APPLICATIONS OF MLSDQ METHOD FOR ANALYSIS OF ELECTROMAGNETIC FIELDS

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Abstract

This paper presents a novel application of the moving least square differential quadrature (MLSDQ) method to the solution of electromagnetic field problems. The MLSDQ is a new method proposed very recently for solution of nonlinear partial differential equations and has been successfully applied to the study of the bending behaviour of plates. In this paper, the applicability and accuracy of the MLSDQ method to electromagnetic problems will be examined. Two examples of electrostatic field and eddy current problems are studied and the numerical results are in excellent agreement with the analytical solutions.

1. INTRODUCTION

The electromagnetic field problems are in general governed by the Maxwell equations. Analytical methods can only solve problems with simple or regular boundary conditions. For practical engineering problems with complicated boundary conditions, various numerical methods, such as the finite element method, finite difference method, boundary element method etc., have been developed to solve the Maxwell equations that govern the electromagnetic field problems. In this paper, a novel numerical method, the moving least square differential quadrature method (MLSDQ) [1], is applied to solve electromagnetic problems.

The differential quadrature (DQ) method, which was introduced by Bellman *et al* [2] in 1972 to solve nonlinear partial differential equations, has gained popularity recently in the analysis of the mechanical behaviour of plate structures [3]. More recently, this method has been applied in the analysis of electromagnetic field problems [4,5]. The DQ method is highly efficient in solving a partial differential equation in a finite domain with a set of predefined boundary conditions. The DQ method discretizes the problem domain by a set of regularly distributed grid points. The derivatives of a function at an arbitrary point in the domain can be represented by a weighted linear combination of the function values at all the discrete points. Although the DQ method is easy to apply, there are three major disadvantages associated with this method. It is difficult for the method to solve problems with material and geometric discontinuities; the method requires the discrete points to be regularly distributed; and the method is only applicable to simple such as rectangular and circular domains.

Liew *et al* [1] proposed very recently a modified DQ method, MLSDQ method, in their study of the bending behaviour of plates. The MLSDQ method employs the moving least square (MLS) technique to replace the normal Lagrangian interpolation scheme of the DQ method in the determination of the weighting coefficients and hence can overcome the abovementioned problems associated with the DQ method.

This paper exams the applicability and accuracy of the MLSDQ method in the analysis of electromagnetic problems. The mathematical formulation of the method is presented. Two examples, electrostatic potential distribution in a rectangular trough and eddy currents in a long rectangular copper bar carrying an ac current, are presented to illustrate the applicability of the method. The convergence of the MLSDQ method is illustrated by the numerical results with different number of grid points and the numerical results are verified by the analytical solutions for the two selected examples.

2. MATHEMATIC FORMULATION

The MLSDQ method [1] is briefly presented in this section. The differential quadrature representations of the derivatives of a function are derived based on the moving least square technique.

Assume that a function $u(\mathbf{x})$ is defined by a partial differential equation in domain Ω, where $\mathbf{x}=x$, $\mathbf{x}=(x,y)$, and $\mathbf{x}=(x,y,z)$ represent the one-, two- and three-dimensional problems, respectively. The domain can be discretized by a set of discrete spatial points $\{\mathbf{x}_i\}_{i=1,2,\dots,N}$, where N is the total number of nodes. Note that the distribution of the discrete points does not need to be regular.

In the original DQ method, the derivatives of $u(\mathbf{x})$ at an arbitrary point $\mathbf{x}$ in the domain can be approximated as a weighted linear sum of the function values at all the discrete points [2]. The MLSDQ method, however, employs the moving least square technique to derive the weighting coefficients of the DQ representations for derivatives of $u(\mathbf{x})$. The approximate function value can be expressed as follows:

$$u^h(\mathbf{x}) = \sum_{i=1}^{m} p_i(\mathbf{x})a_i(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{1}$$

where $u^h(\mathbf{x})$ is the approximate value of $u(\mathbf{x})$, $p_i(\mathbf{x})$ a finite set of basis functions of a complete space, $a_i(\mathbf{x})$ the unknown coefficients, $\mathbf{p}^T(\mathbf{x})=[p_1(\mathbf{x})\ p_2(\mathbf{x}) \dots p_m(\mathbf{x})]$, $\mathbf{a}(\mathbf{x})=[\ a_1(\mathbf{x})\ a_2(\mathbf{x})\ \dots\ a_m(\mathbf{x})]$ and m is the number of basis functions that form a complete space.

Applying the moving least square technique, we can determine the unknown coefficients $\mathbf{a}(\mathbf{x})$, which is a function of the spatial coordinates $\mathbf{x}$, by minimizing the following weighted quadratic form:

$$\prod(\mathbf{a}) = \sum_{i=1}^{n} \varpi(\mathbf{x}-\mathbf{x}_i)(u^h(\mathbf{x}_i)-u_i)^2 = \sum_{i=1}^{n} \varpi_i(\mathbf{x})(\mathbf{p}^T(\mathbf{x}_i)\mathbf{a}(\mathbf{x})-u_i)^2 \tag{2}$$

where n is the number of the discrete points in the neighbourhood of $\mathbf{x}$, u_i the nodal parameter of $u(\mathbf{x})$ at point $\mathbf{x}_i$, and $\varpi_i(\mathbf{x})=\varpi(\mathbf{x}-\mathbf{x}_i)$ a positive weight function which decreases as $\|\mathbf{x}-\mathbf{x}_i\|$ increases. The weight function is equal to unit at the point if $\mathbf{x}_i=\mathbf{x}$ and vanishes when $\mathbf{x}_i$ is beyond a prescribed influence domain of $\mathbf{x}$. The size of the domain of influence, or support size, determines the number of discrete points n required in (2).

Minimizing (2) with respect to $\mathbf{a}(\mathbf{x})$, one can obtain the unknown coefficients as follows:

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{u} \tag{3}$$

where

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{n} \varpi_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i) \tag{4}$$

$$\mathbf{B}(\mathbf{x}) = [\varpi_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1)\ \ \varpi_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2)\ \ \cdots\ \ \varpi_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)] \tag{5}$$

and

$$\mathbf{u} = [u_1\ \ u_2\ \ \cdots\ \ u_n]^T \tag{6}$$

Substituting (3) into (1), the approximate function $u^h(\mathbf{x})$ can be expressed in terms of the nodal values as

$$u^h(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x})u_i \tag{7}$$

where the shape function $\phi_i(\mathbf{x})$ is given by

$$\phi_i(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\varpi_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i) \tag{8}$$

The first order derivatives of $u(\mathbf{x})$ can be directly derived from (7) as follows:

$$\frac{\partial u(\mathbf{x})}{\partial x} \approx \frac{\partial u^h(\mathbf{x})}{\partial x} = \sum_{i=1}^{n} \frac{\partial \phi_i(\mathbf{x})}{\partial x} u_i \tag{9}$$

and

$$\frac{\partial u(\mathbf{x})}{\partial y} \approx \frac{\partial u^h(\mathbf{x})}{\partial y} = \sum_{i=1}^{n} \frac{\partial \phi_i(\mathbf{x})}{\partial y} u_i \tag{10}$$

where $\frac{\partial \phi_i(\mathbf{x})}{\partial x}$ and $\frac{\partial \phi_i(\mathbf{x})}{\partial y}$ are the weighting coefficients of the first-order derivative of $u(\mathbf{x})$ in x and y directions at any spatial point $\mathbf{x}$. The weighting coefficients for the higher derivatives of $u(\mathbf{x})$ can be derived from (7) in similar manners.

3. EXAMPLES

The MLSDQ method is applied in this section to calculate two selected electromagnetic field problems. The weight function used in this study is selected as follows [1]:

$$\varpi_i(\mathbf{x}) = \begin{cases} \dfrac{\exp(-(\|\mathbf{x}-\mathbf{x}_i\|/c)^2 - \exp(-r/c)^2)}{1-\exp(-(r/c)^2)} & \|\mathbf{x}-\mathbf{x}_i\| \le r \\ 0 & \|\mathbf{x}-\mathbf{x}_i\| > r \end{cases} \tag{11}$$

where r is the domain of influence and $c = r/4$. The finite set of basis functions $p_i(\mathbf{x})$ is set to be a 2-D complete polynomial of the second degree in the present study.

3.1 The electrostatic boundary value problem

The first example is an electrostatic boundary-value problem of an infinitely long trough. Figure 1 shows the cross sectional dimensions [6].

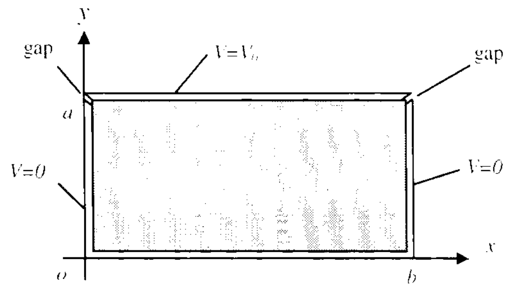

Figure 1. Potential $V(x,y)$ due to a conducting rectangular trough

3.1.1 Analytical solution

This is a two dimensional boundary value problem, and the electrostatic potential $V(x,y)$ is governed by the Laplace's equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{12}$$

with the given boundary conditions:

$$V(x=0, 0 \le y \le a) = 0 \tag{13}$$
$$V(x=b, 0 \le y \le a) = 0 \tag{14}$$
$$V(0 \le x \le b, y=0) = 0 \tag{15}$$
$$V(0 \le x \le b, y=a) = V_0 \tag{16}$$

The analytical solution of this problem is [6]:

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\sin\frac{n\pi x}{b} \sinh\frac{n\pi y}{b}}{n \sinh\frac{n\pi a}{b}} \tag{17}$$

3.1.2 Numerical solution

The MLSDQ method is applied to solve this problem. We assume V_o=100, and a = b = 10. The domain of influence r = 5 is adopted in the calculation. The problem domain is divided by uniformly distributed grid points along the x and y directions. Equations (12) to (16) can be rewritten as

$$\left(\sum_{i=1}^{n} \frac{\partial^2 \phi_i (x_I, y_I)}{\partial x^2} + \sum_{i=1}^{n} \frac{\partial^2 \phi_i (x_I, y_I)}{\partial y^2} \right) \overline{V}_I = 0 \tag{18}$$

$$\sum_{i=1}^{n} \phi_i (x_J, y_J) \overline{V}_J = 0 \tag{19}$$

and

$$\sum_{i=1}^{n} \phi_i (x_K, y_K) \overline{V}_K = V_0 \tag{20}$$

where subscripts I, J and K represent the inner grid points on the left, right, top, and bottom edges, respectively, and $\overline{V}_I$, $\overline{V}_J$ and $\overline{V}_K$ are the nominal potentials at the grid points. Solving (18) to (20), the nominal potentials can be obtained. The potential value V at an arbitrary point in the domain can be calculated by (7).

Figure 2 shows the convergence of potential V at the central point of the trough with the increase of number of grid points from 7×7 to 17×17. It is observed that the potential V approaches the analytical solution monotonically as the number of grid points increases. Good convergence is achieved as the number of grid points reaches 17×17.

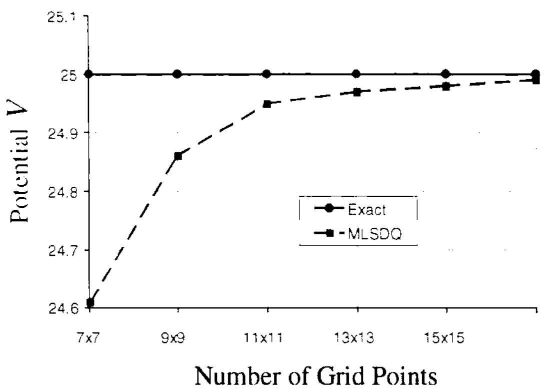

Figure 2. Convergence of potential V at the central point of the trough

Table 1 lists the values of electrical potentials at several selected points obtained by analytical and MLSDQ methods. When calculating the potential V using (17), the number of n is truncated at 101. It is observed that the MLSDQ results are in excellent agreement with the analytical solution.

3.2 The steady state eddy current problem

An eddy current steady state problem is used as another example to illustrate the versatility and accuracy of the method. Consider a conductor carries a time varying current *i(t)* inserted in a rectangular iron slot. Figure 3 shows the conductor cross sectional dimensions.

Table 1. Analytical and MLSDQ solutions of electrical potential *V* in the trough

y (m)	x (m)			
	0.125		0.250	
	Analytical	MLSDQ	Analytical	MLSDQ
0.125	1.709	1.706	3.148	3.146
0.250	3.698	3.696	6.797	6.794
0.375	6.312	6.310	11.540	11.540
0.500	10.070	10.070	18.200	18.200
0.625	15.940	15.940	28.040	28.030
0.750	26.260	26.260	43.200	43.180
0.875	48.290	48.200	66.900	66.830
y (m)	x (m)			
	0.375		0.500	
	Analytical	MLSDQ	Analytical	MLSDQ
0.125	4.100	4.097	4.432	4.429
0.250	8.834	8.830	9.541	9.537
0.375	14.930	14.920	16.090	16.090
0.500	23.290	23.280	25.000	24.990
0.625	35.070	35.060	37.320	37.310
0.750	51.580	51.560	54.050	54.030
0.875	73.640	73.620	75.430	75.410

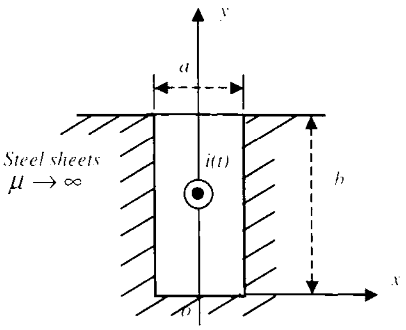

Figure 3. Cross sectional dimensions of a rectangular iron slot.

3.2.1 Analytical solution

To simplify the problem, we first assume the permeability of iron is infinite ($\mu \to \infty$) and thus the flux lines can be regarded as perpendicular to the surface of iron. Secondly, assume the electromagnetic field is one dimensional, i.e. $\mathbf{H} = H_x(y)\mathbf{i}$, and $\mathbf{E} = E_z(y)\mathbf{k}$. Finally, assume the current to be sinusoidal.

From the Maxwell equations the phasor H_x can be presented as

$$H_x = c_1 \cdot e^{\alpha y} + c_2 \cdot e^{-\alpha y} \tag{21}$$

where c_1 and c_2 are constants to be determined by the boundary condition, $\alpha = \frac{1+j}{\delta}$ and $\delta = \sqrt{\frac{2}{\omega\sigma\mu_0}}$ is the skin depth, and ω is the angular frequency. Substituting boundary conditions

$$H_x\big|_{y=0} = 0 = c_1 + c_2 \tag{22}$$

$$H_x\big|_{y=b} = \frac{-I}{a} = c_1 e^{\alpha b} + c_2 e^{-\alpha b} \tag{23}$$

into (21), we have

$$c_1 = -c_2 = \frac{-I}{a} \times \frac{1}{e^{\alpha b} - e^{-\alpha b}} \tag{24}$$

and

$$H_x = \frac{-I}{a} \times \frac{e^{\alpha y} - e^{-\alpha y}}{e^{\alpha b} - e^{-\alpha b}} = \frac{-I}{a} \times \frac{\sinh \alpha y}{\sinh \alpha b} \tag{25}$$

The current density for the one-dimensional case can be derived as:

$$J_z = \frac{-dH_x}{dy} = \frac{\alpha I}{a} \times \frac{\cosh \alpha y}{\sinh \alpha b} \tag{26}$$

3.2.2 Numerical solution

When using the MLSDQ method to address this problem, the two-dimensional case is considered. The governing differential equation for this problem may be derived as follows.

Define a vector magnetic potential **A**. From the Maxwell's equations, the current density can be derived as

$$J_z = -\sigma \frac{\partial A_z}{\partial t} - \sigma \nabla \phi = -\sigma \frac{\partial A_z}{\partial t} + J_s \tag{27}$$

where ϕ is the electrical potential and $-\sigma \frac{\partial A_z}{\partial t}$ is the eddy current density induced by time varying magnetic field. $J_s = -\sigma\nabla\phi$ is the excitation current generated by the applied electric potential.

As the current varies with time sinusoidally, the above equation can be expressed by phaser as

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = j\omega\mu_0\sigma A_z - \mu_0 J_s \tag{28}$$

where J_s is the specified current density and can be calculated as

$$J_s = \frac{I}{ab} \tag{29}$$

At the boundary between the conductor and the iron, the flux lines are perpendicular to the boundary surface. Therefore we have

$$\frac{\partial A_z}{\partial n} = 0 \tag{30}$$

where n is the normal direction (x). At the boundary $y = b$, we have

$$B_y\Big|_{y=b} = 0 = \frac{\partial A_z}{\partial y} \text{ or } A_z = constant \tag{31}$$

The MLSDQ method is employed to solve the problem defined by (28) to (31). The implementation of the MLSDQ method for this problem is similar to the problem as given in Section 3.1.2. The dimensions of the problem domain are $a = 10\ mm$ and $b = 50\ mm$, and the parameters are the frequency of the excitation current $f = 50\ Hz$, the rms value of the excitation current $I = 5000A$, the conductivity σ=2.7x10^7, and the permeability μ_o=4πx10^{-7}, respectively.

Grid points are uniformly distributed in the domain and set to be 11x51. The domain of influence in the MLSDQ method is chosen to be $r = 0.005$. In order to compare the numerical results with the 1-D analytical solution, the constant in (31) is calculated from (26) and (27) and the value is given by

$$A_z = 0.0043014 + 0.0031324j \text{ (A/m)} \tag{32}$$

Table 2 presents the current density values along the y axis obtained by the 1-D analytical and the 2-D MLSDQ methods, respectively. It can be seen that the MLSDQ results are in close agreement with the analytical solution. This confirms the applicability and accuracy of the MLSDQ method in solving the steady state eddy current problem.

Table 2. Current density along the central line of the conductor calculated by the analytical and MLSDQ methods

y (mm)	J_z (A/mm^2)	
	Analytical	MLSDQ
0	2.684	2.684
3	2.686	2.686
6	2.717	2.717
9	2.846	2.846
12	3.170	3.170
15	3.769	3.769
18	4.683	4.683
21	5.928	5.928
24	7.522	7.521
25	8.136	8.136
26	8.796	8.796
29	11.074	11.074
32	13.872	13.872
35	17.315	17.315
38	21.566	21.566
41	26.834	26.833
44	33.377	33.376
47	41.519	41.517
50	51.658	51.658

4. CONCLUSION

This paper studies the applicability and accuracy of the MLSDQ method in solving electromagnetic field problems. Compared with other numerical methods for solving partial differential equations, the MLSDQ method features in simple formulation and fast speed. Two examples of electrostatic and eddy current problems are employed to verify the effectiveness of the MLSDQ method for electromagnetic problems. The numerical results obtained by the MLSDQ method are in excellent agreement with the corresponding analytical solutions for the two selected examples. The application of the MLSDQ method on electromagnetic problems with material discontinuity and irregular domains will be studied in the future.

5. REFERENCES

[1] K.M. Liew, Y.Q. Huang, and J.N. Reddy, "Moving least square differential quadrature method and its application to the analysis of shear deformable plates," International Journal for Numerical Methods in Engineering, vol. 56, pp. 2331-2351, Apr 2003

[2] R. Bellman, B. G. Kashef, and J. Casti, "Differential quadrature: A technique for the rapid solution of nonlinear partial differential

equations," J. Comput. Phys., vol. 10, pp. 40-52, 1972
[3] C. Shu, "Free vibration analysis of composite laminated conical shells by generalized differential quadrature," J. Sound Vibrations, vol. 194, pp. 587-604, 1996
[4] C. Shu, "Analysis of elliptical waveguides by differential quadrature method," IEEE Trans. Microwave Theory Tech., vol. 48, no. 2, pp. 319-322, Feb. 2000
[5] Q.W. Xu and P. Mazumder, "Accurate modeling of lossy nonuniform transmission lines by using differential quadrature methods," IEEE Trans. Microwave Theory Tech., vol. 50, no. 10, pp. 2233-2246, Feb. 2000
[6] M.N.O. Sadiku, Elements of Electromagnetics, third edition, Oxford University Press, Inc., pp. 211-219, 2001