*K*th-Best Algorithm for Fuzzy Bilevel Programming

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Abstract

Organizational decision making often involves two decision levels. When the leader at the upper level attempts to optimize his/her objective, the follower at the lower level tries to find an optimized strategy according to each of possible decisions made by the leader. Furthermore, such bilevel decision making may involve uncertain parameters which appear either in the objective functions or constraints of the leader or the follower. Following our previous work, this study first proposes a fuzzy parameter bilevel programming model and related theories. It then develops an approximation Kth-Best algorithm for solving such fuzzy bilevel programming problems. A numerical example further illustrates the proposed algorithm.

Keywords: Bilevel programming, Kth-Best algorithm, Fuzzy set, Fuzzy optimization, Decision making

1. Introduction

Bilevel decision making (also called bilevel programming, BLP) techniques, first introduced by Von Stackelberg [19], have been developed for mainly solving decentralized planning problems with decision makers in a hierarchical organization [3, 9, 21]. Decision maker at the upper level is termed as the leader, and at the lower level, the follower. Each decision maker (leader or follower) tries to optimize his/her own objective function, but the decision of each level affects the objective value of the other level [4].

Bilevel decision making theory and technology have been applied with remarkable success in different domains, for example, decentralized resource planning, electric power market, logistics, civil engineering, chemical engineering and road network management. [1,2, 10-12]. The vast majority of research on BLP has centered on the linear version of the problem, i.e., linear BLP problems. A set of approaches and algorithms have been well developed such as well known Kuhn-Tucker approach [4,5], *K*th-best approach [6,7] and Branch-and-bound algorithm [8]. However, existing BLP approaches mainly suppose the situation in which the objective functions and constraints are characterized with precise parameters. Therefore, the parameters are required to be fixed at some values in an experimental and/or subjective manner through the experts' understanding of the nature of the parameters in the problem-formulation process. It has been observed that, in most real-world situations, for example, in a logistics planning, the possible values of these parameters are often only imprecisely or ambiguously known to the experts who establish this model. With this observation, it would be certainly more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy sets [22]. A BLP problem in which the parameters, either in objective functions or in constrains of the leader or the follower, are described by fuzzy values is called a fuzzy bilevel programming (FBLP) or a fuzzy bilevel decision making (FBLDM) problem in the study.

The FBLP problem was first researched by Sakawa et al. in 2000 [14]. Sakawa et al. formulates BLP problems with fuzzy parameters from the perspective of experts' imprecision and proposes a fuzzy programming approach for cooperative BLP problems. However, the solution concept for a cooperative BLP problem proposed by Lai [9,18] is different from the solution concept of Bard who deals with uncooperative BLP problem.

Our recent research work has extended Bard's solution concept of BLP by proposing an extended solution concept which can overcome the arbitrary linear form problem indicated above. We then proposed a set of extended approaches based on the new solution concept for solving linear BLP problems [15-17]. This paper is deals with FBLP problems based on the extended solution concept and related theorems [23-7]. In particular, it proposes a general fuzzy number based extended Kth-Best algorithm. This algorithm can solve uncooperative FBLP problems where fuzzy parameters can be expressed by any forms of membership functions. Following the introduction, Section 2 proposes a definition of optimal solution for FBLP problem and related transformation theory. A fuzzy number based approximation *K*th Best algorithm is presented in Section 3. Section 4 shows a numeral example for illustrating the proposed approximation

*K*th Best algorithm. Conclusion and further study are discussed in Section 5.

2. Fuzzy Parameter Linear Bilevel Programming Problem Related Solution Transformation Theory

Definition 3.1 A topological space is compact if every open cover of the entire space has a finite subcover. For example, $[a,b]$ is compact in R (the Heine-Borel theorem) [20].

Consider the following fuzzy linear bilevel programming (FLBLP) problem:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F: X \times Y \to F^*(R)$, and $f: X \times Y \to F^*(R)$,

$$\min_{x \in X} F(x,y) = \widetilde{c}_1 x + \widetilde{d}_1 y \quad (2.1a)$$

$$\text{subject to } \widetilde{A}_1 x + \widetilde{B}_1 y \preceq \widetilde{b}_1 \quad (2.1b)$$

$$\min_{y \in Y} f(x,y) = \widetilde{c}_2 x + \widetilde{d}_2 y \quad (2.1c)$$

$$\text{subject to } \widetilde{A}_2 x + \widetilde{B}_2 y \preceq \widetilde{b}_2 \quad (2.1d)$$

where $\widetilde{c}_1, \widetilde{c}_2 \in F^*(R^n)$, $\widetilde{d}_1, \widetilde{d}_2 \in F^*(R^m)$, $\widetilde{b}_1 \in F^*(R^p)$, $\widetilde{b}_2 \in F^*(R^q)$, $\widetilde{A}_1 = (\widetilde{a}_{ij})_{p\times n}$, $\widetilde{a}_{ij} \in F^*(R)$, $\widetilde{B}_1 = (\widetilde{b}_{ij})_{p\times m}$, $\widetilde{b}_{ij} \in F^*(R)$, $\widetilde{A}_2 = (\widetilde{e}_{ij})_{q\times n}$, $\widetilde{e}_{ij} \in F^*(R)$, $\widetilde{B}_2 = (\widetilde{s}_{ij})_{q\times m}$, $\widetilde{s}_{ij} \in F^*(R)$.

Associated with the (FLBLP) problem, we now consider the following (MLBLP) problem:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F: X \times Y \to F^*(R)$, and $f: X \times Y \to F^*(R)$,

$$\begin{aligned} &\min_{x \in X} (F(x,y))^L_\lambda = c^L_{1\lambda} x + d^L_{1\lambda} y, \quad \lambda \in [0,1] \\ &\min_{x \in X} (F(x,y))^R_\lambda = c^R_{1\lambda} x + d^R_{1\lambda} y, \quad \lambda \in [0,1] \end{aligned} \quad (2.2a)$$

subject to

$$A^L_{1\lambda} x + B^L_{1\lambda} y \leqq b^L_{1\lambda},\ A^R_{1\lambda} x + B^R_{1\lambda} y \leqq b^R_{1\lambda}, \quad \lambda \in [0,1] \quad (2.2b)$$

$$\begin{aligned} &\min_{y \in Y} (f(x,y))^L_\lambda = c^L_{2\lambda} x + d^L_{2\lambda} y, \quad \lambda \in [0,1] \\ &\min_{y \in Y} (f(x,y))^R_\lambda = c^R_{2\lambda} x + d^R_{2\lambda} y, \quad \lambda \in [0,1] \end{aligned} \quad (2.2c)$$

subject to

$$A^L_{2\lambda} x + B^L_{2\lambda} y \leqq b^L_{2\lambda},\ A^R_{2\lambda} x + B^R_{2\lambda} y \leqq b^R_{2\lambda}, \quad \lambda \in [0,1] \quad (2.2d)$$

where $c^L_{1\lambda}, c^R_{1\lambda}$, $c^L_{2\lambda}, c^R_{2\lambda} \in R^n$, $d^L_{1\lambda}, d^R_{1\lambda}$, $d^L_{2\lambda}, d^R_{2\lambda} \in R^m$, $b^L_{1\lambda}, b^R_{1\lambda} \in R^p$, $b^L_{2\lambda}, b^R_{2\lambda} \in R^q$, $A^L_{1\lambda} = (a^L_{ij\lambda}), A^R_{1\lambda} = (a^R_{ij\lambda}) \in R^{p\times n}$, $B^L_{1\lambda} = (b^L_{ij\lambda}), B^R_{1\lambda} = (b^R_{ij\lambda}) \in R^{p\times m}$, $A^L_{2\lambda} = (e^L_{ij\lambda}), A^R_{2\lambda} = (e^R_{ij\lambda}) \in R^{q\times n}$, $B^L_{2\lambda} = (s^L_{ij\lambda}), B^R_{2\lambda} = (s^R_{ij\lambda}) \in R^{q\times m}$.

Theorem 2.1 [25] Let (x^*, y^*) be the solution of the (MLBLP) problem (2.2). Then it is also a solution of the (FLBLP) problem defined by (2.1).

Theorem 2.2 For $x \in X \subset R^n$, $y \in Y \subset R^m$, If all the fuzzy coefficients $\widetilde{a}_{ij}, \widetilde{b}_{ij}, \widetilde{e}_{ij}, \widetilde{s}_{ij}, \widetilde{c}_i$ and $\widetilde{d}_i$ have trapezoidal membership functions of the (MLBLP) problem (2.1).

$$\mu_{\widetilde{z}}(t) = \begin{cases} 0 & t < z^L_\beta \\ \dfrac{\alpha - \beta}{z^L_\alpha - z^L_\beta}(t - z^L_\beta) + \beta & z^L_\beta \leqq t < z^L_\alpha \\ \alpha & z^L_\alpha \leqq t < z^R_\alpha, \\ \dfrac{\alpha - \beta}{z^R_\beta - z^R_\alpha}(-t + z^R_\beta) + \beta & z^R_\alpha \leqq t \leqq z^R_\beta \\ 0 & z^R_\beta < t \end{cases} \quad (2.3)$$

where $\widetilde{z}$ denotes $\widetilde{a}_{ij}, \widetilde{b}_{ij}, \widetilde{e}_{ij}, \widetilde{s}_{ij}, \widetilde{c}_i$ and $\widetilde{d}_i$ respectively. Then, it is the solution of the problem (2.1) that $(x^*, y^*) \in R^n \times R^m$ satisfying

$$\begin{aligned} &\min_{x \in X} (F(x,y))^L_\alpha = c^L_{1\alpha} x + d^L_{1\alpha} y, \\ &\min_{x \in X} (F(x,y))^R_\alpha = c^R_{1\alpha} x + d^R_{1\alpha} y, \\ &\min_{x \in X} (F(x,y))^L_\beta = c^L_{1\beta} x + d^L_{1\beta} y, \\ &\min_{x \in X} (F(x,y))^R_\beta = c^R_{1\beta} x + d^R_{1\beta} y, \end{aligned} \quad (2.4a)$$

subject to

$$A_{1\alpha}^{L}x + B_{1\alpha}^{L}y \leqq b_{1\alpha}^{L},$$
$$A_{1\alpha}^{R}x + B_{1\alpha}^{R}y \leqq b_{1\alpha}^{R},$$
$$A_{1\beta}^{L}x + B_{1\beta}^{L}y \leqq b_{1\beta}^{L},$$
$$A_{1\beta}^{R}x + B_{1\beta}^{R}y \leqq b_{1\beta}^{R}, \tag{2.4b}$$

$$\min_{y\in Y}(f(x,y))_{\alpha}^{L} = c_{2\alpha}^{L}x + d_{2\alpha}^{L}y,$$
$$\min_{y\in Y}(f(x,y))_{\alpha}^{L} = c_{2\alpha}^{L}x + d_{2\alpha}^{L}y,$$
$$\min_{y\in Y}(f(x,y))_{\beta}^{L} = c_{2\beta}^{L}x + d_{2\beta}^{L}y,$$
$$\min_{y\in Y}(f(x,y))_{\beta}^{R} = c_{2\beta}^{R}x + d_{2\beta}^{R}y, \tag{2.4c}$$

subject to

$$A_{2\alpha}^{L}x + B_{2\alpha}^{L}y \leqq b_{2\alpha}^{L},$$
$$A_{2\alpha}^{L}x + B_{2\alpha}^{L}y \leqq b_{2\alpha}^{L},$$
$$A_{2\beta}^{L}x + B_{2\beta}^{L}y \leqq b_{2\beta}^{L},$$
$$A_{2\beta}^{R}x + B_{2\beta}^{R}y \leqq b_{2\beta}^{R}. \tag{2.4d}$$

Prof. We can easy to prove it by Definition of fuzzy number order.

Theorem 2.3 For $x \in X \subset R^n$, $y \in Y \subset R^m$, If all the fuzzy coefficients $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_i$ and $\tilde{d}_i$ have trapezoidal membership functions of the (MLBLP) problem (2.1).

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z_{\alpha_0}^{L} \\ \dfrac{\alpha_1 - \alpha_0}{z_{\alpha_1}^{L} - z_{\alpha_0}^{L}}\left(t - z_{\alpha_0}^{L}\right) + \alpha_0 & z_{\alpha_0}^{L} \leqq t < z_{\alpha_1}^{L} \\ \dfrac{\alpha_1 - \alpha_0}{z_{\alpha_2}^{L} - z_{\alpha_1}^{L}}\left(t - z_{\alpha_1}^{L}\right) + \alpha_1 & z_{\alpha_1}^{L} \leqq t < z_{\alpha_2}^{L} \\ \cdots & \cdots \\ \alpha & z_{\alpha_n}^{L} \leqq t < z_{\alpha_n}^{R} \\ \dfrac{\alpha_n - \alpha_{n-1}}{z_{\alpha_{n-1}}^{R} - z_{\alpha_n}^{R}}\left(-t + z_{\alpha_{n-1}}^{R}\right) + \alpha_{n-1} & z_{\alpha_n}^{R} \leqq t < z_{\alpha_{n-1}}^{R} \\ \cdots & \cdots \\ \dfrac{\alpha_0 - \alpha_1}{z_{\alpha_1}^{R} - z_{\alpha_0}^{R}}\left(-t + z_{\alpha_0}^{R}\right) + \alpha_0 & z_{\alpha_1}^{R} \leqq t \leqq z_{\alpha_0}^{R} \\ 0 & z_{\alpha_0}^{R} < t \end{cases}, \tag{2.5}$$

where $\tilde{z}$ denotes $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_i$ and $\tilde{d}_i$ respectively. Then, it is the solution of the problem (3.1) that $(x^*, y^*) \in R^n \times R^m$ satisfying

$$\min_{x\in X}(F(x,y))_{\alpha_0}^{L} = c_{1\alpha_0}^{L}x + d_{1\alpha_0}^{L}y,$$
$$\vdots$$
$$\min_{x\in X}(F(x,y))_{\alpha_n}^{L} = c_{1\alpha_n}^{L}x + d_{1\alpha_n}^{L}y,$$
$$\min_{x\in X}(F(x,y))_{\alpha_0}^{R} = c_{1\alpha_0}^{R}x + d_{1\alpha_0}^{R}y,$$
$$\vdots$$
$$\min_{x\in X}(F(x,y))_{\alpha_n}^{R} = c_{1\alpha_n}^{R}x + d_{1\alpha_n}^{R}y, \tag{2.6a}$$

subject to $A_{1\alpha_0}^{L}x + B_{1\alpha_0}^{L}y \leqq b_{1\alpha_0}^{L}$,

$$\vdots$$
$$A_{1\alpha_n}^{L}x + B_{1\alpha_n}^{L}y \leqq b_{1\alpha_n}^{L},$$
$$A_{1\alpha_0}^{R}x + B_{1\alpha_0}^{R}y \leqq b_{1\alpha_0}^{R}, \tag{2.6b}$$
$$\vdots$$
$$A_{1\alpha_n}^{R}x + B_{1\alpha_n}^{R}y \leqq b_{1\alpha_n}^{R},$$

$$\min_{y\in Y}(f(x,y))_{\alpha_0}^{L} = c_{2\alpha_0}^{L}x + d_{2\alpha_0}^{L}y,$$
$$\vdots$$
$$\min_{y\in Y}(f(x,y))_{\alpha_n}^{L} = c_{2\alpha_n}^{L}x + d_{2\alpha_n}^{L}y,$$
$$\min_{y\in Y}(f(x,y))_{\alpha_0}^{R} = c_{2\alpha_0}^{R}x + d_{2\alpha_0}^{R}y,$$
$$\vdots$$
$$\min_{y\in Y}(f(x,y))_{\alpha_n}^{R} = c_{2\alpha_n}^{R}x + d_{2\alpha_n}^{R}y, \tag{2.6c}$$

subject to

$$A_{2\alpha_0}^{L}x + B_{2\alpha_0}^{L}y \leqq b_{2\alpha_0}^{L},$$
$$\vdots$$
$$A_{2\alpha_n}^{L}x + B_{2\alpha_n}^{L}y \leqq b_{2\alpha_n}^{L},$$
$$A_{2\alpha_0}^{R}x + B_{2\alpha_0}^{R}y \leqq b_{2\alpha_0}^{R}, \tag{2.6d}$$
$$\vdots$$
$$A_{2\alpha_n}^{R}x + B_{2\alpha_n}^{R}y \leqq b_{2\alpha_n}^{R}.$$

We note

$$\overline{A}_1 x + \overline{B}_1 y \leqq \overline{b}_1 \qquad (2.6b')$$

$$\overline{A}_2 x + \overline{B}_2 y \leqq \overline{b}_2 \qquad (2.6d')$$

where

$$\overline{A}_1 = \begin{pmatrix} A_{1\alpha_0}^L \\ \vdots \\ A_{1\alpha_n}^L \\ A_{1\alpha_0}^R \\ \vdots \\ A_{1\alpha_n}^R \end{pmatrix}, \overline{A}_2 = \begin{pmatrix} A_{2\alpha_0}^L \\ \vdots \\ A_{2\alpha_n}^L \\ A_{2\alpha_0}^R \\ \vdots \\ A_{2\alpha_n}^R \end{pmatrix}, \overline{B}_1 = \begin{pmatrix} B_{1\alpha_0}^L \\ \vdots \\ B_{1\alpha_n}^L \\ B_{1\alpha_0}^R \\ \vdots \\ B_{1\alpha_n}^R \end{pmatrix}, \overline{B}_2 = \begin{pmatrix} B_{2\alpha_0}^L \\ \vdots \\ B_{2\alpha_n}^L \\ B_{2\alpha_0}^R \\ \vdots \\ B_{2\alpha_n}^R \end{pmatrix},$$

$$\overline{b}_1 = \begin{pmatrix} b_{1\alpha_0}^L \\ \vdots \\ b_{1\alpha_n}^L \\ b_{1\alpha_0}^R \\ \vdots \\ b_{1\alpha_n}^R \end{pmatrix}, \overline{b}_2 = \begin{pmatrix} b_{2\alpha_0}^L \\ \vdots \\ b_{2\alpha_n}^L \\ b_{2\alpha_0}^R \\ \vdots \\ b_{2\alpha_n}^R \end{pmatrix}.$$

Then we can re-write (2.6) by using

$$\min_{x\in X} (F(x,y))_{\alpha_0}^L = c_{1\alpha_0}^L x + d_{1\alpha_0}^L y,$$

$$\vdots \qquad (2.6a')$$

$$\min_{x\in X} (F(x,y))_{\alpha_n}^L = c_{1\alpha_n}^L x + d_{1\alpha_n}^L y,$$

$$\min_{x\in X} (F(x,y))_{\alpha_0}^R = c_{1\alpha_0}^R x + d_{1\alpha_0}^R y,$$

$$\vdots$$

$$\min_{x\in X} (F(x,y))_{\alpha_n}^R = c_{1\alpha_n}^R x + d_{1\alpha_n}^R y,$$

subject to $\overline{A}_1 x + \overline{B}_1 y \leqq \overline{b}_1$, (2.4b')

$$\min_{y\in Y} (f(x,y))_{\alpha_0}^L = c_{2\alpha_0}^L x + d_{2\alpha_0}^L y,$$

$$\vdots$$

$$\min_{y\in Y} (f(x,y))_{\alpha_n}^L = c_{2\alpha_n}^L x + d_{2\alpha_n}^L y,$$

$$\min_{y\in Y} (f(x,y))_{\alpha_0}^R = c_{2\alpha_0}^R x + d_{2\alpha_0}^R y,$$

$$\vdots$$

$$\min_{y\in Y} (f(x,y))_{\alpha_n}^R = c_{2\alpha_n}^R x + d_{2\alpha_n}^R y,$$

(2.6c')

subject to $\overline{A}_2 x + \overline{B}_2 y \leqq \overline{b}_2$.

(2.6d')

To solve the problem (2.6'), we can use the method of weighting [12] to this problem, such that it is the following problem:

$$\min_{x\in X}(F(x,y)) = \sum_{i=0}^{n}\left(\left(c_{1\alpha_i}^L x + d_{1\alpha_i}^L y\right) + \left(c_{1\alpha_i}^R x + d_{1\alpha_i}^R y\right)\right) \qquad (2.7a)$$

subject to $\overline{A}_1 x + \overline{B}_1 y \leqq \overline{b}_1$, (2.7b)

$$\min_{y\in Y}(f(x,y)) = \sum_{i=0}^{n}\left(\left(c_{2\alpha_i}^L x + d_{2\alpha_i}^L y\right) + \left(c_{2\alpha_i}^R x + d_{2\alpha_i}^R y\right)\right) \qquad (2.7c)$$

subject to $\overline{A}_2 x + \overline{B}_2 y \leqq \overline{b}_2$. (2.7d)

Definition 2.2

(a) Constraint region of the linear BLP problem:

$$S = \{(x,y) : x \in X, y \in Y, \overline{A}_1 x + \overline{B}_1 y \leqq \overline{b}_1, \overline{A}_2 x + \overline{B}_2 y \leqq \overline{b}_2\}$$

(b) Feasible set for the follower for each fixed $x \in X$:

$$S(x) = \{y \in Y : \overline{B}_2 y \leqq \overline{b}_2 - \overline{A}_2 x\}$$

(c) Projection of S onto the leader's decision space:

$$S(X) = \{x \in X : \exists y \in Y, \overline{A}_1 x + \overline{B}_1 y \leqq \overline{b}_1, \overline{A}_2 x + \overline{B}_2 y \leqq \overline{b}_2\}$$

Follower's rational reaction set for $x \in S(X)$:

$$P(x) = \{y \in Y : y \in \arg\min[(f(x,\hat{y})) : \hat{y} \in S(x)]\}$$

where

$$\arg\min[f(x,\hat{y}) : \hat{y} \in S(x)]$$
$$= \{y \in S(x) : (f(x,y)) \leqq (f(x,\hat{y})), \hat{y} \in S(x)\}$$

Inducible region:

$$IR = \{(x,y) : (x,y) \in S, y \in P(x)\}$$

The rational reaction set $P(x)$ defines the response while the inducible region IR represents the set over which the leader may optimize his objective. Thus in terms of the above notations, the linear BLP problem can be written as

$$\min\{F(x,y) : (x,y) \in IR\}. \qquad (2.8)$$

Theorem 2.4 The inducible region can be written equivalently as piecewise linear equality constraint comprised of supporting hyperplance of constraint region *S*.

Proof. Let us begin by writing the inducible region of Definition 3.1(e) explicitly as follower:

$$IR = \{(x,y) : (x,y) \in S, \overline{d}_2 y = \min[\overline{d}_2 \overline{y} : \overline{B}_i \overline{y} \leqq \overline{b}_i - \overline{A}_i x, i = 1,2, \overline{y} \geq 0]\} \qquad (2.9)$$

where
$\bar{c}_i = c_i + c_{i0}^L + c_{i0}^R, \bar{d}_i = d_i + d_{i0}^L + d_{i0}^R, i = 1,2.$
Now we define

$$Q(x) = \min\{\bar{d}_2 y : \bar{B}_i y \leq \bar{b}_i - \bar{A}_i x, i = 1,2, y \geq 0\} \quad (2.10)$$

Let us define

$$\bar{B} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix}.$$

We rewrite (2.10) as follows

$$Q(x) = \min\{\bar{d}_2 y : \bar{B} y \leq \bar{b} - \bar{A} x, y \geq 0\} \quad (2.11)$$

For each value of $x \in S(X)$, the resulting feasible region to problem (2.5) is nonempty and compact. Thus $Q(x)$, which is a linear program parameterized in x, always has a solution. From duality theory, we get

$$\max\{u(\bar{A}x - b) : u\bar{B} \geq -\bar{d}_2, u \geq 0\}, \quad (2.12)$$

which has the same optimal value as (3.7) at the solution u^*. Let $u^1, \ldots, u^s$ be a listing of all the vertices of the constraint region of (2.12) given by $U = \{u : u\bar{B} \geq -\bar{d}_2, u \geq 0\}$. Because we know that a solution to (2.12) occurs at a vertex of U, we get the equivalent problem

$$\max\{u^j(\bar{A}x - \bar{b}) : u^j \in \{u^1, \cdots, u^s\}\}, \quad (2.13)$$

which demonstrates that $Q(x)$ is a piecewise linear function. Rewriting IR as

$$IR = \{(x, y) \in S : Q(x) - \bar{d}_2 y = 0\}, \quad (2.14)$$

yields the desired result.

Corollary 2.1 The linear BLP problem (2.7) is equivalent to minimizing F over a feasible region comprised of a piecewise linear equality constraint.
Proof. From (2.8) and Theorem 2.3, we have the desired result.

Corollary 2.2 A solution for the linear BLP problem occurs at a vertex of IR.
Proof. A linear BLP programming can be written (2.8). Since F is linear, if a solution exists, one must occur at a vertex of IR. The proof is completed

Theorem 2.5 The solution (x^*, y^*) of the linear BLP problem occurs at a vertex of S.
Proof. Let $(x^1, y^1), \ldots, (x^r, y^r)$ be the distinct vertices of S. Since any point in S can be written a convex combination of these vertices, let $(x^*, y^*) = \sum_{i=1}^{r} \alpha_i (x^i, y^i)$, where $\sum_{i=1}^{r} \alpha_i = 1, \alpha_i \geq 0, i = 1, \ldots, \bar{r}$ and $\bar{r} \leq r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (2.5) at (x^*, y^*) in their piecewise linear form (2.6).

$$\begin{aligned} 0 &= Q(x^*) - \bar{d}_2 y^* \\ &= Q(\sum_i \alpha_i x^i) - \bar{d}_2(\sum_i \alpha_i y^i) \\ &\leq \sum_i \alpha_i Q(x^i) - \sum_i \alpha_i \bar{d}_2 y^i \end{aligned}$$

by convexity of $Q(x)$

$$= \sum_i \alpha_i (Q(x^i) - \bar{d}_2 y^i).$$

But by definition,

$$Q(x^i) = \min_{y \in S(x^i)} \bar{d}_2 y \leq \bar{d}_2 y^i.$$

Therefore, $Q(x^i) - \bar{d}_2 y^i \leq 0, i = 1, \ldots, \bar{r}$. Noting that $\alpha_i \geq 0, i = 1, \ldots, \bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q(x^i) - \bar{d}_2 y^i = 0$ for all i. This implies that $(x^i, y^i) \in IR$, $i = 1, \ldots, \bar{r}$ and (x^*, y^*) can be written as a convex combination of points in IR. Because (x^*, y^*) is a vertex of IR, a contradiction results unless $\bar{r} = 1$.

Corollary 2.3 If x is an extreme point of IR, it is an extreme point of S.
Proof: Let (x^*, y^*) be an extreme point of IR and assume that it is not an extreme point of S. Let $(x^1, y^1), \ldots, (x^r, y^r)$ be the distinct vertices of S. Since any point in S can be written a convex combination of these vertices, let $(x^*, y^*) = \sum_{i=1}^{r} \alpha_i (x^i, y^i)$, where $\sum_{i=1}^{r} \alpha_i = 1, \alpha_i \geq 0, i = 1, \ldots, \bar{r}$ and $\bar{r} \leq r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (2.5) at (x^*, y^*) in their piecewise linear form (2.6).

$$\begin{aligned} 0 &= Q(x^*) - \bar{d}_2 y^* \\ &= Q(\sum_i \alpha_i x^i) - \bar{d}_2(\sum_i \alpha_i y^i) \\ &\leq \sum_i \alpha_i Q(x^i) - \sum_i \alpha_i \bar{d}_2 y^i \end{aligned}$$

by convexity of $Q(x)$

$$= \sum_i \alpha_i (Q(x^i) - \bar{d}_2 y^i).$$

But by definition,

$$Q(x^i) = \min_{y \in S(x^i)} \bar{d}_2 y \le \bar{d}_2 y^i.$$

Therefore, $Q(x^i) - \bar{d}_2 y^i \le 0, i = 1, \ldots, \bar{r}$. Noting that $\alpha_i \ge 0, i = 1, \ldots, \bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q(x^i) - \bar{d}_2 y^i = 0$ for all i. This implies that $(x^i, y^i) \in IR$, $i = 1, \ldots, \bar{r}$ and (x^*, y^*) can be written as a convex combination of points in IR. Because (x^*, y^*) is an extreme point of IR, a contradiction results unless $\bar{r} = 1$. This means that (x^*, y^*) is an extreme point of S. The proof is completed.

Theorem 2.3 and Corollary 2.3 have provided theoretical foundation for our new algorithm. It means that by searching extreme points on the constraint region S, we can efficiently find an optimal solution for a linear BLP problem. The basic idea of our extended properties algorithm is that according to the objective function of the upper level, we descendent order all the extreme points on S, and select the first extreme point to check if it is on the inducible region IR. If yes, the current extreme point is the optimal solution. If not, select the next one and check.

More specifically, let $(x_{[1]}, y_{[1]}), \ldots, (x_{[N]}, y_{[N]})$ denote the N ordered extreme points to the linear programming problem

$$\min\{\bar{c}_1 x + \bar{d}_1 y : (x, y) \in S\}, \quad (2.15)$$

such that

$$\bar{c}_1 x_{[i]} + \bar{d}_1 y_{[i]} \le \bar{c}_1 x_{[i+1]} + \bar{d}_1 y_{[i+1]}, i = 1, \ldots, N-1.$$

Let $\tilde{y}$ denote the optimal solution to the following problem

$$\min(f(x_{[i]}, y) : y \in S(x_{[i]})). \quad (2.16)$$

We only need to find the smallest i ($i \in \{1, \ldots, N\}$) under which $y_{[i]} = \tilde{y}$.

Let write (2.16) as follows

$$\min f(x, y)$$
$$\text{subject to } y \in S(x)$$
$$x = x_{[i]}.$$

From Definition 2.1(a) and (c), we have

$$\min f(x, y) = \bar{c}_2 x + \bar{d}_2 y \quad (2.17a)$$
$$\text{subject to } \bar{A}_1 x + \bar{B}_1 y \le \bar{b}_1 \quad (2.17b)$$
$$\bar{A}_2 x + \bar{B}_2 y \le \bar{b}_2 \quad (2.17c)$$
$$x = x_{[i]} \quad (2.17d)$$
$$y \ge 0. \quad (2.17e)$$

The solving is equivalent to select one ordered extreme point $(x_{[i]}, y_{[i]})$, then solve (2.17) to obtain the optimal solution $\tilde{y}$. If $\tilde{y} = y_{[i]}$, $(x_{[i]}, y_{[i]})$ is the global optimum to (2.7). Otherwise, check the next extreme point.

3. An Approximation *K*-best Algorithm for Solving Fuzzy Linear Bilevel Programming

Based on Theorem 2.5, we present an approximation *K*-best approach for solving fuzzy linear bilevel programming (FLBLP) problem (2.1) as follows:

Step 1 The problem (2.1) is transferred to the problem (2.6)

Step 2 Let the interval [*0*, 1] be decomposed into 2^{l-1} mean sub-intervals with $(2^{l-1}+1)$ nodes $\lambda_j \left(j = 0, \cdots, 2^{l-1}\right)$ which are arranged in the order of $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{2^{l-1}} = 1$ and a range of errors $\varepsilon > 0$.

Step 3 Set $l = 1$, then solve $(\text{MLBLP})_{2^l}$, i.e. (2.6) by using Branch approach when $\beta = 0$ and $\alpha = 1$, we obtain optimization solution $(x, y)_{2^l}$.

Step 4 Put $i \leftarrow 1$. Solve (2.15) with the simplex method to obtain the optimal solution $(x_{[1]}, y_{[1]})$. Let $W = \{(x_{[1]}, y_{[1]})\}$ and $T = \phi$. Go to Step 5.

Step 5 Solve (2.17) with the bounded simplex method. Let $\tilde{y}$ denote the optimal solution to (2.17). If $\tilde{y} = y_{[i]}$, stop; $(x_{[i]}, y_{[i]})$ is the global optimum to (2.5) with $K^* = i$. Otherwise, go to Step 6.

Step 6 Let $W_{[i]}$ denote the set of adjacent extreme points of $(x_{[i]}, y_{[i]})$ such that $(x, y) \in W_{[i]}$ implies $\bar{c}_1 x + \bar{d}_1 y \ge \bar{c}_1 x_{[i]} + \bar{d}_1 y_{[i]}$. Let $T = T \cup \{(x_{[i]}, y_{[i]})\}$ and $W = (W \cup W_{[i]}) \setminus T$. Go to Step 7.

Step 7 Set $i \leftarrow i+1$ and choose $(x_{[i]}, y_{[i]})$ so that

$$fx_{[i]} + gy_{[i]} = \min\{\overline{c}_1 x + \overline{d}_1 y : (x, y) \in W\}$$

. Go to Step 5.

Step 8 Solve $(\text{MLBLP})_{2^{l+1}}$ by Step 4 to Step 7 and we obtain optimization solution $(x, y)_{2^{l+1}}$.

Step 9 If $\|(x, y)_{2^{l+1}} - (x, y)_{2^l}\| < \varepsilon$, then the solution (x^*, y^*) of the fuzzy linear bilevel problem is $(x, y)_{2^{l+1}}$ Otherwise, update l to $2l$ and go back to Step 8.

Step 10 Show the result.

4. An Illustrative Example

This section will give an example to illustrate the proposed algorithm.

Example Consider the following fuzzy linear BLP problem with $x \in R^1$, $y \in R^1$, and $X = \{x \geq 0\}$, $Y = \{y \geq 0\}$,

$$\min_{x \in X} F(x, y) = \tilde{1}x - \tilde{2}y$$
$$\text{subject to } -\tilde{1}x + \tilde{3}y \leq \tilde{4}$$
$$\min_{y \in Y} f_1(x, y) = \tilde{1}x + \tilde{1}y$$
$$\text{subject to } \tilde{1}x - \tilde{1}y \leq \tilde{0}$$
$$-\tilde{1}x - \tilde{1}y \leq \tilde{0}$$

where

$$\mu_{\tilde{1}}(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \leqq t < 1 \\ 2 - t & 1 \leqq t < 2 \\ 0 & 2 \leqq t \end{cases},$$

$$\mu_{\tilde{2}}(t) = \begin{cases} 0 & t < 1 \\ t - 1 & 1 \leqq t < 2 \\ 3 - t & 2 \leqq t < 3 \\ 0 & 3 \leqq t \end{cases},$$

$$\mu_{\tilde{3}}(t) = \begin{cases} 0 & t < 2 \\ t - 2 & 2 \leqq t < 3 \\ 4 - t & 3 \leqq t < 4 \\ 0 & 4 \leqq t \end{cases},$$

$$\mu_{\tilde{4}}(t) = \begin{cases} 0 & t < 3 \\ t - 3 & 3 \leqq t < 4 \\ 5 - t & 4 \leqq t < 5 \\ 0 & 5 \leqq t \end{cases},$$

$$\mu_{\tilde{0}}(t) = \begin{cases} 0 & t < -1 \\ t + 1 & -1 \leqq t < 0 \\ 1 - t^2 & 0 \leqq t < 1 \\ 0 & 1 \leqq t \end{cases}.$$

To solve this fuzzy linear bilevel programming problem by using above the approximation Kuhn-Tucker approach.

Step 1 The FBLP problem is transferred to the following LMMBLP problem by using Theorem 2.2

$$\min_{x \in X} (F(x, y))_\lambda^L = \tilde{1}_\lambda^L x + (-\tilde{2})_\lambda^L y, \quad \lambda \in [0, 1]$$

$$\min_{x \in X} (F(x, y))_\lambda^R = \tilde{1}_\lambda^R x + (-\tilde{2})_\lambda^R y, \quad \lambda \in [0, 1]$$

subject to

$$(-\tilde{1})_\lambda^L x + \tilde{3}_\lambda^L y \leqq \tilde{4}_\lambda^L, (-\tilde{1})_\lambda^R x + \tilde{3}_\lambda^R y \leqq \tilde{4}_\lambda^R, \quad \lambda \in [0, 1]$$

$$\min_{y \in Y} (f(x, y))_\lambda^L = \tilde{1}_\lambda^L x + \tilde{1}_\lambda^L y, \quad \lambda \in [0, 1]$$

$$\min_{y \in Y} (f(x, y))_\lambda^R = \tilde{1}_\lambda^R x + \tilde{1}_\lambda^R y, \quad \lambda \in [0, 1]$$

subject to

$$\tilde{1}_\lambda^L x + (-\tilde{1})_\lambda^L y \leqq \tilde{0}_\lambda^L, \tilde{1}_\lambda^R x + (-\tilde{1})_\lambda^R y \leqq \tilde{0}_\lambda^R, \quad \lambda \in [0, 1]$$

$$(-\tilde{1})_\lambda^L x + (-\tilde{1})_\lambda^L y \leqq \tilde{0}_\lambda^L, (-\tilde{1})_\lambda^R x + (-\tilde{1})_\lambda^R y \leqq \tilde{0}_\lambda^R, \quad \lambda \in [0, 1]$$

Step 2. Let the interval [*0*, 1] be decomposed into 2^{l-1} mean sub-intervals with $(2^{l-1}+1)$ nodes $\lambda_i \left(i = 0, \cdots, 2^{l-1}\right)$ which are arranged in the order of $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{2^{l-1}} = 1$ and a range of errors $\varepsilon = 10^{-6} > 0$.

Step 3-step 7. When $l = 1$, we solve this following multi-objective multi-follower programming by use original *K*-best approach

$$\min_{x \in X} F(x, y) = 3x - 6y$$
$$\text{subject to } -1x + 3y \leq 4$$
$$-2x + 2y \leq 3$$
$$0x + 4y \leq 5$$
$$\min_{y \in Y} f_1(x, y) = 3x + 3y$$

subject to $1x - 1y \le 0$

$$0x - 2y \le -1$$
$$2x - 0y \le 1$$
$$-1x - 1y \le 0$$
$$-2x - 2y \le -1.$$

We found that the optimal solution occurs at the point $(x^*, y^*) = (0, 0.5)$ with

$$\min_{x \in X} F_1(x,y) = 1x - 2y = -1$$
$$\min_{x \in X} F_2(x,y) = 0x - 3y = -1.3$$
$$\min_{x \in X} F_3(x,y) = 2x - 1y = -0.5$$
$$\min_{y \in Y} f_1(x,y) = 0.5$$
$$\min_{y \in Y} f_2(x,y) = 1$$

Step 8. When $l = 2$, we solve another linear multi-objective multi-follower programming problem and get the following result.

$$\min_{x \in X} F(x,y)_1^{L(R)} = 1x - 2y = -1$$
$$\min_{x \in X} F(x,y)_{\frac{1}{2}}^{L} = \frac{\sqrt{2}}{2}x - \frac{3}{2}y = -\frac{3}{4}$$
$$\min_{x \in X} F(x,y)_0^{L} = 0x - 3y = -\frac{3}{2}$$
$$\min_{x \in X} F(x,y)_{\frac{1}{2}}^{R} = \frac{3}{2}x - \frac{5}{2}y = -\frac{5}{4}$$
$$\min_{x \in X} F(x,y)_0^{R} = 2x - 1y = -\frac{1}{2}$$
$$\min_{y \in Y} f_1(x,y)_1^{L(R)} = 1x + 1y = \frac{1}{2}$$
$$\min_{y \in Y} f_1(x,y)_{\frac{1}{2}}^{L} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = \frac{\sqrt{2}}{4}$$
$$\min_{y \in Y} f_1(x,y)_1^{R} = 2x + 2y = 1$$
$$\min_{y \in Y} f_1(x,y)_{\frac{1}{2}}^{R} = \frac{3}{2}x + \frac{3}{2}y = \frac{3}{4}$$
$$x = 0, y = 0.5$$

Step 9. $x = 0, y = 0.5$ is the optimal solution the example because $\|(x,y)_{2^2} - (x,y)_{2^1}\| = 0 < \varepsilon$.

Step 10. Show the solution of the problem is $x = 0, y = 0.5$ such that

$$\min_{x \in X} F(x,y) = \tilde{1}x - \tilde{2}y = -\frac{\tilde{2}}{2}$$
$$\min_{y \in Y} f_1(x,y) = \tilde{1}x + \tilde{1}y = \frac{\tilde{1}}{2}.$$

where

$$\mu_{-\frac{\tilde{2}}{2}}(t) = \begin{cases} 0 & t < -1.5 \\ \frac{t+1.5}{0.5} & -1.5 \le t < -1 \\ \frac{-0.5-t}{0.5} & -1 \le t < -0.5 \\ 0 & -0.5 \le t \end{cases},$$

$$\mu_{\frac{\tilde{1}}{2}}(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{0.5} & 0 \le t < 0.5 \\ \frac{1-t}{0.5} & 0.5 \le t < 1 \\ 0 & 1 \le t \end{cases}.$$

5. Conclusion and Further Study

Following our previous research [21, 22, 30], this paper proposes a fuzzy number based extended *K*th best approach to solve proposed FPBLP problem. A numeral example is shown to illustrate the proposed fuzzy number based extended *K*th best approach. Further study will include the development of fuzzy parameter based multi-follower and multi-objective bilevel programming problems.

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