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OPTIMAL PROCESSES IN IRREVERSIBLE THERMODYNAMICS AND MICROECONOMICS

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SUMMARY

This paper describes general methodology that allows one to extend Carnot efficiency of classical thermodynamic for zero rate processes onto thermodynamic systems with finite rate. We define the class of minimal dissipation processes and show that it represents generalization of reversible processes and determines the limiting possibilities of finite rate systems. The described methodology is then applied to microeconomic exchange systems yielding novel estimates of limiting efficiencies for such systems.

KEY WORDS

finite-time thermodynamics, minimal dissipation process, optimal exchange processes in microeconomics

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LIMITING POSSIBILITIES OF HEAT ENGINES

One of the basic results in thermodynamics, obtained by Sadi Carnot, is the limiting value of heat engine's efficiency. Efficiency understood as the ratio of the mechanical work A to the amount of heat energy Q_+ removed from the hot source [1]. This limit turns out to be equal to

$$\eta^0 = \frac{A}{Q_+} = 1 - \frac{T_-}{T_+}, \tag{1}$$

where T_+ and T_- are absolute temperatures of the hot and cold sources in heat engine cycle.

The value of η^0 was found by Carnot essentially intuitively, because he did not know the energy conservation law and believed in the thermogen theory. Note that the value η^0 does not depend on such characteristics of the engine as its size, the material of heat-exchange surfaces, the equation of state of the working body, etc. If Carnot were to consider this problem from the viewpoint of modern thermodynamics he would cast it as the problem of finding the optimal dependence of working body's temperature on time $T(t)$ subject to the cyclic changes of the working body's state and the optimal cycle's period τ (Figure 1). He would arrive at the following optimal control problem:

$$\eta = \frac{\int_0^{\gamma\tau} q_+(T_+, T)\mathrm{d}t - \int_{\gamma\tau}^{\tau} q_-(T, T_-)\mathrm{d}t}{\int_0^{\gamma\tau} q_+(T_+, T)\mathrm{d}t} \to \max_{\tau, \gamma, T(t)}, \tag{2}$$

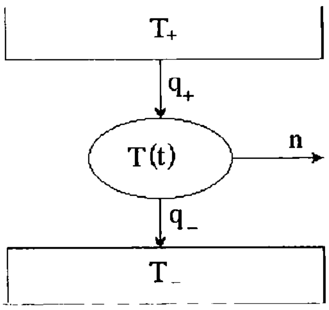

Figure 1. The structure of thermodynamic system (heat engine).

where the time of contact between the working body and the hot reservoir is $\gamma\tau$ and the time of contact between the working body and the cold reservoir is denoted as $(1 - \gamma)\tau$; q_+ and q_- are the heat fluxes from the hot and to the cold reservoirs correspondingly. From the energy conservation and cyclic changes of working body's state it follows that the denominator in (2) is equal to the obtained work A. Because of the same cyclic condition the maximum in (2) is to be found subject to constraint

$$\Delta S_p = \int_0^{\gamma\tau} \frac{q_+(T_+, T)}{T}\mathrm{d}t - \int_{\gamma\tau}^{\tau} \frac{q_-(T, T_-)}{T}\mathrm{d}t = 0, \tag{3}$$

where ΔS_p is the change of the working body's entropy during a cycle and $0 \le \gamma \le 1$.

The problem (2) and (3) does not have a solution, that is, for natural laws of heat transfer q_+ and q_-, η increases monotonically when τ increases, and the optimal temperature of the working body over the interval $\gamma\tau$ tends to $T = T_+$, and over the interval $(1 - \gamma)\tau$ to $T = T_-$. Meanwhile, the value of η is bounded and its limit (supremum) turns out to be equal to η^0.

Thus, Carnot's intuition helped him to bypass succesfully these mathematical difficulties and to obtain the correct result. This result is valid for any q_+ and q_-, which obey the natural requirement that the heat flux is directed from the body with the higher temperature to the body with the lower temperature.

A number of other problems can be formulated for the system, shown in Figure 1. Namely:

1. What is the maximal power, that can be obtained in a heat engine?

$$n = \frac{1}{\tau}\left[\int_0^{\gamma\tau} q_+(T_+,T)\mathrm{d}t - \int_{\gamma\tau}^{\tau} q_-(T,T_-)\mathrm{d}t\right] \rightarrow \max_{\tau,\gamma,T(t)}, \tag{4}$$

   The maximum in (4) is to be found subject to constraint (3).

2. What is the maximal value of a heat engine's efficiency η, when its power n_0 is fixed?

If the solution of the Problem 1, $n_{\max}$, is found then the fixed power n_0 must obey inequalities

$$0 \le n_0 \le n_{\max}, \tag{5}$$

If the left inequality is violated then the cycle is already not a cycle of a heat engine and if the right inequality is violated then the solution does not exist.

The first of these problems was solved by Novikov [2], and then independently and much later by Curson and Albborn [3] for the Newton law of heat transfer

$$q_+ = \alpha_+(T_+ - T), \quad q_- = \alpha_-(T - T_-). \tag{6}$$

The coefficients α_+ and α_- in these expressions implicitly describe the size of the heat engine. Problem 2 was solved for the Newton law and other laws of heat transfer in [4, 5].

Mathematical features of the Problems 1 and 2 are due to the averaging operations in their formulations. Indeed, in the maximal power problem it is required to choose a temperature of the working body $T(t)$, such that the average value of the heat flux q (after taking into account its sign), which is supplied to it, is maximal, subject to zero average change in the working body's entropy. In the maximal efficiency problem it is required to minimize the average value of the flux q_+ subject to constraints on the entropy of the working body and the fixed average power. Such problems are called averaged nonlinear programming problems [6]. As a rule, they have an infinite set of solutions, each of which switches between not more than $k + 1$ so-called "basic" values, where k is the number of averaged conditions. For example, since in the maximal power problem there is only one averaged condition (3), the temperature of the working body in the optimal cycle takes not more than two values T_1 and $T_2 < T_1$. Thus, for any law of heat exchange the optimal cycle includes two isotherms and the instant (adiabatic) temperature switches between them. Here $T_1 < T_+$, and $T_2 > T_-$. The basic values T_1 and T_2 are determined jointly with the fractions γ and $(1 - \gamma)\tau$ of the cycle's period τ (when $T(t) = T_1$ and $T(t) = T_2$ and the working body makes contact with the hot and cold reservoir correspondingly) by solving the auxiliarly nonlinear programming problem (without averaging).

In the maximal efficiency problem with the fixed power $n = n_0$ there are two averaged conditions – one on the average entropy rate of the working body and the other on the average power. Therefore the number of isotherms in the optimal cycle does not exceed three. But for the Newton law of heat exchange (6) this number is two and the cycle has the same form as the maximal power cycle (two isotherms and two adiabats) but with different values of T_1 and T_2 and with different time fractions γ and $1-\gamma$.

Let us write down the solution of the limiting power problem

$$n_{\max} = \alpha_0\left(\sqrt{T_+} - \sqrt{T_-}\right)^2, \tag{7}$$

where

$$\alpha_0 = \frac{\alpha_+\alpha_-}{\left(\sqrt{\alpha_+} + \sqrt{\alpha_-}\right)^2}, \tag{8}$$

The efficiency that corresponds to this power is

$$\eta_{n_{\max}} = 1 - \sqrt{\frac{T_-}{T_+}} < \eta_0. \tag{9}$$

In the limiting efficiency problem with fixed power $n_0 < n_{\max}$ this value is

$$\eta_n = 1 - \frac{1}{2T_+}\left[T_+ + T_- - \frac{\eta_0}{\alpha_0} - \sqrt{(T_+ - T_-)^2 + \left(\frac{\eta_0}{\alpha_0}\right)^2 - 2\frac{\eta_0}{\alpha_0}(T_+ - T_-)}\right], \tag{10}$$

and α_0 corresponds to (8). The optimal fraction γ of the cycle period when the working body stays in contact with the hot reservoir is the same in both problem

$$\gamma^* = \frac{\sqrt{\alpha_-}}{\sqrt{\alpha_+} + \sqrt{\alpha_-}}. \tag{11}$$

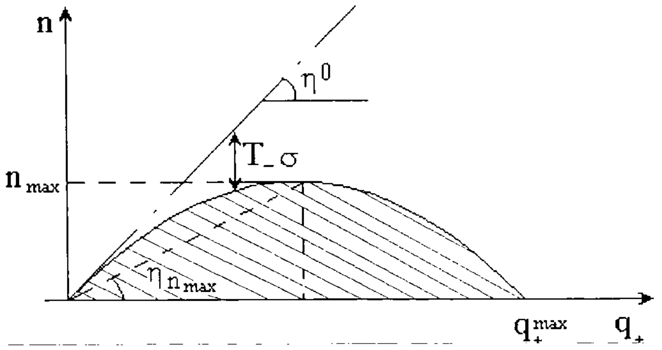

Figure 2. The area of the feasible regimes of the heat engine.

It is interesting to note, that if the power of the heat engine is fixed then it is possible to find not only its maximal but also its minimal efficiency. Here it is possible to construct the feasible area of the heat engine in the 2-D space where the coordinates are $\bar{q}_+$ (the average per cycle value of q_+) and the average power n. This is the dashed area in Figure 2. Its boundary, which is denoted here with the solid line, corresponds to cycles that include two isotherms and two adiabats. The dependence between n and $\bar{q}_+$, which corresponds to the infinite heat transfer coefficients (that is, to an infinitely large engine) is denoted with the dotted-dashed line. Here the processes approach reversible ones and the efficiency (the tangent of the slope of dotted-dashed line) is equal to η^0. The efficiencies, given by the expression (10), correspond to the slopes of the limiting power curve for the point that is located to the left of the maximum. The difference between the dotted-dashed curve and the boundary of the dashed area characterizes the irreversibility of the processes in the heat engine. Later we will show that this difference is equal to the product of the entropy production in the system σ on the temperature T_-. Thus the problem of efficiency maximization corresponds to the minimization of the entropy production (dissipation) in the system

$$\sigma = \frac{\Delta S}{\tau} = \frac{1}{\tau}\left[\frac{1}{T_-}\int_0^{(1-\gamma)\tau} q_-(T,T_-)\mathrm{d}t - \frac{1}{T_+}\int_{(1-\gamma)\tau}^{\tau} q_+(T_+,T)\mathrm{d}t\right], \tag{12}$$

MINIMAL DISSIPATION PROCESSES

The study of the limiting possibilities of heat engines led to the development of the new branch of thermodynamics - finite-time thermodynamics (optimizational thermodynamics). This branch of thermodynamics studies the limiting possibilities of thermodynamic systems of various types where the average rates of heat and mass fluxes are fixed. One of the ways to fix these fluxes is by fixing the duration of the process when the values of all or some of the state variables are fixed. Let us describe the general schema of solving the problem of determining the feasible area in the space of parameters of the thermodynamic system.

The state of a system is described by its internal energy U, the amount of mass (vector) N with components N_i, ($i = 1, \ldots, m$) and the entropy S. In the general case, the system can be open, that is, an exchange of mass, energy and entropy between the system and the environment can takes place.The changes to U, N and S are determined by the mass, energy and entropy balances:

$$\dot{U} = \sum_j g_j h_j + \sum_j g_{\alpha j} h_{\alpha j} + \sum_j q_j - n, \tag{13}$$

$$\dot{N}_i = \sum_j g_j x_{ij} + \sum_j g_{\alpha j} x_{\alpha ij} + \sum_\nu k_{i\nu} W_\nu, \tag{14}$$

$$\dot{S} = \sum_j g_j s_j + \sum_j \frac{g_{\alpha j}}{T_{\alpha j}}\left(h_{\alpha j} - \sum_i \mu_{\alpha ij}\right) + \sum_i \frac{q_j}{T_j} + \sigma, \tag{15}$$

Here g_j and $g_{\alpha,j}$ are the fluxes of mass that are driven by diffusion and conduction; h_j and $h_{\alpha,j}$ are the specific enthalpy of these fluxes; q_j are the heat fluxes; n is the produced work ; x_{ij} is the concentration of the i-th component in the j-th flux; k_{ij} is the stoihiometric coefficient of the i-th component in the ν-th reaction; the rate of this reaction – W_ν; s_j is the specific enthalpy of the j-th flux; $\mu_{\alpha,ij}$ is the chemical potential of the i-th component in the j-th diffusion flux.

If the state of the system does not change, then the right-hand sides of the balance equations (13) – (15) are equal to zero for any t. If the state of the system changes cyclically with the period τ, then the integrals over the time interval $[0, \tau]$ of the right-hand sides of equation (13) – (15) are equal to zero, because

$$U(\tau) = U(0), \quad N(\tau) = N(0), \quad S(\tau) = S(0).$$

Let us emphasize that the last of the balance equations includes the entropy production σ, which is non-negative according to the second law of thermodynamics. Only such trajectories and such stationary states in the state space for which $\sigma \geq 0$ can be realized. This limits the possible states of the system. For example, for a heat engine with two reservoirs of infinity heat capacity, the condition $\sigma \geq 0$ singles out the area below the dashed-dotted line in Figure 2, and thus limits the efficiency to below η^0. Indeed, if there are no exchange fluxes between this system and the environment then the entropy increment is equal to the change of the sources' entropies during the cycle

$$\frac{Q_-}{T_-} - \frac{Q_+}{T_+} = \int_0^{\tau} \sigma \mathrm{d}t = \Delta S\,, \tag{16}$$

The first term here is the entropy increment of the cold reservoir when the heat is supplied to it; the second term is the reduction of the hot reservoir's entropy when heat is removed from it; and σ is the entropy production in the system. From the energy balance it follows that

$$Q_+ - Q_- = A. \tag{17}$$

After elimination of Q_- from (16) and (17), and denoting the ratio A/Q_+ as η, from the condition $\Delta S \geq 0$ it follows that

$$\frac{A}{Q_+} = \eta \leq \frac{T_+ - T_-}{T_+}. \tag{18}$$

If the state of the system is described by the equations (13) – (15) and some additional constraints are imposed on the initial and final states of the system and on the duration of the process τ, then it is possible to find such phase trajectories among all feasible trajectories for which the average entropy production (dissipation) is minimal and equal to $\overline{\sigma_{\min}} > 0$.

The condition

$$\int_0^{\tau} \sigma \mathrm{d}t \geq \tau \overline{\sigma_{\min}}\,, \tag{19}$$

narrows down the area of the system's feasible phase trajectories. Thus, the condition of the fixed power of the heat engine leads to the inequality

$$\Delta \mathrm{S} \geq \tau \overline{\sigma_{\min}}(n_0)\,,$$

and instead of the inequality (18) we get

$$\frac{\mathrm{A}}{\mathrm{Q}_+} = \eta \leq \frac{T_+ - T_-}{T_+} - \frac{\tau \overline{\sigma_{\min}}(n) T_-}{Q_+}, \tag{20}$$

Thus, the estimate of the limiting possibilities of a thermodynamic system is reduced to finding the minimal dissipation in the system subject to given constraints.

In order to find an estimate of the minimal dissipation in the system with the fixed rate of the processes in it, it is reasonable to solve the minimal dissipation problem for each of the possible processes (heat exchange, mass transfer, chemical reactions, throttling, etc.). Then the dissipation in a complex system can be estimated by decomposing its internal processes into a set of minimal dissipation processes. In some sense the class of such processes extends the class of reversible processes by taking into account the non-zero rate of their processes. Let us first describe how the conditions of minimal dissipation are derived for the heat exchange and then we will generalize this derivation to other processes. A heat-exchange process between two bodies with the temperatures T_1 and T_2 is accompanied by entropy production

$$\sigma = q(T_1, T_2)\left(\frac{1}{T_2} - \frac{1}{T_1}\right), \tag{21}$$

where q is the heat exchange law.

Assume that T_2 is the control and that T_1 changes according to the equation

$$\dot{T} = -\frac{q(T_1, T_2)}{C_1(T_1)}, \tag{22}$$

where for simplicity we assume that the heat capacity C_1 is constant. The duration of the process is fixed and it is required to change the temperature T_1 from T_{10} to $\overline{T_1}$ in such a fashion that the system's entropy increment is minimal

$$\Delta S = \int_0^\tau \sigma \mathrm{d}t = \int_0^\tau q(T_1, T_2)\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\mathrm{d}t \to \min_{T_2(t)}, \tag{23}$$

The problem (22) and (23) is an optimal control problem. Its conditions of optimality can be easily derived by using the fact that the sign of the function q coincides with the sign of the difference $T_{10} - \overline{T_1}$ and does not change over the interval $[0, \tau]$. After transformation which replaces the time with the temperature of the first flux, that is, $\mathrm{d}t$ with $\mathrm{d}T_1 = -[C_1(T_1)/q(T_1, T_2)]\mathrm{d}t$, the problem takes the following form

$$\int_{T_{10}}^{\overline{T_1}} C_1(T_1)\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\mathrm{d}T_1 \to \min, \tag{24}$$

subject to the constraint

$$\int_{T_{10}}^{\overline{T_1}} \frac{C_1(T_1)}{q(T_1, T_2)}\mathrm{d}T_1 = \tau. \tag{25}$$

The condition of optimality for the problem (24), (25) has the following form

$$\left[\frac{q(T_1, T_2)}{T_2}\right]^2 \frac{\partial q}{\partial T_2} = M. \tag{26}$$

That is, the optimal temperature T_2 has to depend on T_1 in such a way that for any instance of time the left-hand side of the equation (26) is constant. The value of the constant M is to be found after substitution of the dependence $T_2^*(T_1, M)$ from (26) into the equation (25).

From (26) it follows that for the linear heat exchange (6) in the minimal dissipation process the temperatures ratio T_1/T_2 must be constant. Let us note that in a counter-flux heat-exchanger the minimal dissipation process can be realized, that is, here it is possible to choose a ratio of the fluxes' input temperatures and rates such that for the heat exchange law (6) the heat exchange operates with the minimal dissipation (26) [6].

Let us generalise the above-described approach to the abstract thermodynamic process where there is a key flux J, which depends on intensive variables (temperatures, pressures, chemical potentials, etc) denoted as U_1 and U_2 for the 1st and 2nd contacting systems correspondingly. It is assumed that U_1 and U_2 are scalars that vary over time or over the length of the apparatus. The function $J(U_1, U_2)$ is such that

$$\left.\begin{aligned} &\operatorname{sign} J(U_1, U_2) = \operatorname{sign}(U_1 - U_2), \\ &J(U_1, U_2) = 0 \quad \text{for} \quad U_1 = U_2. \end{aligned}\right\} \tag{27}$$

The difference between the values of U_1 and U_2 generates driving forces $X(U_1, U_2)$, which obey the conditions (27). For example, the driving force in a heat exchange process is the difference $(1/T_2 - 1/T_1)$, and in mass transfer process the driving force is

$$X = \frac{1}{T}[\mu_1(C_1) - \mu_2(C_2)],$$

where C_i is the concentration of the key component and μ_i is the chemical potential, which depends on this concentration. The dissipation here is

$$\bar{\sigma} = \frac{1}{\tau}\int_0^\tau J(U_1,U_2)X(U_1,U_2)\mathrm{d}t \to \min. \tag{28}$$

Because J and X obey the conditions (27), $\bar{\sigma} \geq 0$. Here the equality holds only if $U_1(t) = U_2(t)$.

The minimal dissipation process is defined as a process with the given average rate of flux

$$\frac{1}{\tau}\int_0^\tau J(U_1,U_2)\mathrm{d}t = \bar{J} \tag{29}$$

where $\bar{\sigma}$ is minimal. Here it is assumed that $U_2(t)$ is the control variable and $U_1(t)$ changes in accordance with the properties of the thermodynamic system as

$$\frac{dU_1}{dt} = C(U_1)J(U_1,U_2), \quad U_1(0) = U_{10}. \tag{30}$$

The right-hand side in this equation is sign definite.

Derivations similar to the ones used above yield the following optimality conditions for the problem (28) – (30)

$$\frac{J^2(U_1,U_2)\cdot \partial X/\partial U_2}{\partial J/\partial U_2} = \text{const.} \tag{31}$$

The value of the constant in (31) is to be found from (29).

The conditions of minimal dissipation in heat exchange (26) follow from (31). In particular if the ratio $\partial X/\partial U_2$ to $\partial J/\partial U_2$ is constant then the conditions of minimal dissipation require the dependence $U_2^*(U_1)$ such that the flux is constant.

The minimal dissipation problem can be also solved for vector fluxes, vector driving forces and vector variables U_1 and U_2. In the vicinity of equilibrium the fluxes and driving forces relate to each other via the Onsager equation

$$J = AX^{\mathrm{T}},$$

where A is a positively definite matrix of phenomenological coefficients. It is easy to see that in this case the solution of the minimal dissipation process corresponds to constant driving forces. Note that in vector case it is possible that the averaged values of not all but only some of J components are fixed.

Separation process are among the largest energy consumers. The dependence of the energy consumption for separation of one mole of a binary (two-component) mixture without losses into the environment, in an infinitely large apparatus and in infinitely long time (that is, in a separation process that is close to reversible) on the concentration X of the first component is shown in Figure 3 with the solid line.

The calculations [5] give the characteristic dependence of minimal energy consumption in separation over finite time (which corresponds to the minimal dissipation process) that is shown in Figure 3 with the dashed line. Thus, for poor mixtures ($X \to 0$, $X \to 1$) the condition of the non-zero rate of separation leads to the jump of the lower bound on energy consumption. This is consistent with the fact that the energy consumption for uranium isotope separation exceeds the reversible estimate by a factor of hundreds of thousands [8].

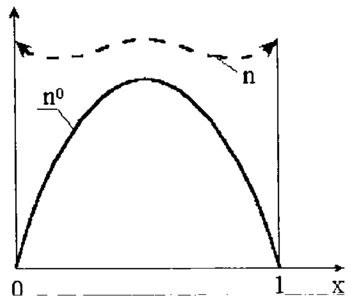

Figure 3. The characteristic dependencies of the reversible and irreversible energy consumption of separation on key component concentration.

OPTIMAL PROCESSES IN MICROECONOMICS

Economic systems, where exchange of commodities and capital between consumers and producers takes place, are in many respects similar to the thermodynamic systems. As in thermodynamics, the system's variables can be divided into two categories: the extensive variables, which are proportional to the system's size, and intensive variables, that do not depend on the size of the system. The capital and stocks of commodities are extensive variables and the prices that describe how valuable the commodities are for the system (commodity estimates) are the intensive ones.

In thermodynamics one considers three classes of systems: systems with finite capacities, reservoirs and working bodies. In a system with finite capacity and constant size the intensive variables depend on its extensive ones. For example, temperature is determined by energy. In reservoirs the amount of energy is so high that their intensive variables can be considered constant (the heat capacity is infinite). And finally the intensive variables of the working bodies can be controlled, for example, by controlling their size. Similarly in economics one can define systems with finite capacities, with infinite capacities and intermediaries. For the finite capacity system the commodity estimate is determined by its current stock; for infinite capacity system this estimate is constant and finally an intermediary itself determines the price it is offering for buying and selling.

The analogy between reversible thermodynamics and economics was emphasised by Samuelson [9], Lihnerowicz [10], Rozonoer [11]. We will consider irreversible process in microeconomics [12, 13] and their optimization.

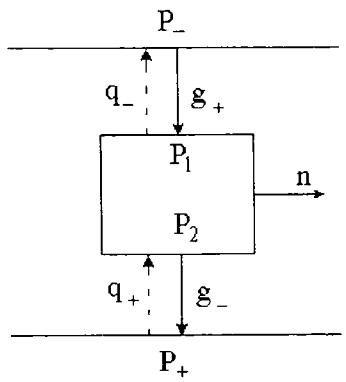

Figure 4. The structure of economic system with intermediary.

LIMITING POSSIBILITIES OF AN INTERMEDIARY

We consider a system (Figure 4) that consists of two markets and an intermediary, which buys commodity on the first market and sells it on the second. The commodity estimates on these markets are P_- and $P_+ > P_-$ correspondingly. P_1 and P_2 denote the prices of purchasing and selling that are offered by the intermediary. q_-, q_+, g_+ and g_- denote the fluxes of capital and commodity. Subscript (+) corresponds to the fluxes that enter the intermediary and (–) to these that leave it. n denotes the flux of the capital produced.

The fluxes g_+ and g_- depend on the relative values of the commodity price and the commodity estimate such that

$$\left.\begin{aligned} \operatorname{sign}[g_+(P_1,P_-)] &= \operatorname{sign}(P_1 - P_-), \\ \operatorname{sign}[g_-(P_+,P_2)] &= \operatorname{sign}(P_+ - P_2), \end{aligned}\right\} \tag{32}$$

and the capital fluxes are

$$q_- = P_1 g_+, \qquad q_+ = P_2 g_-. \tag{33}$$

Let us find the limiting possibilities of the intermediary, that is, the maximal amount of capital per unit of the initial investment and the maximal rate of profit (rate of flux n) that can be achieved in this system. The schema of the solution process here is similar to the one used to solve heat engine's maximal efficiency and maximal power problems.

The balances on the commodity and capital of the intermediary are

$$\left.\begin{aligned} g_+ - g_- = \frac{q_-}{P_1} - \frac{q_+}{P_2} &= 0, \\ q_+ - q_- - n &= 0. \end{aligned}\right\} \tag{34}$$

After elimination of q_+ we get the profit per unit of expenses

$$\eta = \frac{n}{q_-} = \frac{P_2}{P_1} - 1. \tag{35}$$

η attains its maximum η^0 at $P_2 = P_+$, $P_1 = P_-$, so

$$\eta^0 = \frac{P_+}{P_-} - 1 \tag{36}$$

However the fluxes of commodities and therefore the rate of profit n here are infinitely small.

Let us find out the limiting value of the profit flux $n_{\max}$ for the linear dependencies

$$g_+ = \alpha_-(P_1 - P_-), \qquad g_- = \alpha_+(P_+ - P_2). \tag{37}$$

The problem that determines $n_{\max}$ takes the following form

$$n = q_+ - q_- = \alpha_+ P_2(P_+ - P_2) - \alpha_- P_1(P_1 - P_-) \to \max_{P_1,P_2}. \tag{38}$$

subject to constraint

$$\alpha_-(P_1 - P_-) = \alpha_+(P_+ - P_2). \tag{39}$$

The solution of this problem leads to the following result

$$n_{\max} = \frac{\alpha_+\alpha_-}{4(\alpha_+ + \alpha_-)}(P_+ - P_-)^2. \tag{40}$$

$$P_1 = \frac{P\alpha + (P_+ + P)\alpha_+/2}{\alpha_+ + \alpha}, \quad P_2 = \frac{P_+\alpha_+ + (P_+ + P)\alpha/2}{\alpha_+ + \alpha}$$

According to (35)

$$\eta = \frac{(P_+ - P)(\alpha_+ + \alpha_-)}{2P_-\alpha_- + (P_+ + P_-)\alpha_+} < \eta^0,$$

The characteristic dependence $n(q_-)$ is shown in Figure 5.

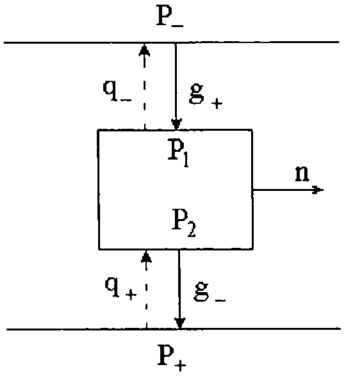

Figure 5. The dependence of the limiting profit on the expenses of the intermediary.

Let us introduce the notion of capital dissipation as the expenses of the intermediary caused by establishing the commodity flux g. Indeed, if the price that is paid by the intermediary for the commodity is P_- then its expenses are $g_+ \cdot P_-$ and in reality the intermediary spends $g_+ \cdot P_1$ for buying and the difference

$$\sigma_1 = g_+(P_1, P_-)\,(P_1 - P_-).$$

represents the trading expenses during buying. Similarly

$$\sigma_2 = g_-(P_+, P_2)\,(P_+ - P_2).$$

describes the trading expenses in a selling process. Because the fluxes of commodity are the same $g_1 = g_2 = g$ the profit is

$$n = g(P_+ - P_-) - (\sigma_1 + \sigma_2) = g(P_+ - P_-) - \sigma. \qquad (41)$$

If α_+ and α_- tend to infinity then the dependence of n on q_- tends to the dashed line in Figure 5. For finite α_+ and α_- an increase in q_- leads to an increase in dissipation σ.

EQUILIBRIUM IN OPEN ECONOMIC SYSTEMS

Exchange fluxes emerge in a non-uniform economic system, which include subsystems with different commodity estimates,. If a non-uniform system is insulated then these fluxes lead it to an equilibrium, where the commodity estimate in all subsystems are the same. If commodity exchange between the system and the environment takes place then in a stationary regime this estimate differs from the estimate at equilibrium. Here the commodity is re-distributed between subsystems so that a new distribution is established. This new distribution is determined by the commodity stocks, the commodity estimates (determined by the stocks) and the commodity fluxes (which are in turn determined by these estimates).

In irreversible thermodynamics the well-known Prigogine theorem [14] states that *in an open thermodynamic system in a stationary state in the vicinity of equilibrium the entropy production is minimal.*

That is, the extensive variables (internal energy, amount of mass, etc) are re-distributed inside the system in such a fashion that the dissipation caused by the fluxes inside the system is minimal. The condition of being near equilibrium is essential because here the dependence of the fluxes on the driving forces can be linearized.

A similar statement is valid for an economic system, namely: *in an equilibrium in an open economic system, where the commodity fluxes linearly depend on the differences in their estimates, the stocks of commodities are distributed between subsystems is such fashion that the dissipation of capital σ is minimal* [15].

Because any non-uniform system can be decomposed into elementary units, which contain sequentially and parallel connected subsystems, we will demonstrate the validity of this statement for such units, as shown in Figure 6.

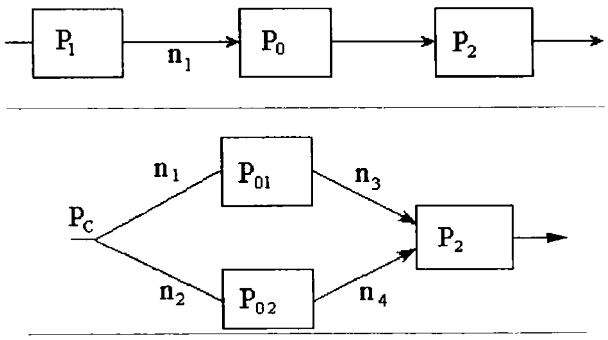

Figure 6. The simplest structures of open microeconomic systems.

For the sequential chain (Figure 6 (a)) the fluxes are

$$n_1 = \alpha_1(P_0 - P_1), \quad n_2 = \alpha_2(P_2 - P_0).$$

and the dissipation is

$$\sigma = \sigma_1 + \sigma_2 = \alpha_1(P_0 - P_1)^2 + \alpha_2(P_2 - P_0)^2 = \frac{n_1^2}{\alpha_1} + \frac{n_2^2}{\alpha_2}. \tag{42}$$

The conditions of σ minimum with respect to P_0, which in turn is determined by the stock of commodity in the middle subsystem, lead to the equation

$$n_1(P_0, P_1) = n_2(P_2, P_0) = n.$$

Thus, the stationarity condition of the system and the condition of minimum of σ coincide.

For the parallel connection (Figure 6 (b)) we get

$$n_1 = \alpha_1(P_{01} - P_c), \quad n_2 = \alpha_2(P_{02} - P_c),$$
$$n_3 = \alpha_3(P_2 - P_{01}), \quad n_4 = \alpha_4(P_2 - P_{02}).$$

Similarly to (49) the dissipation is $\sigma = \sum_{i=1}^{4} n_i^2 / \alpha_i$. The conditions of its minimum on n_1 and n_2 for $n_1 = n_3$, $n_2 = n_4$, $n_1 + n_2 = n$ lead to the equation

$$\frac{n_1}{n_2} = \frac{(\alpha_2 + \alpha_4)\alpha_1\alpha_3}{\alpha_2\alpha_4(\alpha_1 + \alpha_3)}. \tag{43}$$

On the other hand, from the stationarity conditions $n_1 = n_3$ and $n_2 = n_4$ it follows

$$P_{01} = \frac{\alpha_1 C + \alpha_3 P_2}{\alpha_1 + \alpha_3}, \quad P_{02} = \frac{\alpha_2 C + \alpha_3 P_2}{\alpha_2 + \alpha_4},$$

and the fluxes are

$$n_1 = \frac{\alpha_1 \alpha_3}{\alpha_1 + \alpha_3}(P_2 - C), \quad n_2 = \frac{\alpha_2 \alpha_4}{\alpha_2 + \alpha_4}(P_2 - C).$$

Their ratio coincides with the equation (43), which is obtained from the condition of minimal dissipation.

CONCLUSION

The optimal processes in thermodynamic systems (from the viewpoint of energy consumption) correspond to the minimum entropy production. The thermodynamic quality of the processes with given productivity (rate) can be evaluated not by the value of σ but by the difference between the actual and the minimal-feasible entropy production.

The irreversible processes in microeconomics are in many respects similar to the thermodynamic processes. The role of entropy production here is played by the capital dissipation, whose minimum corresponds to the stationary state of an open economic system.

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OPTIMALNI PROCESI U IREVERZIBILNOJ TERMODINAMICI I MIKROEKONOMICI

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SAŽETAK

U ovom radu razmotrena je opća metodologija koja omogućuje proširenje Carnotove učinkovitosti u klasičnoj termodinamici za beskonačno spore procese na termodinamičke procese konačnog trajanja. Defnirana je klasa procesa minimalen disipacije i pokazano da ona predstavlja poopćenje reverzibilnih procesa i određuje krajnje mogućnosti sustava u kojima se odvijaju procesi konačnog trajanja. Opisana metodologija je primijenjena na mikroekonomske sustave izmjene što dovodi do novih procjena graničnih učinkovitosti takvih sustava.

KLJUČNE RIJEČI

termodinamika u konačnom vremenu, procesi minimalne disipacije, procesi optimalne izmjene u ekonomiji