An α-Fuzzy Max Order and Solution of Linear Constrained Fuzzy Optimization Problems

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ABSTRACT – In this paper, we introduce a new concept, the α-fuzzy max order, and then use the concept in the study of fuzzy linear constrained optimization problems. For constraints given by n inequalities involving fuzzy numbers with isosceles triangle membership functions, we prove that the feasible solution space is determined by $3n$ non-fuzzy inequalities. For constraints involving fuzzy numbers with other forms of membership functions, we develop two numerical algorithms respectively for the determination of the feasible solution space and the solution of the fuzzy optimization problem. An illuminative example is also given in this paper to demonstrate the validity of the methods and algorithms developed.

Keywords: Optimisation, Fuzzy mathematics, Constraint, Feasible solution space.

1 Introduction

In optimizing real world systems, one usually ends up with a linear and nonlinear programming problem. For many cases, the coefficients involved in the objective and constraint functions are imprecise in nature and have to be interpreted as fuzzy numbers to reflect the real world situation. The resulting mathematical programming is therefore referred to as fuzzy mathematical programming problem.

In recent years, various attempts have been made to study the solution of fuzzy mathematical programming problems with objective functions involving fuzzy numbers, either from theoretical or computational point of view. Tanaka et al. [12] formulated the fuzzy linear programming (FLP) problem as a parametric linear programming problem, while Luhandjura [7] formulated the (FLP) problem as a semi-finite linear programming problem with infinitely many objective functions. More recently, Maeda [9] formulated the (FLP) problem as a two-objective linear programming problem. However, Maeda's work is applicable to fuzzy numbers with triangular membership functions only. Zhang et. al [16] has thus further developed Maeda's work to formulate the (FLP) problem as a four-objective linear programming problem and the new work is applicable to problems involving fuzzy numbers with any form of membership functions.

This paper is a continuation of our recent work presented in [16]. In the paper, we develop an optimization method for solving a different kind of problems, namely linear optimization problems with constraint inequalities involving fuzzy numbers. The rest of paper is organized as follows. In section two, we give some basic definitions and theorems fundamental to the development to be described in section three. In Section three, we firstly introduce the concept of α-fuzzy max order and use the concept to define a fuzzy optimization problem. Then, an important theorem is developed concerning the determination of the feasible solution space defined by the constraint inequalities involving only fuzzy numbers with isosceles triangle membership functions. In Section four, two numerical algorithms are developed respectively for the solution of the (FLP) problem and the solution space defined by the constraints involving fuzzy numbers with any form of membership functions. In Section five, an illustrative example is given to demonstrate the validity of the methods and the algorithms developed.

2. Preliminaries

In this section, we present some basic concepts, definitions and theorems that are to be used in the subsequent sections. The work presented in this section can also be found from our recent paper in [16].

Let R be the set of all real numbers, R^n be n-dimensional Euclidean space, and $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ be any two vectors, where $x_i, y_i \in R$, $i = 1,2, \ldots, n$ and T denotes the transpose of the vector. Then we denote the inner product of x and y by $\langle x, y \rangle$. For any two vectors x, $y \in R^n$, we write $x \geqq y$ iff $x_i \geqq y_i$, $\forall\, i = 1,2,\cdots,n$; $x \geq y$ iff $x \geqq y$ and $x \neq y$; $x > y$ iff $x_i > y_i$, $\forall\, i = 1,2, \ldots, n$.

Definition 2.1 A fuzzy number $\tilde{a}$ is defined as a fuzzy set on R, whose membership function $\mu_{\tilde{a}}$ satisfies the following conditions:

1. $\mu_{\tilde{a}}$ is a mapping from R to the closed interval [0, 1];
2. it is normal, i.e., there exists $x \in R$ such that $\mu_{\tilde{a}}(x) = 1$;
3. for any $\lambda \in (0, 1]$, $a_\lambda = \{x;\ \mu_{\tilde{a}}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^L, a_\lambda^R]$.

Let $F(R)$ be the set of all fuzzy numbers. By the decomposition theorem of fuzzy set, we have

$$\tilde{a} = \bigcup_{\lambda \in [0,1]} \lambda [a_\lambda^L, a_\lambda^R], \tag{1}$$

for every $\tilde{a} \in F(R)$.

Let $F^*(R)$ be the set of all finite fuzzy numbers on R.

Theorem 2.1 *Let $\tilde{a}$ be a fuzzy set on R, then $\tilde{a} \in F(R)$ if and only if $\mu_{\tilde{a}}$ satisfies*

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 & x \in [m, n] \\ L(x) & x < m \\ R(x) & x > n \end{cases},$$

where $L(x)$ is the right-continuous monotone increasing function, $0 \leqq L(x) < 1$ and $\lim_{x \to -\infty} L(x) = 0$, $R(x)$ is the left-continuous monotone decreasing function, $0 \leqq R(x) < 1$ and $\lim_{x \to \infty} R(x) = 0$.

Corollary 2.1 *For every $\tilde{a} \in F(R)$ and $\lambda_1, \lambda_2 \in [0, 1]$, if $\lambda_1 \leqq \lambda_2$ then $a_{\lambda_2} \subset a_{\lambda_1}$.*

Definition 2.2 For any $\tilde{a}, \tilde{b} \in F(R)$ and $0 \leqq \lambda \in R$, the sum of $\tilde{a}$ and $\tilde{b}$ and the scalar product of λ and $\tilde{a}$ are defined by the membership functions

$$\mu_{\tilde{a}+\tilde{b}}(t) = \sup_{t=u+v} \min\{\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)\}, \tag{2}$$

$$\mu_{\lambda\tilde{a}}(t) = \sup_{t=\lambda u} \mu_{\tilde{a}}(u). \tag{3}$$

Theorem 2.2 *For any $\tilde{a}, \tilde{b} \in F(R)$ and $0 \leqq \alpha \in R$,*

$$\tilde{a} + \tilde{b} = \bigcup_{\lambda \in [0,1]} \lambda [a_\lambda^L + b_\lambda^L, a_\lambda^R + b_\lambda^R],$$

$$\alpha\tilde{a} = \bigcup_{\lambda \in [0,1]} \lambda [\alpha a_\lambda^L, \alpha a_\lambda^R].$$

Definition 2.3 Let $\tilde{a}_i \in F(R), i = 1,2,\cdots,n$. We define $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$

$$\mu_{\tilde{a}} : R^n \to [0,1]$$
$$x \mapsto \bigwedge_{i=1}^{n} \mu_{\tilde{a}_i}(x_i),$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, and $\tilde{a}$ is called an n-dimensional fuzzy number on R^n. If $\tilde{a}_i \in F^*(R), i = 1,2,\cdots,n$, $\tilde{a}$ is called an n-dimensional finite fuzzy number on R^n.

Let $F(R^n)$ and $F^*(R^n)$ be the set of all n-dimensional fuzzy numbers and the set of all n-dimensional finite fuzzy numbers on R^n respectively.

Proposition 2.1 *For every $\tilde{a} \in F(R^n)$, $\tilde{a}$ is normal.*

Proof. Since $\tilde{a} \in F(R^n)$, there exist $\tilde{a}_i \in F(R), i = 1,2,\cdots,n$ such that $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$. As $\tilde{a}_i$ $(i = 1,2,\cdots,n)$ is normal, it follows that there exists $x_i \in R$ $(i = 1,2,\cdots,n)$ such that $\mu_{\tilde{a}_i}(x_i) = 1$ $(i = 1,2,\cdots n)$. Let $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, then

$$\mu_{\tilde{a}}(x) = \bigwedge_{i=1}^{n} \mu_{\tilde{a}_i}(x_i) = 1,$$

which implies that $\tilde{a}$ is normal.

Proposition 2.2 *For every $\tilde{a} \in F(R^n)$, the λ-section of $\tilde{a}$ is an n-dimensional closed rectangular region for any $\lambda \in [0,1]$.*

Proof. Since $\tilde{a} \in F(R^n)$, there exist $\tilde{a}_i \in F(R), i = 1,2,\cdots,n$ such that $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$. As the λ-section of $\tilde{a}_i$ $(i = 1,2,\cdots,n)$ is a closed interval $[a_{i\lambda}^L, a_{i\lambda}^R]$ $(i = 1,2,\cdots,n)$, we have

$$\begin{aligned} a_\lambda &= \{x; \mu_{\tilde{a}}(x) \geqq \lambda\} \\ &= \{x; \bigwedge_{i=1}^{n} \mu_{\tilde{a}_i}(x_i) \geqq \lambda\} \\ &= \{x; \mu_{\tilde{a}_i}(x_i) \geqq \lambda, i = 1,2,\cdots,n\} \\ &= \{x = (x_1, x_2, \cdots, x_n)^T; x_i \in a_{i\lambda}, i = 1,2,\cdots,n\} \\ &= \{x = (x_1, x_2, \cdots, x_n)^T; x_i \in [a_{i\lambda}^L, a_{i\lambda}^R], i = 1,2,\cdots,n\}, \end{aligned}$$

for any $\lambda \in (0, 1]$. This implies that the λ-section of $\tilde{a}$ is an n-dimensional closed rectangular region for any $\lambda \in [0, 1]$.

Proposition 2.3 *For every $\tilde{a} \in F(R^n)$ and $\lambda_1, \lambda_2 \in [0,1]$, if $\lambda_1 \leqq \lambda_2$, then $a_{\lambda_2} \subset a_{\lambda_1}$.*

Proof. Obvious.

Definition 2.4 For any n-dimensional fuzzy numbers $\tilde{a}, \tilde{b} \in F(R^n)$, we define

1. $\tilde{a} \succeqq \tilde{b}$ iff $a_{i\lambda}^L \geqq b_{i\lambda}^L$ and $a_{i\lambda}^R \geqq b_{i\lambda}^R$, $i = 1,2,\cdots,n$, $\lambda \in (0,1]$;
2. $\tilde{a} \succeq \tilde{b}$ iff $a_{i\lambda}^L \geq b_{i\lambda}^L$ and $a_{i\lambda}^R \geq b_{i\lambda}^R$, $i = 1,2,\cdots,n$, $\lambda \in (0,1]$;
3. $\tilde{a} \succ \tilde{b}$ iff $a_{i\lambda}^L > b_{i\lambda}^L$ and $a_{i\lambda}^R > b_{i\lambda}^R$, $i = 1,2,\cdots,n$, $\lambda \in (0,1]$.

We call the binary relations $\succeqq, \geq$ and $\succ$ a fuzzy max order, a strict fuzzy max order and a strong fuzzy max order, respectively.

3. Fuzzy linear programming problem and α-fuzzy max order

Consider the following fuzzy linear programming (FLP) problem:

$$(\text{FLP})\quad \begin{cases} \text{maximize} & \langle c,x\rangle_F = \sum_{i=1}^{n} c_i x_i \\ \text{subject to} & \widetilde{A}x \preceq \widetilde{b},\ x \geqq 0. \end{cases} \tag{4}$$

where

$$\widetilde{c} = (\widetilde{c}_1, \widetilde{c}_2, \cdots, \widetilde{c}_n)^T \in F^*(R^n), \qquad \widetilde{b} = (\widetilde{b}_1, \widetilde{b}_2, \cdots, \widetilde{b}_m)^T \in F^*(R^m), \qquad \widetilde{A} = (\widetilde{a}_{ij}),$$

in which $\widetilde{a}_{ij} \in F^*(R), i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$. Based on the definition of the fuzzy max order (Definition 2.5), the (FLP) problem (5) is equivalent to the following (FLP$_\lambda$) problem.

$$(\text{FLP}_\lambda)\quad \begin{cases} \text{maximize} & \langle c,x\rangle_F = \sum_{i=1}^{n} c_i x_i \\ \text{subject to} & A_\lambda^L x \leqq b_\lambda^L,\ A_\lambda^R x \leqq b_\lambda^R,\ x \geqq 0,\ \forall \lambda \in [0,1] \end{cases} \tag{5}$$

Obviously, a feasible solution must satisfy the constraints for all $\lambda \in [0, 1]$. However, in general, this requirement is too strong. Now consider a typical coefficient c_i represented by a fuzzy number $\widetilde{c}_i$. The possibility of such a parameter c_i taking value in the range $[c_{i\lambda}^L, c_{i\lambda}^R]$ is λ or above. While the possibility of c_i taking value beyond $[c_{i\lambda}^L, c_{i\lambda}^R]$ is less than λ. Thus, one would generally be more interested in solutions obtained using coefficients c_i taking values in $[c_{i\lambda}^L, c_{i\lambda}^R]$ with $\lambda \geq \alpha > 0$. As a special case, if the coefficients involved are either real numbers or fuzzy numbers with triangular membership functions, then, we will have the usual non-fuzzy optimization problem suppose that we choose $\alpha = 1$. To formulate this idea, we introduce the following definitions.

Definition 3.1 For any n-dimensional fuzzy numbers $\widetilde{a}, \widetilde{b} \in F(R^n)$, we define

4. $\widetilde{a} \succeq_\alpha \widetilde{b}$ iff $a_{i\lambda}^L \geqq b_{i\lambda}^L$ and $a_{i\lambda}^R \geqq b_{i\lambda}^R$, $i = 1, 2, \cdots, n$, $\lambda \in [\alpha, 1]$;
5. $\widetilde{a} \succeq_\alpha \widetilde{b}$ iff $a_{i\lambda}^L \leq b_{i\lambda}^L$ and $a_{i\lambda}^R \leq b_{i\lambda}^R$, $i = 1, 2, \cdots, n$, $\lambda \in [\alpha, 1]$;
6. $\widetilde{a} \succ_\alpha \widetilde{b}$ iff $a_{i\lambda}^L > b_{i\lambda}^L$ and $a_{i\lambda}^R > b_{i\lambda}^R$, $i = 1, 2, \cdots, n$, $\lambda \in [\alpha, 1]$.

We call the binary relations $\succeq_\alpha$, $\succeq_\alpha$ and $\succ_\alpha$ a α-fuzzy max order, a strict α-fuzzy max order and a strong α-fuzzy max order, respectively.

With Definition 3.1, we turn our interest to the solution of the following problem:

$$(\text{FLP}_\alpha)\quad \begin{cases} \text{maximize} & \langle c,x\rangle_F = \sum_{i=1}^{n} c_i x_i \\ \text{subject to} & \widetilde{A}x \preceq_\alpha \widetilde{b},\ x \geqq 0. \end{cases} \tag{6}$$

Associated with the (FLP$_\alpha$) problem, we now consider the following problem.

$$(\text{FLP}_{\alpha\lambda})\quad \begin{cases} \text{maximize} & \langle c,x\rangle = \sum_{i=1}^{n} c_i x_i \\ \text{subject to} & A_\lambda^L x \leqq b_\lambda^L,\ A_\lambda^R x \leqq b_\lambda^R,\ x \geqq 0,\ \forall \lambda \in [\alpha, 1] \end{cases} \tag{7}$$

where $c = (c_i)_{1\times n}$, $A_\lambda^L = \left(a_{ij\lambda}^{\ L}\right)_{m\times n}$, $A_\lambda^R = \left(a_{ij\lambda}^{\ R}\right)_{m\times n}$, $b_\lambda^L = (b_{i\lambda}^{\ L})_{m\times 1}$ and $b_\lambda^R = (b_{i\lambda}^{\ R})_{m\times 1}$.

Theorem 3.1 Let x^* be the solution of the (FLP$_{\alpha\lambda}$) problem (7). Then it is also a solution of the (FLP$_\alpha$) problem defined by (6).

Proof. The proof is obvious from Definition 3.1.

Theorem 3.2 If all the fuzzy coefficients $\tilde{a}_{ij}$ and $\tilde{b}_i$ have isosceles triangle membership functions

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z - h_{\tilde{z}} \\ \dfrac{t - z + h_{\tilde{z}}}{h_{\tilde{z}}} & z - h_{\tilde{z}} \leqq t < z \\ \dfrac{-t + z + h_{\tilde{z}}}{h_{\tilde{z}}} & z \leqq t < z + h_{\tilde{z}} \\ 0 & z + h_{\tilde{z}} \leqq t \end{cases}, \tag{8}$$

where $\tilde{z}$ denotes $\tilde{a}_{ij}$ or $\tilde{b}_i$, z and $h_{\tilde{z}}$ are the centre and the deviation parameter of $\tilde{z}$ respectively. Then, the space of feasible solutions X is defined by the set of $x \in R^n$ with x_i, for $i = 1,2,\ldots,n$, satisfying

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \leqq b_i \\ \sum_{j=1}^{n} [a_{ij} - (1-\alpha) h_{\tilde{a}_{ij}}] x_j \leqq b_i - (1-\alpha) h_{\tilde{b}_i} \\ \sum_{j=1}^{n} [a_{ij} + (1-\alpha) h_{\tilde{a}_{ij}}] x_j \leqq b_i + (1-\alpha) h_{\tilde{b}_i} \\ x_i \geqq 0 \end{cases} \tag{9}$$

Proof.

From Theorem 3.1, X is defined by

$$X = \{x \in R^n \mid \sum_{j=1}^{n} a_{ij\lambda}^{\ L} x_j \leqq b_{i\lambda}^{\ L},\ \sum_{j=1}^{n} a_{ij\lambda}^{\ R} x_j \leqq b_{i\lambda}^{\ R},\ x \geqq 0, \forall \lambda \in [\alpha,1] \text{ and } i = 1,2,\cdots,m\}. \tag{10}$$

That is, X is the set of $x \in R^n$ with $x \geqq 0$ and satisfying

$$I_{i\lambda} = \sum_{j=1}^{n} a_{ij\lambda}^{\ L} x_j - b_{i\lambda}^{\ L} \leqq 0, \quad J_{i\lambda} = \sum_{j=1}^{n} a_{ij\lambda}^{\ R} x_j - b_{i\lambda}^{\ R} \leqq 0, \forall \lambda \in [\alpha,1] \text{ and } i = 1,2,\cdots,m. \tag{11}$$

For fuzzy numbers with isosceles triangle membership functions, we have

$$a_{ij\lambda}^{\ L} = a_{ij} - h_{\tilde{a}_{ij}}(1-\lambda), \quad a_{ij\lambda}^{\ R} = a_{ij} + h_{\tilde{a}_{ij}}(1-\lambda), \tag{12}$$

$$b_{i\lambda}^{\ L} = b_i - h_{\tilde{b}_i}(1-\lambda), \quad b_{i\lambda}^{\ R} = b_i + h_{\tilde{b}_i}(1-\lambda). \tag{13}$$

Substituting (12) and (13) into (11), we have

$$I_{i\lambda} = \sum_{j=1}^{n} [a_{ij} - h_{\tilde{a}_{ij}}(1-\lambda)] x_j - [b_i - h_{\tilde{b}_i}(1-\lambda)], \tag{14}$$

$$J_{i\lambda} = \sum_{j=1}^{n} [a_{ij} + h_{\tilde{a}_{ij}}(1-\lambda)] x_j - [b_i + h_{\tilde{b}_i}(1-\lambda)]. \tag{15}$$

Now, our problem becomes to show that $I_{i\lambda} \leqq 0, J_{i\lambda} \leqq 0, \forall\lambda \in [\alpha,1]$ and $i = 1,2,\cdots,m$ if (9) is satisfied. From $(9)_1$, we have

$$\sum_{j=1}^{n} a_{ij}x_j - b_i \leqq 0. \tag{16}$$

From $(9)_{2,3}$, we obtain

$$\sum_{j=1}^{n}(1-\alpha)(h_{\tilde{a}_{ij}}x_j - h_{\tilde{b}_i}) \geqq \sum_{j=1}^{n} a_{ij}x_j - b_i \ , \tag{17}$$

$$\sum_{j=1}^{n}(1-\alpha)(h_{\tilde{a}_{ij}}x_j - h_{\tilde{b}_i}) \leqq -(\sum_{j=1}^{n} a_{ij}x_j - b_i). \tag{18}$$

Thus, from (14) and (15) and using (16) - (18), we have, for any $\lambda \in [\alpha, 1]$ and $i = 1,2,\ldots,m$,

$$\begin{aligned} I_{i\lambda} &= (\sum_{j=1}^{n} a_{ij}x_j - b_i) - (\sum_{j=1}^{n} h_{\tilde{a}_{ij}}x_j - h_{\tilde{b}_i})(1-\lambda) \\ &< (\sum_{j=1}^{n} a_{ij}x_j - b_i) - (\sum_{j=1}^{n} a_{ij}x_j - b_i)\frac{1-\lambda}{1-\alpha} \\ &= (\sum_{j=1}^{n} a_{ij}x_j - b_i)\frac{\lambda-\alpha}{1-\alpha} \\ &\leqq 0. \end{aligned}$$

$$\begin{aligned} J_{i\lambda} &= (\sum_{j=1}^{n} a_{ij}x_j - b_i) + (\sum_{j=1}^{n} h_{\tilde{a}_{ij}}x_j - h_{\tilde{b}_i})(1-\lambda) \\ &< (\sum_{j=1}^{n} a_{ij}x_j - b_i) - (\sum_{j=1}^{n} a_{ij}x_j - b_i)\frac{1-\lambda}{1-\alpha} \\ &\leqq 0. \end{aligned}$$

The proof is complete.

4 Numerical algorithm

It should be emphasized that for problems involving fuzzy numbers with nonlinear membership functions. Theorem 3.2 will not be applicable. However, based on Theorem 3.1, we can derive a numerical algorithm for the determination of the space of feasible solutions and an algorithm for the numerical solution of the ($FLP_{\alpha\lambda}$) problem defined by (7). For simplicity in presentation, we define

$$X_\lambda = \{x \in R^n \mid A_\lambda^L x \leqq b_\lambda^L, A_\lambda^R x \leqq b_\lambda^R, x \geqq 0 \ \forall\lambda \in [\alpha,1]\}.$$

Algorithm for the space of feasible solutions X:

Let the interval $[\alpha, 1]$ be divided into m sub-intervals with $(m+1)$ nodes λ_i ($i = 0, m$) arranged in the order $\alpha = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_m = 1$.

Step 1: Set $m = 2$, then determine $X^m = \bigcap_{i=0}^{m} X_{\lambda_i}$;

Step 2: Determine $X^{2m} = \bigcap_{i=0}^{2m} X_{\lambda_i}$;

Step 3: If ($X^{2m} \sim X^m$), then $X \approx X^{2m}$. Otherwise, set m to $2m$ and go to Step 2, where $X^{2m} \sim X^m$ means that the space X^{2m} is close to X^m, namely

$$[x_i^L, x_i^R]^{2m} \approx [x_i^L, x_i^R]^m, \ \forall i = 1,2,\cdots,n,$$

in which $[x_i^L, x_i^R]^{2m}$ represents the interval of x_i obtained by using $2m$ sub-intervals.

Algorithm for the (FLP_α) problem:

Let the interval $[\alpha, 1]$ be divided into m sub-intervals with $(m-1)$ nodes λ_i ($i = 0, m$) arranged in the order $\alpha = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_m = 1$ and denote

$$(FLP_{\alpha\lambda})_m : \begin{cases} \max & \langle c, x \rangle \\ \text{subject to} & x \in X^m \end{cases}.$$

Step 1: Set $m = 2$, then solve the $(FLP_{\alpha\lambda})_m$ problem for $(x)_m$, where $(x)_m = (x_1, x_2, \ldots, x_n)_m$ and the subscript m indicates that the result is obtained subject to constraint $x \in X^m$;

Step 2: Solve the $(FLP_{\alpha\lambda})_{2m}$ problem for $(x)_{2m}$;

Step 3: If $\|(x)_{2m} - (x)_m\| <$ Tol, the solution of the ($FLP_{\alpha\lambda}$) problem is $x^* = (x)_{2m}$. Otherwise, update m to $2m$ and go to Step 2.

5 An illustrative example

To conclude this paper, we give an example in this section.

Example

$$(FLP) \quad \begin{cases} \text{maximize} & f(x,y) = 19x + 7y & (19) \\ \text{subject to} & \tilde{7}x + \tilde{6}y \preceq_{\alpha} \tilde{42} & \\ & \tilde{5}x + \tilde{9}y \preceq_{\alpha} \tilde{45} & (20) \\ & x - y \preceq 4 & \\ & x \geq 0, \ y \geq 0 & \end{cases}$$

where α=0.5, $\tilde{5}$, $\tilde{6}$, $\tilde{7}$ and $\tilde{9}$ are fuzzy numbers with membership functions given by

$$\mu_{\tilde{c}}(x) = \begin{cases} 0 & x < c-1 \\ x-c+1 & c-1 \leq x < c \\ 1+c-x & c \leq x < c+1 \\ 0 & c-1 \leq x \end{cases},$$

where c denotes a non-fuzzy value satisfying $\mu_{\tilde{c}}(c) = 1$. While, $\tilde{42}$ and $\tilde{45}$ are fuzzy numbers with membership functions given by

$$\mu_{\tilde{c}}(x) = \begin{cases} 0 & x < c-2 \\ \dfrac{x-c+2}{2} & c-2 \leq x < c \\ \dfrac{2+c-x}{2} & c \leq x < c+2 \\ 0 & c-2 \leq x \end{cases}.$$

If all fuzzy numbers $\tilde{c}_i$ are replaced by non-fuzzy value c_i satisfying $\mu_{\tilde{c}_i}(c_i)=1$, then the (FLP) problem becomes the normal linear mathematical programming problem. The solution in this case is $(x^*, y^*) = (5\frac{1}{13}, 1\frac{1}{13})$ with objective function value -104.000. If fuzziness has to be considered, by Theorem 3.2, the problem becomes a usual linear programming problem subject to 9 constraint inequalities. The solution in this case is $(x^*, y^*) = (4.9286, 0.9286)$ with objective function value -100.143. As a validation of the algorithms presented in Section 4, the feasible solution space X and the $(FLP)_{\alpha\lambda}$ problem are also solved by the algorithm for X and the algorithm for the $(FLP)_{\alpha\lambda}$ problem respectively. The solutions obtained are exactly the same as those from Theorem 3.2.

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