MULTI-FOLLOWER LINEAR BILEVEL PROGRAMMING: MODEL AND KUHN-TUCKER APPROACH

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ABSTRACT

The majority of research on bilevel programming has centered on the linear version of the problem in which only one leader and single follower are involved. This paper proposes a general model and Kuhn-Tucker approach for linear bilevel programming problems in which one leader and multiple follower(s) are involved, and there may (not) be sharing variables among the followers. Finally, a numeric example is given to show how the Kuhn-Tucker approach is applied to solve multi-follower linear bilevel problems.

KEYWORDS

Linear bilevel programming, Kuhn-Tucker approach, Optimization, Decision making.

1. INTRODUCTION

Bilevel programming (BLP) was motivated by the game theory of Von Stackelberg [1] in the context of unbalanced economic market. In a basic BLP model, the control for decision variables is partitioned amongst the players. The upper-level is termed as the leader and the lower-level is termed as the follower. The leader goes first and attempts to optimize his/her objective function. The follower observes the leader's decision and makes his/her decision. Because the set of feasible choices available to either player is interdependent, the leader's decision affects both the follower's payoff and allowable actions, and vice versa [2].

The majority of research on BLP has centered on the linear version of the problem in which only one follower is involved. There have been nearly two dozen algorithms, such as, the K^{th} best approach [3,4], Kuhn-Tucker approach [5,6], complementarity pivot approach [7], penalty function approach [8], proposed for solving linear BLP problems since the field being caught the attention of researchers in the mid-1970s. The most popular one is Kuhn-Tucker approach [2]. Kuhn-Tucker approach has been proven to be a valuable analysis tool with a wide range of successful applications for linear BLP [2,6].

Our previous work presented a new definition of solution and related theorem for linear BLP problems in which one follower is involved, thus overcame the fundamental deficiency of existing linear BLP theory [9]. We also described theoretical properties of linear BLP, developed an extended *K*th-best approach for linear BLP [10], an extended Kuhn-Tucker approach [11] and an extended branch and bound algorithm for linear BLP. We also identified five kinds of relationships among the followers through building a bilevel multi-follower framework. Particularly, it proposed a model and Kuhn-Tucker approach for linear bilevel multi-follower problems (BLMFP) in which there are not sharing variables among followers. We explored theoretical properties of linear BLMFP, developed a *K*th-best approach and branch and bound algorithm for the model. This paper proposes a general model and Kuhn-Tucker approach for linear BLP problems in which one leader and one or multiple follower(s) are involved, and the followers may (not) share variables except the leader's for multi-follower problems. Following the introduction, this paper proposes a general model for linear BLP in Section 2. A general Kuhn-Tucker approach for this model is described in Section 3. A numeric example for this approach is given in Section 4. A conclusion is given in Section 5.

2. A GENERAL MODEL FOR LINEAR BILEVEL PROGRAMMING

Let us consider a BLP problem in which one leader and one or multiple follower(s) are involved. In our BLP model, the control for the decision variables is partitioned amongst the players who seek to minimize their individual payoff objective functions. Perfect information is assumed so that all players know the objective and feasible choices available of the others. The leader goes first and attempts to optimize her/his objective function. In order to that, the leader must anticipate all possible responses of her/his opponents. Each follower executes simultaneously her/his policies after, and in view of, decisions of the leader. Because the set of feasible choices available to each player is interdependent, one player's decision affects both the payoff and allowable actions of all of others.

For $x \in X \subset R^n, y_i \in Y_i \subset R^{m_i}$, $F: X \times Y_1 \times \ldots \times Y_K \to R^1$, and $f_i: X \times Y_1 \times \ldots \times Y_K \to R^1$, $i = 1,2,\ldots,K$, a linear BLP problem in which $K \geq 1$ is given:

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s \tag{1a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leq b \tag{1b}$$

$$\min_{y_i \in Y_i} f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \tag{1c}$$

$$\text{subject to } A_i x + \sum_{s=1}^{K} C_{is} y_s \leq b_i \tag{1d}$$

where $c \in R^n$, $c_i \in R^n$, $d_i \in R^{m_i}$, $e_{is} \in R^{m_s}$, $b \in R^p$, $b_i \in R^{q_i}$, $A \in R^{p \times n}$, $B_i \in R^{p \times m_i}$, $A_i \in R^{q_i \times n}$, $C_{is} \in R^{q_i \times m_s}$, $i, s = 1,2,\ldots,K$.

Definition 1 A topological space is compact if every open cover of the entire space has a finite subcover. For example, $[a,b]$ is compact in R (the Heine-Borel theorem) [12].

Corresponding to (1), we give following basic definition for linear BLP solution.

Definition 2

(a) Constraint region of the linear BLP problem:

$$S = \{(x, y_1, \ldots, y_K) \in X \times Y_1 \times \ldots \times Y_k, Ax + \sum_{s=1}^{K} B_s y_s \leq b,$$
$$A_i x + \sum_{s=1}^{K} C_{is} y_s \leq b_i, i = 1,2,\ldots,K\}.$$

The linear BLP problem constraint region refers to all possible combinations of choices that the leader and follower(s) may make.

(b) Projection of S onto the leader's decision space:

$$S(X) = \{x \in X : \exists y_i \in Y_i, Ax + \sum_{s=1}^{K} B_s y_s \leq b, A_i x + \sum_{s=1}^{K} C_{is} y_s \leq b_i, i = 1,2,\ldots,K\}.$$

Unlike the rules in non-cooperative game theory where each player must choose a strategy simultaneously, the definition of BLP model requires that the leader moves first by selecting an x in attempt to minimize his objective subjecting to constraints of both upper and lower level.

(c) Feasible set for each follower $\forall x \in S(X)$:

$$S_i(x) = \{y_i \in Y_i : (x, y_1, \ldots, y_K) \in S\}.$$

The feasible region for each follower is affected by the leader's choice of x, and the allowable choices of each follower are the elements of S.

(d) Each follower's rational reaction set for $x \in S(X)$:

$$P_i(x) = \{y_i \in Y_i : y_i \in \arg\min[f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \neq i) : \hat{y}_i \in S_i(x)]\},$$

where $i = 1,2,\ldots,K$, $\arg\min[f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \neq i) : \hat{y}_i \in S_i(x)] =$

$$\{y_i \in S_i(x) : f_i(x, y_1, \ldots, y_K) \leq f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \neq i), \hat{y}_i \in S_i(x)\}.$$

The followers observe the leader's action and simultaneously react by selecting y_i from their feasible set to minimize their objective functions.

(e) Inducible region:

$$IR = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, y_i \in P_i(x), i = 1,2,\ldots,K\}.$$

Thus in terms of the above notations, the linear BLP problem can be written as

$$\min\{F(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in IR\}. \qquad (2)$$

We propose the following theorem to characterize the condition under which there is an optimal solution for a linear BLP problem.

Theorem 1 If S is nonempty and compact, there exists an optimal solution for a linear BLP problem.

Proof: Obvious.

3. A GENERAL KUHN-TUCKER APPROACH FOR LINEAR BILEVEL PROGRAMMING

Let write a linear programming (LP) as follows.

$$\min f(x) = cx$$

subject to $Ax \geq b \qquad x \geq 0$

where c is an n-dimensional row vector, b an m-dimensional column vector, A an $m \times n$ matrix with $m \leq n$, and $x \in R^n$.

Let $\lambda \in R^m$ and $\mu \in R^n$ be the dual variables associated with constraints $Ax \geq b$ and $x \geq 0$, respectively. Bard [2] gave the following proposition.

Proposition 3.1 A necessary and sufficient condition that (x^*) solves above LP is that there exist (row) vectors λ^*, μ^* such that (x^*, λ^*, μ^*) solves:

$$\lambda A - \mu = -c$$
$$Ax - b \geq 0$$
$$\lambda(Ax - b) = 0$$
$$\mu x = 0$$
$$x \geq 0, \lambda \geq 0, \mu \geq 0.$$

Proof: (See reference [2] PP. 59-60)

Let $u_i \in R^p$, $v_i \in R^{q_1+q_2+\ldots+q_K}$ and $w_i \in R^{m_i}$ ($i = 1,2,\ldots,K$) be the dual variables associated with constraints ($Ax + \sum_{s=1}^{K} B_s y_s \leq b$), ($A^{'}x + \sum_{s=1}^{K} C^{'}_s y_s \leq b^{'}$) and $y_i \geq 0 \quad (i = 1,\ldots,K)$, respectively, where $A^{'} = (A_1, A_2, \ldots, A_K)^T$, $C^{'}_i = (C_{i1}, C_{i2}, \ldots, C_{iK})^T$, $b^{'} = (b_1, b_2, \ldots, b_K)^T$. We have a following theorem.

Theorem 2 A necessary and sufficient condition that $(x^*, y_1^*, \ldots, y_K^*)$ solves the linear BLP problem (1) is that there exist (row) vectors $u_1^*, u_2^*, \ldots, u_K^*$, $v_1^*, v_2^*, \ldots, v_K^*$ and $w_1^*, w_2^*, \ldots, w_K^*$ such that $(x^*, y_1^*, \ldots, y_K^*, u_1^*, \ldots, u_K^*, v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ solves:

$$\min F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s \tag{3a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \le b \tag{3b}$$

$$A' x + \sum_{s=1}^{K} C_s' y_s \le b' \tag{3c}$$

$$u_i B_i + v_i C_i' - w_i = -e_{ii} \tag{3d}$$

$$u_i (b - Ax - \sum_{s=1}^{K} B_s y_s) + v_i (b' - A' x - \sum_{s=1}^{K} C_s' y_s) + w_i y_i = 0 \tag{3e}$$

$$x \ge 0, y_j \ge 0, u_j \ge 0, v_j \ge 0, w_j \ge 0,\ j = 1,2,\ldots,K, \tag{3f}$$

where $i = 1,2\ldots,K$.

Proof: 1. Let us get an explicit expression of (2).
Rewrite (2) as follows:

$$\min F(x, y_1, \ldots, y_K)$$
$$\text{subject to } (x, y_1, \ldots, y_K) \in IR.$$

We have

$$\min F(x, y_1, \ldots, y_K)$$
$$\text{subject to } (x, y_1, \ldots, y_K) \in S$$
$$y_i = P_i(x),$$

where $i = 1,2,\ldots,K$, by Definition 2(e). Then, we have

$$\min F(x, y_1, \ldots, y_K)$$
$$\text{subject to } (x, y_1, \ldots, y_K) \in S$$
$$y_i \in \arg\min[f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \ne i) : \hat{y}_i \in S_i(x)],$$

where $i = 1,2,\ldots,K$, by Definition 2(d). We rewrite it as:

$$\min F(x, y_1, \ldots, y_K)$$
$$\text{subject to } (x, y_1, \ldots, y_K) \in S$$
$$\min f_i(x, y_1, \ldots, y_K)$$
$$\text{subject to } y_i \in S_i(x),$$

where $i = 1,2,\ldots,K$. We have

$$\min F(x, y_1, \ldots, y_K)$$
$$\text{subject to } (x, y_1, \ldots, y_K) \in S$$
$$\min_{y_i \in Y_i} f_i(x, y_1, \ldots, y_K)$$
$$\text{subject to } (x, y_1, \ldots, y_K) \in S,$$

where $i = 1,2,\ldots,K$, by Definition 2(c). Consequently, we can have

$$\min F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s \tag{4a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \le b \tag{4b}$$

$$A_j x + \sum_{s=1}^{K} C_{is} y_s \le b_i ,\ j = 1,2,\ldots,K \tag{4c}$$

$$\min_{y_i \in Y_i} f_i(x, y_i, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \tag{4d}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \le b \tag{4e}$$

$$A_j x + \sum_{s=1}^{K} C_{is} y_s \le b_i ,\ j = 1,2,\ldots,K , \tag{4f}$$

where $i = 1,2,\ldots,K$, by Definition 2(a).

This simple transformation has shown that solving the linear BLP (1) is equivalent to solving (4). 2. Necessity is obvious from (4). 3. Sufficiency. If $(x^*, y_1^*, \ldots, y_K^*)$ is the optimal solution of (1), we need to show that there exist (row) vectors $u_1^*, u_2^*, \ldots, u_K^*$, $v_1^*, v_2^*, \ldots, v_K^*$ and $w_1^*, w_2^*, \ldots, w_K^*$ such that $(x^*, y_1^*, \ldots, y_K^*, u_1^*, \ldots, u_K^*,\ v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ to solve (3). Going one step farther, we only need to proof that there exist (row) vectors $u_1^*, u_2^*, \ldots, u_K^*$, $v_1^*, v_2^*, \ldots, v_K^*$ and $w_1^*, w_2^*, \ldots, w_K^*$ such that $(x^*, y_1^*, \ldots, y_K^*, u_1^*, \ldots, u_K^*, v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ satisfies the follows

$$u_i B_i + v_i C_i' - w_i = -e_{ii} \tag{5a}$$

$$u_i (b - Ax - \sum_{s=1}^{K} B_s y_s) = 0 \tag{5b}$$

$$v_i (b' - A'x - \sum_{s=1}^{K} C_s' y_s) = 0 \tag{5c}$$

$$w_i y_i = 0 \tag{5d}$$

where $u_i \in R^p$, $v_i \in R^{q_1+q_2+\ldots+q_K}$, $w_i \in R^{m_i}$, $i = 1,2,\ldots,K$ and they are not negative variables. Because $(x^*, y_1^*, \ldots, y_K^*)$ is the optimal solution of (1), we have $(x^*, y_1^*, \ldots, y_K^*) \in IR$, by (2). Thus we have $y_i^* \in P_i(x^*)$, where $i = 1,2,\ldots,K$, by Definition 2(e). Consequently, $(y_1^*, y_2^*, \ldots, y_K^*)$ is the optimal solution to the following problem

$$\min(f_i(x^*, y_1, \ldots, y_K) : y_i \in S_i(x^*)) , \tag{6}$$

where $i = 1,2,\ldots,K$, by Definition 2(d). Rewrite (6) as follows

$$\min f_i(x, y_1, \ldots, y_K)$$

$$\text{subject to } y_i \in S_i(x)$$

$$x = x^*$$

$$y_j = y_j^* ,\ j = 1,2,\ldots,K, j \ne i ,$$

where $i = 1,2,\ldots,K$. From Definition 2(c), we have

$$\min f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \tag{7a}$$

subject to $Ax + \sum_{s=1}^{K} B_s y_s \le b$ (7b)

$$A_j x + \sum_{s=1}^{K} C_{js} y_s \le b_j \,,\; j = 1,2,\ldots,K \quad (7c)$$

$$x = x^* \quad (7d)$$

$$y_i \ge 0 \quad (7e)$$

$$y_j = y_j^*,\; j = 1,2,\ldots,K,\; j \ne i\,, \quad (7f)$$

where $i = 1,2,\ldots,K$. Let us define:

$A^{'} = (A_1, A_2, \ldots, A_K)^{-1}$, $b^{'} = (b_1, b_2, \ldots, b_K)^{-1}$, $C_i^{'} = (C_{i1}, C_{i2}, \ldots, C_{iK})^{-1}$, $i = 1,2\ldots,K$. To simplify (7c), we can have

$$\min\; f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \quad (8a)$$

subject to $Ax + \sum_{s=1}^{K} B_s y_s \le b$ (8b)

$$A^{'} x + \sum_{s=1}^{K} C_s^{'} y_s \le b^{'} \quad (8c)$$

$$x = x^* \quad (8d)$$

$$y_i \ge 0\,, \quad (8e)$$

$$y_j = y_j^*,\; j = 1,2,\ldots,K,\; j \ne i\,, \quad (8f)$$

where $i = 1,2,\ldots,K$. Thus simplify (8), we can have

$$\min\; f_i(y_i) = e_{ii} y_i \quad (9a)$$

subject to $-\begin{pmatrix} B_i \\ C_i^{'} \end{pmatrix} y_i \ge -\begin{pmatrix} b - Ax^* - \sum_{s=1, s\ne i}^{K} B_s y_s^* \\ b_i^{'} - A^{'} x^* - \sum_{s=1, s\ne i}^{K} C_s^{'} y_s^* \end{pmatrix}$ (9b)

$$y_i \ge 0\,, \quad (9c)$$

where $i = 1,2,\ldots,K$.

Now we see that y_i^* is the optimal solution of (9) which is a LP problem. By Proposition 3.1, there exists vector λ_i^*, μ_i^* , $i = 1,2,\ldots,K$ that satisfy a system below

$$\lambda_i \begin{pmatrix} B_i \\ C_i^{'} \end{pmatrix} - \mu_i = -e_{ii} \quad (10a)$$

$$\lambda_i (-\begin{pmatrix} B_i \\ C_i^{'} \end{pmatrix} y_i + \begin{pmatrix} b - Ax^* - \sum_{s=1, s\ne i}^{K} B_s y_s^* \\ b^{'} - A^{'} x^* - \sum_{s=1, s\ne i}^{K} C_s^{'} y_s^* \end{pmatrix}) = 0 \quad (10b)$$

$$\mu_i y_i = 0\,, \quad (10c)$$

where $\lambda_i \in R^{p+q_1+\ldots+q_K}$, $\mu_i \in R^{m_i}$, $i = 1,2,\ldots,K$.

Let $u_i \in R^p$, $v_i \in R^{q_i+q_2+\ldots+q_K}$, $w_i \in R^{m_i}$ and define

$$\lambda_i = (u_i, v_i)$$

$$w_i = \mu_i,$$

where $i = 1,2,\ldots,K$.

Thus we have $(x^*, y_1^*,\ldots, y_K^*, u_1^*,\ldots,u_K^*, v_1^*,\ldots,v_K^*, w_1^*,\ldots,w_K^*)$ that satisfy (5). Our proof is completed.

Theorem 2 means that the most direct approach to solving (1) is to solve the equivalent mathematical program given in (3). One advantage that it offers is that it allows for a more robust model to be solved without introducing any new computational difficulties.

4. A NUMERIC EXAMPLE FOR THE KUHN-TUCKER APPROACH

Let us give the following example to show how the Kuhn-Tucker approach works.

Example 1 Consider the following linear BLP problem with $x \in R^1$, $y \in R^1$, and $X = \{x \geq 0\}$, $Y = \{y \geq 0\}$.

$$\min_{x\in X} F(x,y) = x - 4y$$

subject to $x + y \geq 3$

$$-3x + 2y \geq -4$$

$$\min_{y\in Y} f(x,y) = x + y$$

subject to $-2x + y \leq 0$

$$2x + y \leq 12 \; .$$

According to our approach, let us write all the inequalities but $x \geq 0$ of Example 1 as follows:

$$g_1(x,y) = x + y - 3 \geq 0 \tag{11a}$$

$$g_2(x,y) = -3x + 2y + 4 \geq 0 \tag{11b}$$

$$g_3(x,y) = 2x - y \geq 0 \tag{11c}$$

$$g_4(x,y) = -2x - y + 12 \geq 0 \tag{11d}$$

$$g_5(x,y) = y \geq 0. \tag{11e}$$

From (3), we have

$$\min = (x - 4y) \tag{12a}$$

subject to $-x - y \leq -3$ (12b)

$$-3x + 2y \geq -4 \tag{12c}$$

$$-2x + y \leq 0 \tag{12d}$$

$$2x + y \leq 12 \tag{12e}$$

$$-u_1 - 2u_2 + u_3 + u_4 - u_5 = -1 \tag{12f}$$

$$u_1 g_1(x,y) + u_2 g_2(x,y) + u_3 g_3(x,y) + u_4 g_4(x,y) + u_5 g_5(x,y) = 0 \tag{12g}$$

$$x \geq 0, y \geq 0, u_1 \geq 0, u_2 \geq 0, u_3 \geq 0, u_4 \geq 0, u_5 \geq 0. \tag{12h}$$

From (12f) and (12h), we can have following three possibilities.

Case 1: $(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (1,0,0,0,0)$

Case 2: $(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (0,0.5,0,0,0)$

Case 3: $(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (0,0,0,0,1)$

From Case1, (12g) and (11a), we have

$$g_1(x, y) = x + y - 3 = 0$$

Consequently, (12) can be rewritten as follows:

$$\begin{aligned} &\min(x - 4y) \\ &\text{subject to } -x - y = -3 \\ &\qquad -3x + 2y \ge -4 \\ &\qquad -2x + y \le 0 \\ &\qquad 2x + y \le 12 \\ &\qquad x \ge 0, y \ge 0 \ . \end{aligned}$$

Using simplex algorithm, a solution occurs at the point $(x^*, y^*) = (1,2)$ with $F^* = -7$ and $f^* = 3$.

By using the same way as that of Case 1, we found that a solution occurs at the point $(x^*, y^*) = (4,4)$ with $F^* = -12$ and $f^* = 8$ for Case 2; it is infeasible for Case 3. By examining above procedure, we found that the optimal solution occurs at the point $(x^*, y^*) = (4,4)$ with $F^* = -12$ and $f^* = 8$. This result is identical with that in [13].

5. CONCLUSION AND FURTHER STUDY

This paper proposes a general model and Kuhn-Tucker approach for linear bilevel programming problems in which one leader and one or multiple follower(s) are involved, and there may (not) be sharing variables among followers except leader's for multi-follower problems. Numeric examples are given to show how the Kuhn-Tucker approach works. The further study of the research is to explore solution algorithms for the general model.

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