INTEGRATING CONCEPTS AND PROCESSES IN EARLY MATHEMATICS: THE AUSTRALIAN PATTERN AND STRUCTURE MATHEMATICS AWARENESS PROJECT (PASMAP)

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A broad descriptive study of 103 first graders and 16 longitudinal case studies found that children's perception and representation of structure[1] *generalised across a wide range of mathematical domains. Children's strategies showing use of pattern and structure were determined from task-based interviews. A high positive correlation (0.944) was found between children's performance on forty Pattern and Structure Assessment (PASA) tasks, and four stages of structural development: pre-structural, emergent, partial, and structural. Multiplicative structure, including unitising and partitioning, and 'spatial structuring', were found as critical to development of pattern and structure.*

BACKGROUND

The development of mathematical concepts involves the recognition of patterns and structural relationships within and between mathematical objects and situations. Mathematical patterns encountered in school range from number sequences and spatial arrays to algebraic generalisations and geometrical theorems. Broadly, a *pattern* may be defined as a numerical or spatial regularity, and the relationship between the various components of a pattern constitute its *structure*. Pattern and structure may be regarded as inherent or constructed from, brought to or imposed on mathematical systems. Research on children's development of mathematical concepts and their representations (e.g., counting, grouping, unitising, partitioning, estimating, base ten and multiplicative structure, and algebraic reasoning) has highlighted the role of pattern and structure. Goldin (2002) described the development of structure in children's representations and found that it leads ultimately to the construction of autonomous representational systems. However, there have been few studies with young children that have described general characteristics of structural development and how pattern and structure are integral to concept development.

In our PME 28 report (Mulligan, Prescott & Mitchelmore, 2004) we described how the mathematical structure present in children's representations generalised across five mathematical domains: time (clockface), number, space and algebra (triangular pattern), measurement (unitising area and length) and data (picture graph). Individual profiles of responses were reliably coded as one of four broad stages of structural

[1] In this paper we refer to the term 'structure' to encompass our definition of both pattern and structure.

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development: *pre-structural, emergent,* and *partial structural stages,* followed by *a stage of structural development.*

At PME 29 (Mulligan, Prescott & Mitchelmore, 2005) we reported the consistency of this structural development across tasks for eight high achieving and eight low achieving individuals, who were tracked over a two-year period. A fifth stage, an *advanced stage of structural development* was identified for high achievers, where the child's structural 'system' depicted an increased level of abstraction. However developmental patterns for the low-achieving cases were inconsistent; the transition from a pre-structural to an emergent stage was somewhat haphazard and some children reverted to earlier, primitive images after a year of schooling. There was further new and compelling evidence that structural development was impeded because children fail to perceive structure initially and thus they continue to produce increasingly crowded and chaotic responses that often rely on replication of superficial, non-mathematical features.

In this paper we report the primary analyses of structural development for 103 first graders who participated in the first year of the study. An aim of the study was to investigate the consistency of children's strategies for solving a wide range of mathematical tasks that incorporated common features of pattern and structure. The use of multiplicative structure and unitising were key features of the tasks. We provide evidence that early mathematics achievement is strongly linked with the child's development of mathematical structure; mathematical structure is an underlying characteristic that generalises across content domains. We build further upon previous analyses (Goldin, 2004, *in communication*; Goldin, 2002; Gray, Pitta & Tall, 2000; Tall, 2005, *in communication*; Thomas, Mulligan & Goldin, 2002), with the aim of making as explicit as possible the bases for our identification of developmental stages of mathematical structure. The implications of this research for classroom-based research using a Pattern and Structure Assessment (PASA) interview instrument and a Pattern and Structure Mathematics Awareness Program (PASMAP) are outlined.

THEORETICAL FRAMEWORK

Our studies on the role of structure in early mathematics have integrated a number of theoretical perspectives that can be traced to previous work on multiplicative reasoning (Mulligan & Mitchelmore, 1997). These studies were based largely on theories of Fischbein ('intuitive models') and Vergnaud ('conceptual fields'). Further research on children's representations of multiplicative situations and the structure of the numeration system led us to adapt Goldin's model of cognitive representational systems (Goldin, 2002; Thomas, Mulligan & Goldin, 2002). We also took into account more explicitly, theories on imagery and 'procepts' to explain qualitative differences in low-achieving students' use of imagery and concept development (Gray & Tall, 2000; Pitta-Pantazi, Gray & Christou (2004). The study of two- and three-dimensional structures (Battista, Clements, Arnoff, Battista & Borrow, 1998), and measurement concepts (Outhred & Mitchelmore, 2000) directed us to include the study of 'spatial structuring' as a critical feature, as it involved the process of constructing an organisation or form. This drew our attention to construction of multiplicative features shown in groups, arrays, grids, equal-sized units and graphs.

Further development of our research project complements other recent studies of early mathematics aimed at describing underlying conceptual bases of abstraction and generalization and the role of mathematical modelling and reasoning. For example, studies such as the Measure Up (MU) project (Slovin & Dougherty, 2004) where children approach mathematics through measurement and algebraic representations or those by English and Watters' (2005) that focus on structural characteristics such as patterns, and relationships rather than superficial features of problem-solving situations. We also integrate some features from studies of early algebraic reasoning (Blanton & Kaput; Schliemann, Carraher, Brizuela, Goodrow, & Peled, 2003; Warren, 2005) focused on number patterns and functional thinking.

Number	**Measurement**	**Space/Graphs/Patterns**
Subitising: visualise array 2 × 3	Length: use informal equal sized units	Pattern/visual memory: reconstruct triangular pattern of dots
Rote counting: multiples of 2, 5 & 3	Length: partitioning halves and thirds (continuous)	2 Dimensional space: use one unit to calculate area of 2D shape
Perceptual counting: multiples of 2 (1–30)	Length: construct units on 'empty' ruler	2 Dimensional /3 Dimensional: units of volume in 2D net and box
Counting: represent multiples (2, 5 & 3) on numeral track (1–30)	Area/Unitising: visualise and calculate area using one unit	Angles: represent and draw corners of a square
Ten as a unit using currency	Area: drawing units in partial grid	Picture graph: use grid and table
Partitioning 2x8x2 grid	Mass: unitising, comparing informal units of mass	Picture graph: construct picture graph from table
Partitive & quotitive sharing	Volume: use one unit in 2D net and box	Create/ draw self generated patterns
Combinatorial: 2x3	Time: draw o'clock on 'empty' clockface	

Table 1: Framework of pattern and structure assessment (PASA) tasks

METHOD AND ANALYSES

Task-based videotaped interviews were conducted with 103 first graders representative of a wide range of mathematical abilities and diverse socio-economic and cultural backgrounds. (For method see Mulligan et al, 2004; 2005). Forty individual tasks representing thirty different mathematical concepts and sub-categories were integrated into an initial assessment framework (see Table 1).

These were representative of key concepts and processes that had been the subject of investigation in related studies usually focused on a single mathematical content domain such as counting or unitising. The assessment included tasks that were beyond mathematics curriculum expectations. Each task required children to use elements of mathematical structure such as equal groups or units, spatial structure such as rows or columns, or numerical and geometrical patterns. Children were required to explain their strategies and draw representations such as reconstructing from memory, a triangular pattern and to visualise, then draw and explain their mental images. The analyses of data involved both qualitative and quantitative methods involving systematic coding of videotaped interviews, and interpretation of children's drawn and written representations. The primary analysis of the first interview data focused on the reliable coding of responses as correct/incorrect for all forty tasks and the matrix examined for patterns. A composite score was compiled for each student to gain a general picture of the performance data and item difficulty. Subsequently, individual children's responses to all forty tasks (individual profiles) were assigned a strategy indicating evidence of structural features.

As a result of this process, each child was assigned a stage of structural development. It was found that the children could be unambiguously sorted into four broad groups and correlations were generated for student performance by grouping (pre-structural, emergent structure, partial structure, structure). The presence of structural features shown in the drawn representations to five of these tasks *(clock face, triangular pattern, area, length* and *picture graph*) were analysed in depth because they gave the most convincing evidence of the child's use of structure. However, it was not assumed that this would be consistent with the child's performance data or that it would be consistent across most tasks.

DISCUSSION OF RESULTS

Between 50% and 70% of the children could solve most of the tasks, but these were solved with a wide range of strategies depicting the relative use of structural features. Several tasks proved most difficult: counting in multiples of three, a quotition problem without the use of materials, using ten as a unit of currency, a combinatorial problem and showing thirds on a continuous length. Most students completed the graphs' tasks showing the correct quantity but were unable to construct a graph with appropriate alignment. Most children could recognise corners in the angles task but could not draw a matching angle. The pattern (visual memory) task proved very difficult for students (see Mulligan et. al., 2005).

Table 2 and Figure 1 show students grouped by stage of structural development across the four levels of structural development. The correlation between level of structural development and the composite PASA score was 0.944, significant at the 0.01 level.

Children at the emergent stage represented larger variability than the other groups. Although there were indicators of emergent structure within 80% of the children's responses, the quality and type of structural features was not consistent across individuals, for example, the inconsistent use of equal sized units in both the area and length tasks. The categorisation of this group may well reflect several sub-categories depicting different forms of emergent structure that are context or task dependent.

Group	Stage	Percentage students	No. students	Composite PASA score
1	Pre-structural (PRS)	11	11	3 – 9
2	Emergent (ES)	38	39	10 – 19
3	Partial (PS)	27	28	20 – 25
4	Structure (S)	24	25	26 – 33

Table 2: Classification of students by stage of structural development.

The children in Groups 2 and 3 were less consistent in their responses in terms of assigning a level of structural development: there was more variability in responses of children in these groups: some 20% of responses showed pre-structural or partial structural responses. For example, a child at the emergent stage could score well on counting tasks but was generally unaware of the presence of structural features in other areas. Similarly some 20% responses at the partial structural stage were more likely to show structural rather than emergent features.

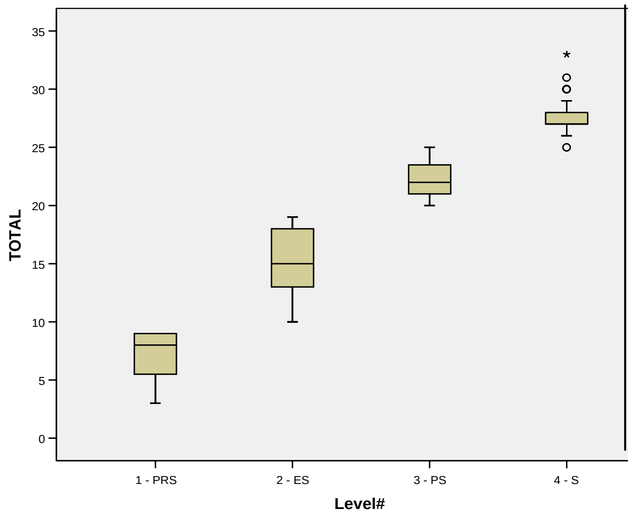

Figure 1: Performance (total score) by level of structural development

All the low achieving children fell into Group 1 (pre-structural). Conversely, the high achieving children all fell into Group 4 (structural) and readily expressed mathematical structure in all or almost all of the tasks. The children in Groups 1 and 4 were all identified on classroom-based assessment measures and other independent psychometric tests to be considered as having low or high mathematical ability respectively.

CONCLUSIONS & LIMITATIONS

It is not conclusive from our data whether the awareness and appropriate use of pattern and structure is a good predictor, or a consequence of, successful acquisition of basic mathematical concepts and skills. What we can conclude from the qualitative analyses is that children at the pre-structural stage did not perceive mathematical structure in most of their responses. For example, even in a simple counting task of multiples of two, these children were able to count aloud using the pattern correctly but could not show the corresponding pattern in units partitioned on a numeral track. Similarly partitioning and visualising in equal sized units proved to be difficult across a range of tasks. Children who had an advanced awareness of pattern and structure excelled across most conceptual areas and showed strong indications of early algebraic reasoning.

Our findings support our initial hypothesis that the more that a child's internal representational system has developed structurally, the more coherent, well-organised, and stable in its structural aspects will be their external representations, and the more mathematically competent the child will be. We extend Goldin's (2002) model to include two substages of developing structure and an advanced stage of structural development that was not expected from such young children. With a larger and more diverse sample and a broader range of tasks we may well find further substages within these stages. But rather than focusing on validation of stage-based developmental theories, we find it more important to identify and describe common structural characteristics across these stages that can enhance the development of mathematically coherent representations and well formed conceptual ideas.

In support of Goldin's theoretical stages of structural development, our analyses shows that mathematical structure does not develop in isolation. It develops from an emergent (inventive/semiotic) stage or stages in which characters or configurations in a new system (or new concept or task) are first given meaning in relation to previously constructed structural features. For example, the notion of equal-sized groups (multiplicative structure) is found across counting patterns, representations of these patterns on numeral tracks; in partitioning and sharing problems, in constructing and counting units of length, area and volume. We have also identified that children who operate at a pre-structural level may not necessarily progress to an emergent stage because they do not perceive some structural features with which to construct new ideas. With the advance of new concepts and skills in formal schooling young children's transition from a pre-structural stage to an emergent stage becomes problematic, somewhat impeded and increasingly chaotic over time, as seen in the many examples of superficial and non-mathematical aspects of pre-structural children's drawn representations.

Imagery, visual memory, and recognising similarity and difference, each play an important role in the development of pattern and structure. But the development of multiplicative structures including the base ten system, unitising and partitioning are critical to building structural relationships. Spatial structuring was found to play a key role in visualising and organising these structures. Our findings show that young children are capable of developing more complex mathematical structures, rather than relying on unitary counting and additive structures, and informal units of measure. We aim to provide an integrated theoretical perspective on the underlying bases of early mathematical development: the development of pattern and structure is generic to a well-connected conceptual framework in early mathematics.

FURTHER RESEARCH AND IMPLICATIONS FOR PRACTICE

There is a considerable body of research showing that low-achieving students of all ages have a poor grasp of mathematical patterns and structures. Rather than dismissing this finding as a characteristic of an immutable "low ability", we believe that it gives the clue to preventing difficulties in learning mathematics. Our recent classroom research suggests that young students can be taught to seek and recognise mathematical patterns and structures, and that the effect on their overall mathematics achievement can be substantial.

In 2003, a school-based numeracy initiative, including 683 elementary school students aged from 5 to 12 years, and 27 teachers, was trialled over a 9-month period using the PASA instrument and the Pattern and Structure Mathematics Awareness Program (PASMAP). Many PASMAP activities developed students' visual memory as they observed, recalled and represented numerical and spatial structures in processes such as counting, partitioning, subitising, grouping and unitising. Activities were regularly repeated in varied form to encourage generalisation. For example, Year 1 students learnt that in a 2 x 3 rectangular grid of squares, the squares are of equal size, they touch each other along their sides, there are the same number in each row and in each column, and the total number can be counted in multiples or patterns. In one lesson, students who initially copied the grid using a scattering of open circles later used squares of a reasonable size showing some structure. This occurred once the teacher had focused the students' attention on the importance of the structure of the grid.

PASMAP was further developed in 2005 to reflect more explicitly, aspects of early algebraic reasoning. PASMAP was trialled consistently in a design study of one first grade classroom over a nine-month period employing 28 children representing a wide range of mathematical abilities. The effectiveness of this initiative reflected the strong commitment of the recent graduate teacher under mentorship of the first researcher. Both initiatives aimed at developing teachers' pedagogical knowledge about the awareness of children's use of pattern and structure across key mathematical concepts. So far we have sufficient empirical and qualitative evidence to warrant an independent evaluation of the PASMAP program. Currently we are evaluating the effects of a PASMAP intervention for younger low-achieving children, aged 4 years 6 months to 6 years, in the first year of formal schooling.

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