0 by

$$u_0 = \varepsilon F^{-1}\left[e^{-\frac{b\xi^2+m}{1+a\xi^2}t}\left\{\left[\cos(\sigma_\xi t) + \frac{b\xi^2+m}{1+a\xi^2}\cdot\frac{\sin(\sigma_\xi t)}{\sigma_\xi}\right]\widehat{\varphi}(\xi)\right.\right.$$
$$\left.\left. + \frac{\sin(\sigma_\xi t)}{\sigma_\xi}\widehat{\psi}(\xi)\right\}\right], \tag{8}$$

where F^{-1} represents the inverse Fourier transform. Letting $k = \min\{m, \frac{b}{a}\}$, we have

$$\frac{b\xi^2+m}{1+a\xi^2} \geq k, \tag{9}$$

and consequently

$$\exp\left[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)\right] \leq \exp\left[-k(t-\tau\right]. \tag{10}$$

Thus, we have from (8)-(9) that

$$\|u_0\|_{s+1} \leq e^{-kt}C\varepsilon\left(\|\varphi\|_{s+1} + \|\psi\|_s\right). \tag{11}$$

Now, by letting $A = C(\|\varphi\|_{s+1} + \|\psi\|_s)$, we have, from (11), that

$$\|u_0\|_{s+1} \leq \varepsilon A e^{-kt} \leq 2\varepsilon A e^{-kt}. \tag{12}$$

The sequence $\{\widehat{u}_n\}$ can thus be constructed as follows

$$\widehat{u}_n(\xi,t) = \widehat{u}_0 - \frac{\beta\xi^2}{(1+a\xi^2)\sigma_\xi}\int_0^t \exp\left[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)\right]$$
$$\times \sin\left[\sigma_\xi(t-\tau)\right]\widehat{f(u_{n-1})}(\xi,\tau)d\tau,$$
$$n = 1, 2, 3, \ldots. \tag{13}$$

Since

$$
\begin{aligned}
&\frac{\beta\xi^2}{(1+a\xi^2)\sigma_\xi}\int_0^t \exp\left[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)\right]d\tau \\
&\leq \frac{\beta\xi^2}{\sqrt{ac\xi^6+(a+c-b^2)\xi^4+pa\xi^2+\xi^2+p^2}}\frac{1+a\xi^2}{b\xi^2+m}\left(1-e^{-\frac{b\xi^2+m}{1+a\xi^2}t}\right) \\
&\leq \frac{2\beta(1+a\xi^2)}{\sqrt{ac\xi^6+(a+c-b^2)\xi^4+pa\xi^2+\xi^2+p^2}} \quad \leq \quad C,
\end{aligned}
\tag{14}
$$

by letting

$$
\begin{aligned}
B(\xi,t) = \frac{\beta\xi^2}{(1+a\xi^2)\sigma_\xi}\int_0^t \exp[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)] \\
\times \sin[\sigma_\xi(t-\tau)]\widehat{f(u)}(\xi,\tau)d\tau,
\end{aligned}
\tag{15}
$$

we have

$$
\begin{aligned}
&\mid B(\xi,t)\mid \leq \mid \frac{\beta\xi^2}{\sigma_\xi(1+a\xi^2)} \mid \int_0^t \mid \exp[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)] \\
&\times \sin[\sigma_\xi(t-\tau)]\widehat{f(u)}(\xi,\tau) \mid d\tau, \\
&\leq \mid \frac{\beta\xi^2}{\sigma_\xi(1+a\xi^2)}\left[\int_0^t \exp\left[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)\right]d\tau\right]^{\frac{1}{2}} \mid \\
&\times [\int_0^t \exp[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)] \mid \widehat{f(u)}(\xi,\tau)\mid^2 d\tau]^{\frac{1}{2}} \\
&\leq C[\int_0^t \exp\left[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)\right] \mid \widehat{f(u)}(\xi,\tau)\mid^2 d\tau]^{\frac{1}{2}},
\end{aligned}
\tag{16}
$$

and consequently

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} <\xi>^{2(s+1)} |B(\xi,t)|^2 d\xi \leq C\int_{-\infty}^{\infty}\int_0^t \exp\left[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)\right] \\
&\times <\xi>^{2(s+1)} \mid \widehat{f(u)}(\xi,\tau)\mid^2 d\tau d\xi, \\
&\leq \int_{-\infty}^{\infty}\int_0^t \exp[-k(t-\tau)] <\xi>^{2(s+1)} \mid \widehat{f(u)}(\xi,\tau)\mid^2 d\tau d\xi, \\
&\leq \int_0^t \exp[-k(t-\tau)]\|f(u)\|_{s+1}^2 d\tau.
\end{aligned}
\tag{17}
$$

From (14)-(17), we have

$$
\|u_n\|_{s+1} \leq \|u_0\|_{s+1} + C(\int_0^t \exp[-k(t-\tau)]\|f(u_{n-1})\|_{s+1}^2 d\tau)^{\frac{1}{2}}. \tag{18}
$$

Therefore, for $n = 1$ and $p > 1$, inequality (18) together with (12) yields

$$
\begin{aligned}
\|u_1\|_{s+1} &\leq \|u_0\|_{s+1} + C\Big(\int_0^t \exp\left[-k(t-\tau)\right] \|f(u_0)\|^2_{s+1} d\tau\Big)^{\frac{1}{2}}, \\
&\leq \|u_0\|_{s+1} + C\Big(\int_0^t \exp\left[-k(t-\tau)\right] \|u_0\|^{2p}_{s+1} d\tau\Big)^{\frac{1}{2}}, \\
&\leq \|u_0\|_{s+1} + C(2\varepsilon A)^p \int_0^t \exp\left[-k(t-\tau)\right] e^{-2pk\tau} d\tau\Big)^{\frac{1}{2}}, \\
&\leq \|u_0\|_{s+1} + C(2\varepsilon A)^p\Big(\int_0^t \exp\left[-k(t-\tau)\right] e^{-2pk\tau} d\tau\Big)^{\frac{1}{2}}, \\
&\leq \|u_0\|_{s+1} + C(2\varepsilon A)^p e^{-kt}, \\
&\leq e^{-kt}(\varepsilon A + C(2\varepsilon A)^{p-1}(2\varepsilon A)).
\end{aligned} \tag{19}
$$

Choosing ε sufficiently small such that

$$2C(2\varepsilon A)^{p-1} < 1, \tag{20}$$

we have, from (19) and (20), that

$$\|u_1\|_{s+1} < e^{-kt}(2\varepsilon A). \tag{21}$$

For $n = 2$, proceeding as for the proof of (21), we have

$$\|u_2\|_{s+1} \leq e^{-kt}(2\varepsilon A). \tag{22}$$

Thus by induction, we get

$$\|u_n\|_{s+1} < e^{-kt}(2\varepsilon A). \quad n = 1, 2, 3, \cdots . \tag{23}$$

Following the same procedure as that for deriving (23), we obtain

$$|||u_n||| < 2\varepsilon A, \quad n = 1, 2, 3, \cdots . \tag{24}$$

From (13) and the assumption of the nonlinear term f, we get

$$|||u_n - u_{n-1}||| \leq C \mid G(|||u_n|||, |||u_{n-1}|||)(|||u_n||| - |||u_{n-1}|||) \mid, \tag{25}$$

where $G(x, y)$ is an $(p-1)th$ order homogeneous polynomial with respect to x and y. By using (24), it follows that

$$\mid G(|||u_n|||, |||u_{n-1}|||) \mid \leq C(2\varepsilon A)^{p-1}. \tag{26}$$

From (25) and (26), we get

$$|||u_n - u_{n-1}||| \leq C[(2\varepsilon A)^{p-1}]^{n-1} \ (4\varepsilon A). \tag{27}$$

Since

$$\begin{aligned} |||u_n||| &= |||u_0 + \sum_1^n (u_i - u_{i-1})||| \\ &\leq |||u_0||| + \sum_1^n |||u_i - u_{i-1}||| \\ &< \varepsilon A + \sum_1^n [C(2\varepsilon A)^{p-1}]^{i-1} \ (4\varepsilon A), \end{aligned} \tag{28}$$

by choosing $C(2\varepsilon A)^{p-1} < \frac{1}{2}$, it is clear that $\{u_n\}$ is uniformly convergent in the space $X_s(\infty)$. Therefore there must exist a function $u(x,t) \in X_s(\infty)$ such that u_n uniformly converges to $u(x,t)$, i.e., a solution of the problem defined by (4) and (5). Moreover,

$$\begin{aligned} \|u_n\|_{s+1} &= \|u_0 + \sum_1^n (u_i - u_{i-1})\|_{s+1} \\ &\leq |||u_0 + \sum_1^n (u_i - u_{i-1})||| \\ &\leq |||u_0||| + \sum_1^n |||u_i - u_{i-1}||| \\ &< \varepsilon A + \sum_1^n [C(2\varepsilon A)^{p-1}]^{i-1} \ (4\varepsilon A). \end{aligned} \tag{29}$$

From (29), we know that $\{u_n\}$ is also uniformly convergent in the sense of the norm $\| \ \|_{s+1}$. Using inequality (23), we have

$$\|u\|_{s+1} < e^{-kt}(2\varepsilon A), \tag{30}$$

which implies that the solution decays exponentially and $\|u\|_{s+1}$ tends to zero as $t \to +\infty$.

(b) *Uniqueness.* Now we prove that the global solution $u(x,t) \in X_s(\infty)$ of the problem defined by(4)-(5) is unique. Assuming that there exist two solutions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ of the problem, we know that both of them are uniformly bounded in the space $X_s(\infty)$. Letting

$$w(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t)$$

we have

$$\widehat{w}(\xi,t) = -\frac{\beta\xi^2}{\sigma_\xi(1+a\xi^2)}\int_0^t \exp[-\frac{b\xi^2+m}{1+a\xi^2}(t-\tau)] \times \sin\left[\sigma_\xi(t-\tau)\right]\widehat{f}_1(\xi,\tau)d\tau, \quad (31)$$

where

$$f_1(x,t) = f(u^{(1)}(x,t)) - f(u^{(2)}(x,t)).$$

It follows from (31) that

$$|\widehat{w}(\xi,t)| \leq \left|\frac{\beta\xi^2}{\sigma_\xi(1+a\xi^2)}\right| \{\int_0^t \exp[-\frac{2b\xi^2}{1+a\xi^2}(t-\tau)]d\tau\}^{\frac{1}{2}} \times [\int_0^t (\widehat{f}_1(\xi,\tau))^2 d\tau]^{\frac{1}{2}} \leq C\left[\int_0^t |\widehat{f}_1(\xi,\tau)|^2\, d\tau\right]^{\frac{1}{2}},$$

and

$$\begin{aligned}\int_{-\infty}^{+\infty} < \xi >^{2(s+1)} |\widehat{w}(\xi,t)|^2 d\xi &\leq C\int_{-\infty}^{+\infty}\int_0^t < \xi >^{2(s+1)} |\widehat{f}_1(\xi,\tau)|^2\, d\tau d\xi \\ &\leq C\int_0^t \{G(|||u^{(1)}|||,|||u^{(2)}|||)(|||u^{(1)}-u^{(2)}|||)\}^2 d\tau \\ &\leq C\int_0^t (2\varepsilon A)^{p-1}\,|\,(|||u^{(1)}-u^{(2)}|||)\,|^2\, d\tau \\ &\leq C\int_0^t |||w|||d\tau.\end{aligned}$$

Thus, we can immediately obtain

$$|||w|||^2 \leq c_1\int_0^t |||w|||^2 d\tau. \quad (32)$$

By the Growall inequality, we have $w(x,t) \equiv 0$ $(w(x,t) \in X_s(\infty))$. That is, $u^{(1)}(x,t) \equiv u^{(2)}(x,t)$ and hence there exists a unique solution $u(x,t) \in X_s(\infty)$ $\left(s > \frac{1}{2}\right)$ of the problem defined by (4)-(5). □

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