Bayesian Student-t stochastic volatility models

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Abstract: This paper considers a Student-*t* stochastic volatility (SV) model using full Bayesian approach. A new two-stage scale mixtures representation for the Student-*t* density is proposed and used to speed up the efficiency of the Markov chain Monte Carlo sampling schemes.

Keywords: Scale mixtures of normal (SMN) distribution; Scale mixtures of uniforms (SMU) distribution; Gibbs sampler; outlier detection.

1 Introduction

In financial world, volatility of an asset (or return of an asset) is an important measure of financial risk and many time series models have been studied to model the volatility. Allowing the volatility to vary stochastically, stochastic volatility (SV) models have been widely used in modelling financial risk. However, likelihood-based analysis is difficult and impractical for SV-type models because parameter estimation and forecasting involves high dimensional integration. Therefore, attention has been shifted to the Bayesian approach. The advancement in Bayesian computational techniques provides an efficient way for implementation. Jacquier *et al.* (1994) adopt the Gibbs sampling scheme while Shephard and Pitt (1997) employ the Metropolis-Hastings scheme.
Although a number of improvements have been proposed to make statistical inference possible and simple, a deviation from the normality assumption for the time series data does impose an increasing computational burden to the analysis. In fact, many financial data exhibit a thick tail behaviour and this tempts statisticians and econometricians to model asset returns using heavy-tailed distributions such as Student-*t* distribution. See Jacquier *et al.* (1994). However, the extension from normal to heavy-tailed distributional assumption increases the computational load dramatically. Therefore, this paper aims to provide a more efficient Gibbs sampler scheme than Jacquier *et al.* (1994) by expressing the Student-*t* density function into a two-stage

scale mixture. For illustrative purpose, an exchange rate data set is analyzed in detail.

2 Two-stage Scale Mixtures Representation for Student-*t* Distribution

The mixtures of normal (SMN) distribution was characterized by Andrew and Mallows (1974) using Laplace transformation approach.
Let θ and σ be the location and scale parameters of the SMN random variable X. Then the probability density function of X is given by

$$f(x) = \int_{R^+} N(x|\theta, \lambda\sigma^2)\, g(\lambda) d\lambda$$

where $N(\cdot|\cdot)$ denotes the normal density, $g(\cdot)$ is a probability density function defined on $\Re^+ = (0, \infty)$ and λ is referred to as a mixing parameter which is commonly used as a global diagnostic check for outliers. See Choy and Smith (1997) for details. In Bayesian framework, the mixture density in (2.1) can be expressed into a two-stage hierarchy of the form

$$X|\theta, \sigma^2, \lambda \sim N(\theta, \lambda\sigma^2) \qquad \text{and} \qquad \lambda \sim g(\lambda).$$

The Student-t distribution with degrees of freedom α is a member of the SMN distribution with a gamma mixing distribution

$$\lambda^{-1} \sim Ga\,(\alpha/2, \alpha/2)$$

where $Ga(a, b)$ is the gamma density function with parameters a and b. To facilitate an efficient computation for the SV models, we express the normal density function into the following scale mixtures of uniform (SMU) representation.

$$N(x|\theta, \sigma^2) = \int_{\theta-\sigma\sqrt{u}}^{\theta+\sigma\sqrt{u}} \frac{1}{2\sigma\sqrt{u}}\, Ga\,(u\,|3/2, 1/2)\, du$$

See Walker and Gutiérrez-Peña (1999) for details about SMU distributions. Now the Student-t distribution with degrees of freedom α is expressed hierarchically as

$$\begin{aligned} X|\theta, \sigma^2, \lambda, u &\sim U(\theta - \sigma\lambda^{1/2}\, u^{1/2}, \theta + \sigma\lambda^{1/2}\, u^{1/2}) \\ \lambda^{-1} &\sim Ga\,(\alpha/2, \alpha/2) \\ u &\sim Ga\,(3/2, 1/2) \end{aligned}$$

where $U(a, b)$ is a uniform distribution defined on the interval (a, b).

3 Bayesian SV Model

Let r_t be the asset value of an equity at time $t = 0, 1, 2, \ldots, n$. The mean adjusted asset return y_t at time t is defined as

$$y_t = \ln\left(r_t/r_{t-1}\right) - \frac{1}{n}\sum_{i=1}^{n} \ln\left(r_i/r_{i-1}\right), \qquad t = 1, 2, \ldots, n.$$

Let H_t and h_t be the volatilities and log-volatilities, respectively. The standard (traditional) SV model for the asset return, y_t is defined as

$$\begin{aligned} y_t &= \beta\, H_t^{1/2}\, \epsilon_t, \qquad t = 1, 2, \ldots, n \\ h_t &= \begin{cases} \sigma\eta_1 \big/ \sqrt{1-\phi^2} & t = 1 \\ \phi\, h_{t-1} + \sigma\eta_t & t > 1 \end{cases} \end{aligned}$$

where $\{\epsilon_t\}$ and $\{\eta_t\}$ are independent standard Gaussian processes. β is a constant factor that represents the model instantaneous volatility which is usually set to one in many literatures. σ is the standard deviation of the log-volatilities and $\phi \in (-1, 1)$ is the persistence of the volatility.

As the objective of this paper is to develop an efficient simulation scheme for the SV model, the mean adjusted returns, y_t, are modelled by a Student-t distribution while the log-volatility is assumed to follow a normal distribution. In fact, we can also use heavy-tailed distribution for modelling the log-volatility to achieve a higher degree of robustness.

In the current setup, the Bayesian model for the SV model is given by

$$\begin{aligned} y_t | h_t, \lambda_t, u_t &\sim U\left(-\beta H_t^{1/2}\lambda_t^{1/2}u_t^{1/2}, \beta H_t^{1/2}\lambda_t^{1/2}u_t^{1/2}\right) \\ \lambda_t^{-1} &\sim Ga\left(\alpha/2, \alpha/2\right) \\ u_t &\sim Ga\left(3/2, 1/2\right) \\ h_t | h_{t-1}, \phi, \sigma^2 &\sim N\left(\phi h_{t-1}, \sigma^2\right) \\ \frac{\phi+1}{2} &\sim Be\left(\alpha_\phi, \beta_\phi\right) \\ \sigma^{-2} &\sim Ga\left(a_\sigma, b_\sigma\right) \end{aligned}$$

for $t = 1, 2, \ldots, n$. This model is implemented by the Gibbs sampler. Details will be provided upon request.

4 Example

Figure 1 presents 1000 mean adjusted daily closing exchange rate returns of US dollars to Sterling pounds from January 2, 1981. Obviously, some trading days produce extreme returns. In the simulation study, we set $\beta = 1$

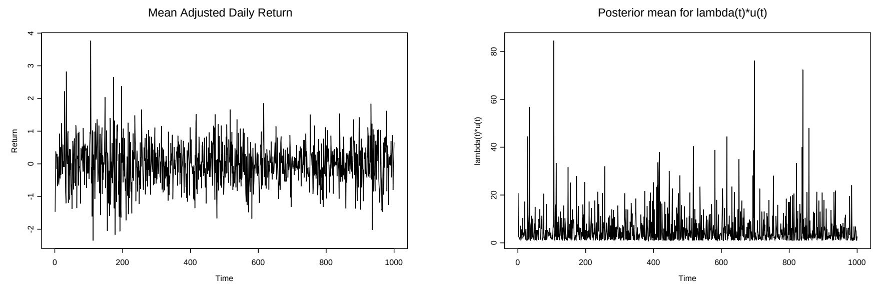

FIGURE 1. *(Left) Time series plot of mean adjusted daily exchange rate returns. (Right) Bayes estimates of the products of mixing parameters, $\lambda_i\, u_i$ for the Cauchy SV model. Large values correspond to extreme daily returns.*

for simplicity and $a_\sigma = b_\sigma = 0.001$ to reflect the non-informative prior knowledge about the variance of the log-volatility σ^2. $(\phi+1)/2$ is, however, assigned a beta $Be(20, 1)$ informative prior. We ran the Gibbs sampler for a single series of 30000 iterations after a 'burn-in' period of 5000 iterations. A random sample of 1000 drawings is taken for providing posterior summaries.

Obviously, the data set contains some outlying observations. From the robustness point of view, heavy-tailed distributions can protect inference against outliers. Bayes estimates of volatilities H_t are more accurate under the Cauchy distribution than the normal distribution. (Results are available upon request). Apart from parameter estimation, the mixing parameters, λ_i and u_i, of the SMN and SMU representations can be used as global indicators for outliers detection. Figure 1 displays the posterior means of $\lambda_i \times u_i$. Large values associate with outliers. The five most influential daily returns are on Days 106, 697, 840, 34 and 858. The advantages of using two-stage scale mixtures distributions are fully demonstrated here.

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