COMPUTATION OF ROBUST $H_\infty$ CONTROLLERS FOR TIME-DELAY SYSTEMS USING GENETIC ALGORITHMS

H. Du,* N. Zhang,* and J. Lam**

Abstract

This paper presents an evolutionary computation approach to design robust $H_\infty$ controllers for linear uncertain time-delay systems via a combination of genetic algorithms (GAs) and linear matrix inequalities (LMIs). Both state feedback and static output feedback controllers can be designed with this approach. It is demonstrated by numerical examples that the controllers designed by this approach can allow larger delay size than previous results for the same $H_\infty$ performance bound. Hence, this approach is less conservative than existing methods.

Key Words

Genetic algorithms, linear matrix inequalities, state feedback, output feedback, time-delay systems

1. Introduction

Considerable research efforts have been devoted to the problem of delay-dependent robust $H_\infty$ controller design for linear uncertain time-delay systems. The main objective of the delay-dependent $H_\infty$ control is to obtain a controller such that the maximal delay size for a fixed $H_\infty$ performance bound is allowed or the minimal $H_\infty$ performance bound for a fixed delay size is achieved. Hence, the conservatism in the delay-dependent $H_\infty$ control is measured by the allowable delay size or the minimal performance bound obtained. During the last decades, various approaches have been proposed to reduce the conservatism of delay-dependent conditions by using new bounding for cross terms or choosing new Lyapunov–Krasovskii functional [1–7]. In particular, a less conservative delay-dependent $H_\infty$ control was proposed in [5] for linear systems with a state delay based on a new Lyapunov–Krasovskii functional. It was also shown that the proposed method is much less conservative than previously existing results presented in [1, 3]. Further improved conditions for the solvability of the delay-dependent $H_\infty$ control are given in [7] where newly obtained results with reduced conservatism are established.

In most of the previously published research works, however, the $H_\infty$ controller synthesis conditions are presented in terms of nonlinear matrix inequalities to reduce the conservatism. Although iterative algorithms have been developed to solve the nonlinear matrix inequalities due to the nonconvex feasibility problem, conservatism can be significant and the iterative algorithms can only locate suboptimal solutions. Moreover, these works are only focused on state feedback control. At present, not a lot of efforts have been given to designing the static output feedback controllers for time-delay systems although the realization of the static output feedback controllers is more practical.

This paper develops an algorithm for designing both state feedback and static output feedback $H_\infty$ controllers for the linear uncertain time-delay systems. To circumvent the nonlinear matrix inequality problems involved in the delay-dependent conditions, the genetic algorithm (GA), which has been extensively applied in many areas [8, 9] due to its high potentialities in global optimization, is employed to search for the possible solutions. Specifically, in each generation of the GA, potential controller candidates are created for the verification of the closed-loop $H_\infty$ performance via its linear matrix inequality (LMI) characterization. Numerical examples show that the proposed approach can achieve a significant reduction in the conservatism when compared with previous results.

2. Problem Formulation

Consider the following state-delayed systems:

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t - \tau(t)) + [B + \Delta B(t)]u(t) + B_w w(t)$$
in designing a memoryless controller are said to be admissible. In this paper, we are interested such parameter uncertainties \(\Delta\) to be a continuous initial function; \(\tau(t)\) is the time-varying delay of the system state and is assumed to satisfy \(0 < \tau(t) \leq \tau, \hat{\tau}(t) \leq \mu\), where \(\tau\) and \(\mu\) are known constant scalars; \(\phi(t)\) is a continuous initial function; \(A, A_1, B, B_w, C_0, C_1, D, D_w, C\) and \(C\) are known constant matrices with appropriate dimensions; \(\Delta A(t), \Delta A_1(t), \) and \(\Delta B(t)\) are time-varying uncertain matrices of the form:

\[
[\Delta A(t) \quad \Delta A_1(t) \quad \Delta B(t)] = MF(t)[N_1 \quad N_4 \quad N_6] (2)
\]

where \(M, N, N_1, \) and \(N_6\) are constant matrices, and \(F(t) \in \mathbb{R}^{k \times k}\) is an unknown matrix satisfying \(F(t)F(t)^T \leq I\). Such parameter uncertainties \(\Delta A(t), \Delta A_1(t), \) and \(\Delta B(t)\) are said to be admissible. In this paper, we are interested in designing a memoryless controller \(K\) to be a \(\gamma\)-suboptimal \(H_\infty\) controller.

1. The closed-loop system is asymptotically stable for all admissible uncertainties.
2. The closed-loop system guarantees, under zero initial condition, \(\|z\|_2 < \gamma\|w\|_2\), where \(\gamma > 0\) is a prescribed constant, for any nonzero \(w(t) \in L_2[0, \infty)\).

A controller \(K\) satisfying the above conditions is said to be a \(\gamma\)-suboptimal \(H_\infty\) controller.

Now, we provide the following theorem based on the recently published result in [7] with proof omitted.

**Theorem 1.** If there exist matrices \(P > 0, Q > 0, Z > 0, Y, W, \) and a scalar \(\varepsilon > 0\) such that the following matrix inequality holds:

\[
\begin{bmatrix}
\Phi_1 + \Gamma & \Phi_2 + \Psi & \bar{\tau}Y & PB_w & \bar{\tau} A^T Z + \bar{\tau} (BKC)^T Z & C_0^T & 0 & (K C)^T & I & \varepsilon I
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Phi_3 + \varepsilon N_1^T N_3 & \bar{\tau} W & 0 & \bar{\tau} A_1^T Z & 0 & C_1^T & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\star & \star & \star & -\gamma^2 I & \bar{\tau} B_w^T Z & D_w^T & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon I
\end{bmatrix}
\]

\[
\begin{bmatrix}
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon I
\end{bmatrix}
\]

where \(\Phi_1 = PA + PBKC + (PA + PBKC)^T - Y - Y^T + Q, \)

\(\Phi_2 = PA_1 + Y - W^T, \Phi_3 = W + W^T - (1 - \mu) Q, \Gamma = \varepsilon [N^T + \]

\((KC)^T N_3^T]\] [\(N^T + (KC)^T N_3^T]\] [\(N^T + (KC)^T N_3^T]\], \(\Psi = \varepsilon [N^T + (KC)^T N_3^T]\), then there exists a \(\gamma\)-suboptimal \(H_\infty\) controller for all time delay satisfying \(0 < \tau(t) \leq \tau\) and \(\hat{\tau}(t) \leq \mu\).

### 3. Computational Algorithm

The matrix inequality condition in Theorem 1 is not an LMI in terms of the decision matrix variables. An iterative algorithm has been used to solve (4) in most of the published works. However, only local solutions can be found and only state feedback controllers can be obtained (i.e., the matrix \(C\) is invertible in (1)). In the following, an algorithm which combines the random search capability of GA with the solvability of LMI will be proposed. Specifically, we find a desirable controller \(K\) by solving the following maximization problem:

\[
\max_{K \in \mathbb{R}^{m \times p}} \tau \text{ subject to LMI (4)} (5)
\]

In this problem, GA is used to randomly generate a matrix \(K \in \mathbb{R}^{m \times p}\) initially which changes thereafter within the evolution procedure according to objective (5). If (5) is feasible for an evolved \(K\), which has the maximum \(\tau\), then this \(K\) satisfies the specifications and thus constitutes a solution to the design problem. Note that the matrix inequality (4) is LMI once the control gain matrix \(K\) is known, and this LMI can be solved efficiently by using efficient convex optimization algorithm. Furthermore, non-square matrix \(C\) cannot affect the characteristic of LMI in (4) so that the static output feedback controller can be designed as well by defining an appropriate \(C\) matrix.

As the standard GAs can be found in most related textbooks, an outline of our algorithm, which is similar to that used in [10], is given below:

**Step 1.** Use a binary string to encode the feedback gain matrix \(K\).

**Step 2.** Generate randomly an initial population of \(N_p\) individuals.

**Step 3.** Evaluate the objective (5) and assign fitness to every individual. Decode the initial population produced
in Step 2 into real values for every controller gain matrix $K_j$, $j = 1, 2, \ldots, N_p$. For every $K_j$, use the bisection method to search for the maximum delay $\tau_j$ such that with such a delay $\tau_j$ and $K_j$, LMI (4) is feasible. Take every delay $\tau_j$ as the objective value corresponding to $K_j$ and associate every $K_j$ with a suitable fitness value according to rank-based fitness assignment approach, and then go to Step 4. If for a $K_j$, there is no feasible delay can be found such that LMI (4) is feasible, the objective value corresponding to $K_j$ will be assigned a large value to reduce its opportunity to be survived in the next generation.

Step 4. Use tournament selection approach to choose the offspring.

Step 5. Perform uniform crossover with probability $p_c$ to produce new offspring.

Step 6. Mutate bits for individuals in the population with a small mutation probability $p_m$.

Step 7. Retain the best chromosomes in the population with elitist reinsertion method.

Steps 3–7 correspond to one generation. The evolution process will repeat for $N_g$ generations or being ended when the search process converges with a given accuracy. The best chromosome is decoded into real values to produce again the control gain matrix.

### 4. Numerical Examples

The basic GA parameters used in this paper are as follows: $N_p = 80$, $p_c = 0.8$, $p_m = 0.01$, $N_g = 30$. As GA is a probabilistic search procedure based on the mechanism of natural selection and natural genetics, to show a fair evaluation, in the following, we will carry out 50 runs independently for every case.

**Example 1.** Consider the example as appeared in [1–7] with the following system matrices:

$$
A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & -1 \\ 0 & 0.9 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
$$

$$
B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0.1, \quad D_w = 0
$$

As considered in the aforementioned references, the time delay is constant and hence $\mu = 0$.

In terms of the given $H_\infty$ performance bound $\gamma$, the $C$ matrix, and the search range for the controller gain, we search for the maximal delay size for several cases using the algorithm (referred to the “GA approach” in the sequel) presented in the last section. The statistical results for 50 runs are given in Table 1, where the percentages within the brackets indicate the probabilities of the obtained values in the 50 runs, i.e., each percentage in the bracket represents the percentage of solutions that gives minima/maxima and the number of all the feasible solutions in each of the 50 runs. In the table, “not found” means that, for such a combination of performance and delay size, a solution may not exist or if exists, the algorithm cannot find it in the computational setting. Fig. 1 shows the evolution process of the maximal delay for 50 runs when $\gamma = 0.1015$, $C = I_{2 \times 2}$. It can be seen from Fig. 1 that every run can converge to a stable value within 30 generations.

The results listed in Table 1 are obtained based on certain specified search ranges for the controller gains. In fact, under the same $H_\infty$ performance bound $\gamma$, different search range will affect the obtained maximal delay. For

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\tau$ Minimum</th>
<th>$\tau$ Maximum</th>
<th>$C$ Matrix</th>
<th>Search Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1015</td>
<td>1.3908 (2.04%)</td>
<td>1.4137 (22.45%)</td>
<td>$I_{2 \times 2}$</td>
<td>$[-800 \ 800]$</td>
</tr>
<tr>
<td>0.1015</td>
<td>Not found</td>
<td>Not found</td>
<td>$[1 \ 0]$</td>
<td>$[-800 \ 800]$</td>
</tr>
<tr>
<td>0.1015</td>
<td>1.4134 (61.7%)</td>
<td>1.4137 (38.3%)</td>
<td>$[0 \ 1]$</td>
<td>$[-800 \ 800]$</td>
</tr>
<tr>
<td>0.1287</td>
<td>1.4137 (100%)</td>
<td>1.4137 (100%)</td>
<td>$I_{2 \times 2}$</td>
<td>$[-40 \ 40]$</td>
</tr>
<tr>
<td>0.1287</td>
<td>Not found</td>
<td>Not found</td>
<td>$[1 \ 0]$</td>
<td>$[-400 \ 400]$</td>
</tr>
<tr>
<td>0.1287</td>
<td>1.4137 (100%)</td>
<td>1.4137 (100%)</td>
<td>$[0 \ 1]$</td>
<td>$[-40 \ 40]$</td>
</tr>
<tr>
<td>19.12</td>
<td>36.7195 (2.33%)</td>
<td>68.7245 (2.33%)</td>
<td>$I_{2 \times 2}$</td>
<td>$[-100 \ 100]$</td>
</tr>
<tr>
<td>19.12</td>
<td>0.8991 (2%)</td>
<td>0.9873 (80%)</td>
<td>$[1 \ 0]$</td>
<td>$[-800 \ 800]$</td>
</tr>
<tr>
<td>19.12</td>
<td>1.4137 (100%)</td>
<td>1.4137 (100%)</td>
<td>$[0 \ 1]$</td>
<td>$[-100 \ 100]$</td>
</tr>
</tbody>
</table>

Figure 1. Evolution process for the maximal time delay within 50 runs, $\gamma = 0.1015$. 

Table 1

Statistical Results of GA Approach with 50 Runs: Example 1
example, when $\gamma = 0.1287$, we search for the state feedback controller within different search range. The results are given in Table 2 from which we can see that under the same performance bound $\gamma$, a larger controller gain is necessary to ensure a larger delay bound for the same system.

To illustrate the reduced conservatism of the obtained results, we compare the optimal results for the state feedback cases obtained in Table 1 with previous results in Table 3. We note that the results published in [7] are less conservative than previous results for the state feedback controllers under the same performance bound in spite of their simplicity. In this sense, the GA approach is less conservative than the methods compared. It can be seen from Table 3 that for the same performance bound $\gamma$, the GA approach can find the state feedback controllers that allow larger delay size than existing methods. In this sense, the GA approach is less conservative than the methods compared. The GA approach is based on the improved matrix inequalities presented in [7], but the obtained results are much less conservative than those reported. This shows that the GA approach has reduced the conservatism caused by using the iterative algorithm. Furthermore, it is noted that the state feedback gains obtained by the GA approach are all smaller than the corresponding ones presented in [5–7] as we naturally constrain the search range for the controller gains in the GA approach. A smaller controller for the same $\gamma$ will reduce the possibility of input saturation.

Part of the optimal results obtained in Table 1 for the static output control cases are listed in Table 4. It is observed that even for some obtained static output feedback controllers, they can also allow larger delay size than the previous results for the state feedback controllers under the same performance bound in spite of their simplicity.

### Example 2.
Now, consider the system matrices as:

$$
A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & -1 \\ 0 & 0.9 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
$$

$$
B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0, \quad D_w = 0,
$$

$$
M = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad N = N_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad N_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

Notice that $\| \triangle A(t) \| \leq 0.2, \| \triangle A_1(t) \| \leq 0.2$. This case has also been studied in [1, 3, 5–7].
Table 5
Statistical Results of GA Approach with 50 Runs: Example 2

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\tau$</th>
<th>$C$ Matrix</th>
<th>Search Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>1.3512 (5.41%)</td>
<td>$I_{2\times2}$</td>
<td>$[-90 \ 90]$</td>
</tr>
<tr>
<td>0.03</td>
<td>Not found</td>
<td>Not found</td>
<td>$[1 \ 0]$</td>
</tr>
<tr>
<td>0.03</td>
<td>1.0384 (100%)</td>
<td>1.0384 (100%)</td>
<td>$[0 \ 1]$</td>
</tr>
<tr>
<td>0.01</td>
<td>1.3481 (2%)</td>
<td>1.3512 (12%)</td>
<td>$I_{2\times2}$</td>
</tr>
<tr>
<td>0.01</td>
<td>Not found</td>
<td>Not found</td>
<td>$[1 \ 0]$</td>
</tr>
<tr>
<td>0.01</td>
<td>1.0378 (100%)</td>
<td>1.0378 (100%)</td>
<td>$[0 \ 1]$</td>
</tr>
</tbody>
</table>

Figure 2. Evolution process for the maximal time delay within 50 runs, $\gamma = 0.03$.

As in Example 1, the maximal delay sizes for several cases are searched using the GA approach in terms of a given $H_\infty$ performance bound $\gamma$ for different $C$ matrices. The statistical results for 50 runs are given in Table 5. Fig. 2 shows the evolution process of the maximal delay for 50 runs when $\gamma = 0.03$, $C = I_{2\times2}$.

To illustrate the reduced conservatism of the obtained results, we compare the optimal results for the state feedback cases obtained in Table 5 with previous results in Table 6. It can be seen from Table 6 that even for the case of $\gamma = 0.01$, $\tau = 1.3512$, a desired robust $H_\infty$ state feedback controller can be computed as $[-33.4319 \ -73.3984]$ using the GA approach. This controller achieves a much smaller $\gamma$ for much bigger $\tau$ which implies that the GA approach is much more effective than the existing methods.

Part of the optimal results obtained in Table 5 for the static output control cases are listed in Table 7. It is again observed that the static output feedback controllers can allow larger delay size than the previous results for the state feedback controllers under the same performance.

5. Conclusion

This paper presents an algorithm for designing robust $H_\infty$ controllers for uncertain time-delay systems. By using a GA to search for possible controller gains and solving an

Table 6. Comparison of $\gamma$-Suboptimal $H_\infty$ Controller (State Feedback)

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\gamma$</th>
<th>$\tau$</th>
<th>Feedback Gain Matrix $K$</th>
<th>$C$ Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>de Souza and Li [1]</td>
<td>1.95</td>
<td>0.3</td>
<td>Not provided</td>
<td>$I_{2\times2}$</td>
</tr>
<tr>
<td>Lee et al. [5]</td>
<td>0.05</td>
<td>0.8</td>
<td>$[-0.0337 \ -64.9821]$</td>
<td>$I_{2\times2}$</td>
</tr>
<tr>
<td>Xu et al. [7]</td>
<td>0.03</td>
<td>0.9</td>
<td>$[0.2536 \ -99.0807]$</td>
<td>$I_{2\times2}$</td>
</tr>
<tr>
<td>GA approach</td>
<td>0.03</td>
<td>1.3527</td>
<td>$[-29.9264 \ -65.6218]$</td>
<td>$I_{2\times2}$</td>
</tr>
<tr>
<td>GA approach</td>
<td>0.01</td>
<td>1.3512</td>
<td>$[-33.4319 \ -73.3984]$</td>
<td>$I_{2\times2}$</td>
</tr>
</tbody>
</table>

Table 7. Optimal Results for $\gamma$-Suboptimal $H_\infty$ controller (Static Output Feedback)

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\tau$</th>
<th>Feedback Gain Matrix $K$</th>
<th>$C$ Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>1.0384</td>
<td>$-33.6042$</td>
<td>$[0 \ 1]$</td>
</tr>
<tr>
<td>0.01</td>
<td>1.0378</td>
<td>$-55.8804$</td>
<td>$[0 \ 1]$</td>
</tr>
</tbody>
</table>
LMI, the required controller can be determined. Moreover, appropriate structural constraints can be imposed on the controllers. Based on the recently developed matrix inequality condition, the proposed approach reduces the conservatism resulting from the iterative algorithm when designing for the $H_\infty$ controller. The improvement over previous methods by allowing a larger delay size under the same $H_\infty$ performance bound has been demonstrated via numerical examples.

References


Biographies

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