A Highly Accurate Algorithm for the Estimation of the Frequency of a Complex Exponential in Additive Gaussian Noise

Sam Reisenfeld

Abstract—**A new algorithm for the precise estimation of the frequency of a complex exponential signal in additive, complex, white Gaussian noise is presented. The algorithm has low computational complexity and is well suited for numerous real time applications. The DFT based algorithm performs an initial coarse frequency estimation using the peak search of an N point complex Fast Fourier Transform. The algorithm forms a frequency estimate using a functional mapping from two modified DFT coefficients which are one half DFT frequency bin below and above largest magnitude FFT coefficient. Recursion is used to provide frequencies of the modified DFT coefficients which minimize the variance of the frequency estimation error. For large N and large signal to noise ratio, the frequency estimation error variance obtained is 0.063 dB above the Cramer-Rao Bound. This excellent performance is achieved with low computational complexity. The algorithm provides exact frequency determination in the noiseless case.**

Index Terms—**Fast Fourier Transform, Discrete Fourier Transform, frequency estimation, frequency tracking, exact frequency determination.**

I. Introduction

There has been substantial prior work in the use of the Fast Fourier Transform (FFT) to estimate the frequency of a time sampled complex exponential signal in additive white Gaussian noise [1–4]. A recursive technique for frequency estimation which uses two DFT coefficients was described in [9]. This paper introduces a new FFT based method to obtain a very accurate frequency estimate [5-6]. In particular, the coarse estimate is the frequency corresponding to the maximum amplitude FFT coefficient. Two modified DFT coefficients are then defined which are functionally related to the index of the maximum FFT coefficient. The frequency estimate is a function of the two modified DFT coefficients. The functional form is provided in this paper. The analysis of the frequency estimation performance of this algorithm and the comparison to the Cramer-Rao Bound are also included. Simulation results for the performance of the algorithm are also provided. The algorithm provides exact frequency determination in the noiseless case.

II. Coarse Frequency Estimation

The received signal plus noise, $r[n]$, is given by

$$r[n] = s[n] + \eta[n], \quad \text{for } n = 0, 1, 2, \ldots, N-1, \qquad (1)$$

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where $s[n]$ is a complex exponential signal with frequency f given by $s[n] = Ae^{j2\pi fnT_s}$ and $\eta[n]$ is a sequence of independent, identically distributed complex Gaussian random variables with zero mean and variance σ^2. It is desired to process $r[n]$, $n = 0, 1, \ldots, N-1$, to obtain an estimate of f, where f is a fixed and unknown parameter, $f \in [0, f_s)$. The sampling frequency is f_s, and $T_s = \frac{1}{f_s}$. The signal to noise ratio (SNR) is defined as

$$\mathrm{SNR} = \frac{A^2}{\sigma^2}. \qquad (2)$$

Rife and Boorstyn [1] described a technique for frequency estimation using the Fast Fourier Transform (FFT). The frequency corresponding to the maximum amplitude FFT coefficient is chosen as a frequency quantized approximation to the maximum likelihood estimate. Define

$$\mathbf{r} = \begin{bmatrix} r[0] \\ r[1] \\ \cdot \\ \cdot \\ \cdot \\ r[N-1] \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} Y[0] \\ Y[1] \\ \cdot \\ \cdot \\ \cdot \\ Y[N-1] \end{bmatrix}$$

where $\mathbf{Y} = \mathrm{FFT}(\mathbf{r})$ and $\mathrm{FFT}(\cdot)$ is the N point complex FFT operator. Following Rife and Boorstyn [1], a coarse frequency estimate, $\hat{f}_0$, may be obtained from

$$k_{\max} = \max{}^{-1}\left[|Y[k]| : 0 \le k \le N-1\right]$$

and

$$\hat{f}_0 = \frac{k_{\max}}{N} f_s. \qquad (3)$$

Assuming that the SNR is sufficiently high, it is highly probable that $f \in [\hat{f}_0 - \frac{f_s}{2N}, \hat{f}_0 + \frac{f_s}{2N}]$. This is an above threshold condition [1]. A fine interpolation may be obtained to improve the frequency accuracy.

III. Modified DFT Coefficients

Define the modified discrete Fourier transform (DFT) coefficients α and β as

$$\alpha = Y\left(k_{\max} - \frac{1}{2}\right) = \sum_{n=0}^{N-1} r[n] e^{-j2\pi n \frac{k_{\max} - \frac{1}{2}}{N}} \qquad (4)$$

and

$$\beta = Y\left(k_{\max} + \frac{1}{2}\right) = \sum_{n=0}^{N-1} r[n] e^{-j2\pi n \frac{k_{\max} + \frac{1}{2}}{N}} \qquad (5)$$

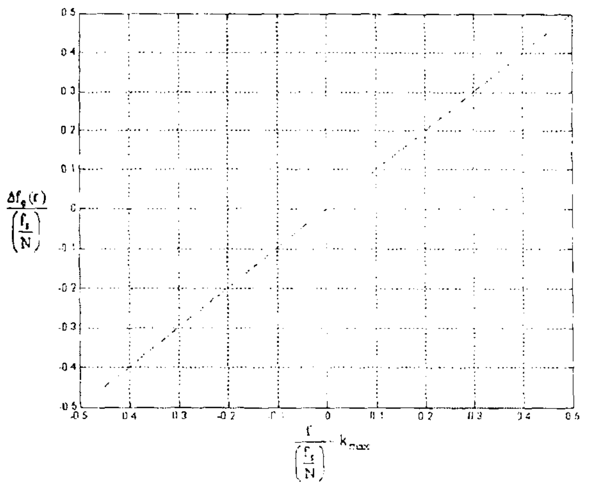

Fig. 1: Normalized frequency discriminant for $\gamma = 1$ and $N = 16$ as a function of normalized frequency.

A number of functional forms, which map α and β to an estimate of the error in $\hat{f}_0$, are possible. Functional forms contained in a particular class [5] may be characterized by

$$\Delta f_0(\mathbf{r}) = \frac{1}{2N\gamma}\frac{|\beta|^\gamma - |\alpha|^\gamma}{|\beta|^\gamma + |\alpha|^\gamma} f_s, \quad \text{for } \gamma > 0. \tag{6}$$

The parameter γ defines the shape of the frequency discriminant function as a function of $\frac{f - \frac{k_{\max}}{N} f_s}{\frac{f_s}{N}}$.

The frequency discriminant function shown in (6) with $\gamma = 1$ may be used for frequency estimation. This function is

$$\Delta f_0(\mathbf{r}) = \frac{1}{2N}\frac{|\beta| - |\alpha|}{|\beta| + |\alpha|} f_s \tag{7}$$

Fig. 1 shows $\frac{\Delta f_0(\mathbf{r})}{(\frac{f_s}{N})}$ as a function of $\frac{f}{(\frac{f_s}{N})} - k_{\max}$ for $\gamma = 1$ and $N = 16$.

From Fig. 1, it is obvious that the frequency discriminant function described in (7) provides an almost linear discriminant which may be used to obtain the estimate of the frequency of the complex exponential signal [5-6].

This paper introduces an improved frequency discriminant function, denoted $\Delta F_m(\mathbf{r})$ [5-6], relative to the almost linear discriminant given in (7).

IV. Recursive Algorithm

The algorithm is defined as

$$\hat{f}_0 = \frac{k_{\max}}{N} f_s \tag{8}$$

For $m = 0, 1, 2, \ldots$, define

$$\alpha_m = \sum_{n=0}^{N-1} r[n] e^{-j2\pi n\left(\frac{\hat{f}_m}{f_s} - \frac{1}{2N}\right)} \tag{9}$$

$$\beta_m = \sum_{n=0}^{N-1} r[n] e^{-j2\pi n\left(\frac{\hat{f}_m}{f_s} + \frac{1}{2N}\right)} \tag{10}$$

$$D_m = \frac{|\beta_m| - |\alpha_m|}{|\beta_m| + |\alpha_m|} \tag{11}$$

$$\hat{f}_{m+1} = \hat{f}_m + \frac{1}{\pi}\tan^{-1}[D_m \tan(\frac{\pi}{2N})]\, f_s \tag{12}$$

Recursion is done in which the index m is incremented. The estimate of the frequency is $\hat{f}_\infty$. The new frequency discriminant, $\Delta F_m(\mathbf{r})$ is shown in (12), where,

$$\Delta F_m(\mathbf{r}) = \frac{1}{\pi}\tan^{-1}[D_m \tan(\frac{\pi}{2N})]\, f_s \tag{13}$$

Then putting (13) into (12) yields,

$$\hat{f}_{m+1} = \hat{f}_m + \Delta F_m(\mathbf{r}) \tag{14}$$

The frequency discriminant $\Delta F_m(\mathbf{r})$ is used in the recursion.

V. Truncation of the Recursion Algorithm to a Finite Number of Iterations

$\hat{f}_\infty$ is the explicit solution of $D_\infty = 0$, $\hat{f}_m$ is the mth recursive estimate of frequency, $m = 1, 2, 3, \ldots$. In practice, $\hat{f}_2$ is sufficiently close to $\hat{f}_\infty$ to be used as the frequency estimate. The recursion may be truncated to the evaluation of $\hat{f}_2$. Therefore, the algorithm is rapidly converging and the frequency estimate may be obtained with only the computation of the FFT and four additional modified DFT coefficients.

VI. The Mapping from the Error Discriminant to the Frequency in the Noiseless Case

In the noiseless case, the exact frequency may be obtained from the identification of the FFT coefficient with the maximum absolute value and then forming α and β as shown in (4) and (5), respectively. D may be computed from

$$D = \frac{|\beta| - |\alpha|}{|\beta| + |\alpha|} \tag{15}$$

It is proven in Appendix A that for the noiseless case,

$$\begin{aligned} f &= \hat{f}_0 + \frac{1}{\pi}\tan^{-1}[D \tan(\frac{\pi}{2N})]\, f_s \\ &= \{\frac{k_{\max}}{N} + \frac{1}{\pi}\tan^{-1}[D \tan(\frac{\pi}{2N})]\} f_s \end{aligned} \tag{16}$$

Therefore, (16) is an exact functional mapping from the FFT output and two modified DFT coefficients to the frequency of a complex exponential signal in the noiseless case.

VII. Reduction of the RMS Error using Recursion

The rms frequency estimation error due to additive noise, using the proposed frequency discriminant, decreases sharply for $\frac{(f-\hat{f})}{(\frac{f_s}{N})} \approx 0$. Since the iterative solution adaptively produces a frequency estimate which is close to this condition, the recursive use of the discriminant produces a frequency estimate with small rms error. By means of the recursion, the discriminant converges to a minimum rms error condition.

Fig. 2, which was obtained by simulation, shows the normalized rms frequency error, $\sqrt{E\{|(\hat{f}_1 - f)T_s|^2\}}$, as a function of the normalized frequency $\frac{f}{(\frac{f_s}{N})} - k_{\max}$.

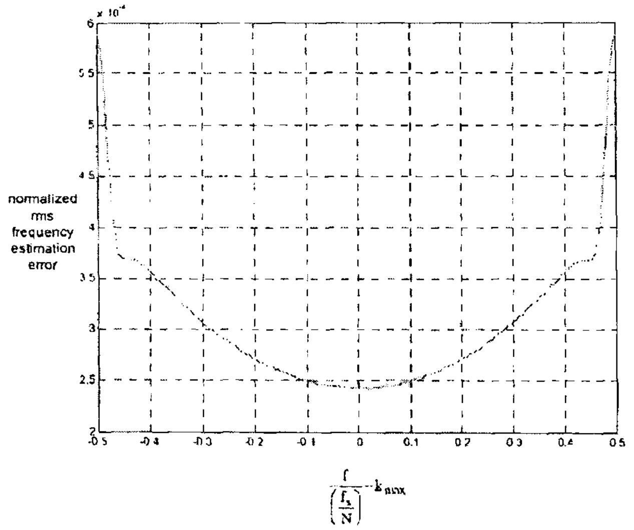

Fig. 2: Normalized rms frequency estimation error as a function of the normalized signal frequency relative to the FFT bin center frequency for SNR = 10dB, $N = 64$.

As can be seen from Fig. 2, the rms frequency error in the noisy case sharply decreases as the signal frequency approaches the mid-point between the frequencies of the two modified DFT coefficients. This justifies the use of the recursion to obtain an estimate that is close to this minimum rms frequency error condition. For the recursion described in (8), (9), (10), (11) and (12), the final estimate approaches the minimum rms error condition because the modified DFT coefficient frequencies approach one half DFT frequency bin width above and below the actual frequency. The initial estimate is

$$\hat{f}_0 = \frac{k_{max}}{N} f_s$$

which is obtained from the FFT peak search. The second estimate is

$$\hat{f}_1 = \hat{f}_0 + \frac{1}{\pi} \tan^{-1}[D_0 \tan(\frac{\pi}{2N})] f_s$$

where, $D_0 = \frac{|\beta_0| - |\alpha_0|}{|\beta_0| + |\alpha_0|}$, will be in a region of low rms estimation error. The third estimate is

$$\hat{f}_2 = \hat{f}_1 + \frac{1}{\pi} \tan^{-1}[D_1 \tan(\frac{\pi}{2N})] f_s$$

where, $D_1 = \frac{|\beta_1| - |\alpha_1|}{|\beta_1| + |\alpha_1|}$, is obtained from $\hat{f}_1$. The third estimate, $\hat{f}_2$, is characterized by extremely low estimation error. The recursion essentially converges for $\hat{f}_2$.

VIII. PERFORMANCE OF THE ALGORITHM

As shown in Appendix B, using the Taylor Series [7] at high signal to noise ratios, the performance of the algorithm is

$$\sigma_f^2 = \frac{N \sin^2(\frac{\pi}{2N}) \tan^2(\frac{\pi}{2N})}{4\ SNR\ \pi^2} \quad (17)$$

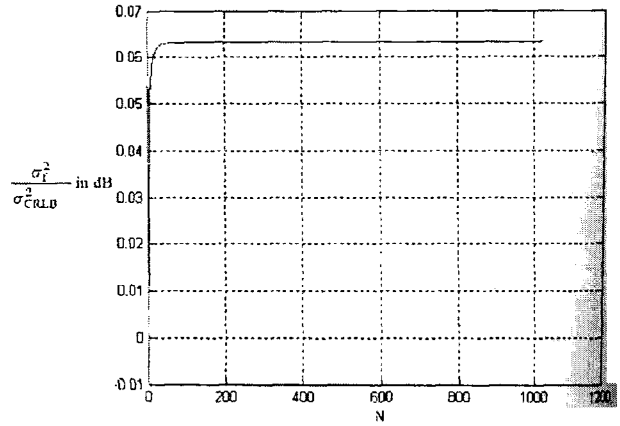

Fig. 3: $\frac{\sigma_f^2}{\sigma_{CRLB}^2}$ in dB as a function of N.

where, σ_f^2 is the mean square value of $(\hat{f}_\infty - f)T_s$ and where f is a uniformly distributed random variable in the interval $[0, f_s)$.

The Cramer-Rao lower bound on the unbiased frequency estimation mean square error [1] is

$$\sigma_{CRLB}^2 = \frac{6}{(2\pi)^2 N(N^2-1)\ \mathrm{SNR}} \quad (18)$$

Therefore, the performance of the frequency estimation algorithm compared to the Cramer-Rao Lower Bound is

$$\frac{\sigma_f^2}{\sigma_{CRLB}^2} = \frac{N^2(N^2-1)\ \sin^2(\frac{\pi}{2N})\ \tan^2(\frac{\pi}{2N})}{6} \quad (19)$$

For large SNR and large N,

$$\lim_{N \to \infty} 10 \log_{10}(\frac{\sigma_f^2}{\sigma_{CRLB}^2}) = 10 \log_{10}(\frac{\pi^4}{96}) = 0.0633 \text{ dB} \quad (2[illegible])$$

This limiting value was also obtained by a different meth[illegible] in [8].

Fig. 3 shows a plot of (19) in which the convergence to [illegible] asymptotic limit with increasing N is apparent. For very sm[illegible] values of N, the degradation of the estimation error rela[illegible] to the Cramer-Rao Bound is less than 0.0633 dB. For sm[illegible] N, less information is discarded by not using all the [illegible] coefficients, and the performance degradation is less than [illegible] large N.

IX. SIMULATION OF THE PERFORMANCE

Figs. 4 and 5 show the rms estimation error, σ_f, as a fu[illegible] of the SNR in dB for the cases of a 64 point and a [illegible] point FFT, respectively. The simulation results agree w[illegible] analysis for the asymptotic value of the degradation fr[illegible] Cramer-Rao Lower Bound of 0.0633 dB.

X. CONCLUSION

An algorithm for frequency estimation has been intro[illegible] For the noiseless case, the algorithm provides an ana[illegible] relationship from the FFT output and two modified DF[illegible] ficients to the exact frequency of a complex exponential [illegible]

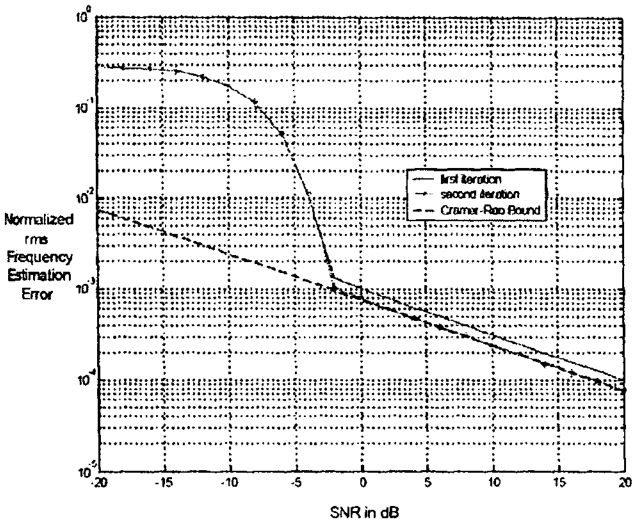

Fig. 4: The rms frequency estimation error as a function of the SNR in dB. Results are shown for one and two iterations. The results are compared to the Cramer-Rao Lower Bound. The FFT length is $N = 1024$.

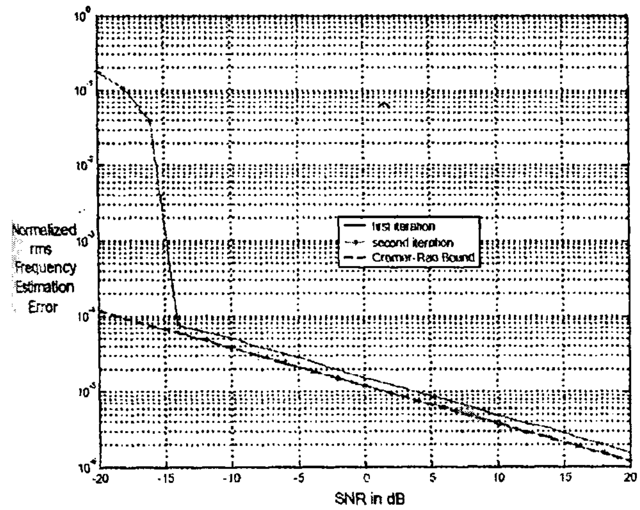

Fig. 5: The rms frequency estimation error as a function of the SNR in dB. Results are shown for one and two iterations. The results are compared to the Cramer-Rao Lower Bound. The FFT length is $N = 64$.

For the case of signal plus Gaussian noise, one iteration of the recursion provides sufficient frequency estimation accuracy for many applications. For applications requiring performance close to the theoretical limit, two iterations of the recursion provide frequency estimation performance which is only 0.063 dB above the Cramer-Rao Lower Bound on frequency estimation error. The algorithm has low computational complexity and is suitable for real time digital signal processing applications including communications, radar, sonar, and spectral analysis.

APPENDIX

EXACT FREQUENCY DETERMINATION FOR THE NOISELESS CASE

Theorem 1: In the noiseless case, $\varepsilon = (f - \hat{f}_m)T_s = \frac{1}{\pi}\tan^{-1}[D_m \tan(\frac{\pi}{2N})]$, and in particular, $f = \{\frac{k_{\max}}{N} + \frac{1}{\pi} \tan^{-1}[D_0 \tan(\frac{\pi}{2N})]\} f_s$ where,

$$D_0 = \frac{|\beta_0| - |\alpha_0|}{|\beta_0| + |\alpha_0|}, \tag{21}$$

$$\alpha_0 = \sum_{n=0}^{N-1} r[n] e^{-j2\pi n(\frac{f_0}{f_s} - \frac{1}{2N})}, \tag{22}$$

and

$$\beta_0 = \sum_{n=0}^{N-1} r[n] e^{-j2\pi n(\frac{f_0}{f_s} + \frac{1}{2N})} \tag{23}$$

Proof: For the noiseless case,

$$|\beta_m| = \frac{A|\cos(\pi N\epsilon)|}{|\sin[\pi(\epsilon - \frac{1}{2N})]|}$$

and

$$|\alpha_m| = \frac{A|\cos(\pi N\epsilon)|}{|\sin[\pi(\epsilon + \frac{1}{2N})]|}$$

$$D_m = \frac{|\beta_m| - |\alpha_m|}{|\beta_m| + |\alpha_m|} = \frac{\tan(\pi\epsilon)}{\tan(\frac{\pi}{2N})}$$

Then,

$$\epsilon = \frac{1}{\pi}\tan^{-1}[D_m \tan(\frac{\pi}{2N})]$$

and, for the first iteration in the recursion,

$$f = \hat{f}_0 + \frac{1}{\pi}\tan^{-1}[D_0 \tan(\frac{\pi}{2N})] f_s = \{\frac{k_{\max}}{N} + \frac{1}{\pi}\tan^{-1}[D_0 \tan(\frac{\pi}{2N})]\} f_s$$

VARIANCE OF THE NORMALIZED FREQUENCY ESTIMATION ERROR

Theorem 2: For sufficiently high signal to noise ratio, such that the probability of error in the FFT peak search is negligible, and for sufficiently large m such that $\hat{f}_\infty \cong \hat{f}_m$,

$$\sigma_f^2 = Var[(f - \hat{f}_m)T_s] = \frac{N \sin^2(\frac{\pi}{2N}) \tan^2(\frac{\pi}{2N})}{4 \text{ SNR } \pi^2}$$

Proof: For high SNR and $\hat{f}_m \cong f$,

$$\mu_\alpha = E[|\alpha_m|] \cong \frac{A}{\sin(\frac{\pi}{2N})}$$

and

$$\mu_\beta = E[|\beta_m|] \cong \frac{A}{\sin(\frac{\pi}{2N})}$$

Also, $|\alpha_m|$ and $|\beta_m|$ are uncorrelated random variables.

Furthermore,

$$\sigma_\alpha^2 = Var[|\alpha_m|] = \frac{1}{2}N\sigma^2$$

and

$$\sigma_\beta^2 = Var[|\beta_m|] = \frac{1}{2}N\sigma^2$$

Using the standard technique of finding the variance at the output of a nonlinear function of two variables by expanding the nonlinearity in a two dimensional Taylor Series Expansion [7],

$$Var[D_m] = \frac{4(\mu_\beta^2\sigma_\alpha^2 + \mu_\alpha^2\sigma_\beta^2)}{(\mu_\alpha + \mu_\beta)^4} = \frac{N\ \sin^2(\frac{\pi}{2N})}{4\ \text{SNR}}$$

Then,

$$\epsilon = (f - \hat{f}_m)T_s = \frac{1}{\pi}\tan^{-1}\left[D_m \tan(\frac{\pi}{2N})\right],$$

$E[D_m] = 0$ and therefore, $E[\epsilon] = 0$.

For small D_m, which is the case for $f \cong \hat{f}_m$,

$$\epsilon \cong \frac{1}{\pi}D_m \tan(\frac{\pi}{2N})$$

Therefore,

$$Var[\epsilon] \cong \frac{1}{\pi^2}Var[D_m]\ \tan^2(\frac{\pi}{2N}) = \frac{N\sin^2(\frac{\pi}{2N})\tan^2(\frac{\pi}{2N})}{4\ \text{SNR}\ \pi^2}.$$

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