Optimized Linear Physical-Layer Network Coding of Full-Rate Full-Diversity in MIMO Two-Way Relay Networks

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***Abstract*—In multiple-input multiple-output (MIMO) two-way relay networks (TWRN), linear physical-layer network coding (LPNC) was proposed to boost the throughput by using spatial multiplexing at source nodes. How to design optimal LPNC for full-rate full-diversity MIMO TWRN is still an open problem. In this paper, we put forth a full-rate full-diversity (FRFD) LPNC scheme. In this scheme, two source nodes, each with two antennas, transmit full-rate universal space-time codes to a two-antenna relay simultaneously. Then, the relay applies LPNC to compute multiple network-coded (NC) messages. In particular, we explicitly solve the optimal LPNC mapping to minimize decoding errors of NC messages in the FRFD LPNC scheme. Our analytical results verify that the optimal FRFD LPNC scheme guarantees the full-diversity and full-rate transmission at the same time. Simulation results are consistent with the analytical results and further demonstrate that our optimal FRFD LPNC scheme outperforms the conventional MIMO LPNC scheme.**

***Index Terms*—Linear physical-layer network coding, MIMO, full diversity, full rate, universal space-time coding.**

I. Introduction

In relay networks, spectral efficiency can be drastically degraded because of interference at the relay induced by the simultaneous transmission from multiple source nodes. Linear physical-layer network coding (LPNC) was originally proposed to boost the network throughput by exploiting the interference [1]–[3]. The core of LPNC is characterized by a designed LPNC mapping at the relay. From the simultaneously received signals, the relay computes a network-coded (NC) message that is a linear combination of messages from the source nodes. The optimized LPNC systems can minimize decoding errors of NC message under different modulations and channel realizations. Recent literature has shown that the framework of LPNC is scalable with high-order modulations and linear channel coding [4]–[10].

The natural way to illustrate the concept of LPNC is through two-way relay network (TWRN). The aim of TWRN is to exchange the information of source nodes via a relay. For nonchannel-coded single-antenna TWRN, optimal LPNC mapping in finite fields was proposed to maximize the reliability of information exchange in the high SNR regime [4], [5]. In [6], the authors formulated a complex LPNC mapping in hybrid finite ring with 4-ary and 5-ary quadrature amplitude modulations. For channel-coded TWRN, the benefit of LPNC mapping facilitates the channel decoding process at the relay. In [7], linear channel coding was integrated with LPNC to maximize the achievable rate in TWRN, assuming the same finite field operation as LPNC. Generally, the channel-coded LPNC mapping can be regarded as a practical realization of compute-and-forward framework [8].

The applications of LPNC are not restricted in TWRN, but raise many new promising and challenging issues in various modern communication systems. For example, [9] proposed an LPNC based protocol for distributed uplink multiple-input and multiple-output (MIMO) systems, where multiple BSs first compute received signals from multiple users by the LPNC mapping and then forward the NC messages to a common central unit. Moreover, LPNC also manifests another advantage by its scalability with multiple-antenna networks. Recently, LPNC was generalized to MIMO TWRN, where both relay and each source node are equipped with multiple antennas [10]. In this scheme, without knowledge of channel state information at transmitters, source nodes with multiple antennas transmit simultaneously in a spatial multiplexing way. Then, the multi-antenna relay computes multiple NC messages from the received signals, referred to as the conventional MIMO LPNC mapping. However, it is widely accepted that spatial multiplexing cannot harvest full diversity for multi-antenna systems. Therefore, in spite of full-rate transmission, conventional MIMO LPNC with spatial multiplexing cannot minimize decoding errors of NC messages.

From an information theoretic perspective, the optimal diversity and multiplex tradeoff can be achieved in MIMO systems by the linear dispersion coding across transmit antennas and integer-forcing linear receiver [11]. Going forward, achieving full diversity while retaining the full-rate transmission in MIMO TWRN can be guaranteed. However, the design of practical network coding based scheme remains challenging. To address this problem, we put forth a full-rate full-diversity (FRFD) LPNC scheme for MIMO TWRN. In this scheme, each source node simultaneously transmits a full-rate universal space-time block code (STBC) to the relay, where *full-rate* means that the rate is M for M transmit antennas. Upon receiving ST coded symbols from the source nodes, the relay adopts the LPNC mapping to compute multiple NC messages. Note that our FRFD LPNC formulation builds on top of conventional MIMO LPNC in [10], but the introduction of space-time coding at source nodes make the optimal LPNC design even more challenging. In particular, we explicitly solve the optimal LPNC mapping to minimize decoding errors of NC messages for arbitrary channel realizations in FRFD LPNC systems. Our analytical results further verify that our FRFD LPNC scheme guarantees the full-diversity and full-rate transmission at the same time.

II. Linear PNC in Space-time coded TWRN

A. System Model

Consider a two-way relay network (TWRN) where source nodes A and B exchange their information with each other via a relay R but they cannot communicate with each other directly. Suppose that both the relay and each node are equipped with two antennas and work in a half-duplex model. The information exchange in TWRN consists of two phases: In the multiple access (MAC) phase, nodes A and B transmit their packets $\{w_{A,l}\}_{l=1,2,\ldots,L}$ and $\{w_{B,l}\}_{l=1,2,\ldots,L}$ to relay R, respectively. In the broadcast (BC) phase, upon receiving the packets from the users, the relay broadcasts a processed packet to the two users. Then, node A decodes $\{w_{B,l}\}$ and node B decodes $\{w_{A,l}\}$, respectively. Here we only illustrate the real-valued system model. [1]

MAC Phase: Consider that $w_{A,l}, w_{B,l} \in \{0, 1, 2, \ldots, q\}$, i.e., $w_{A,l}, w_{B,l} \in \mathbb{F}_q$, where $\mathbb{F}_q$ is a finite field with a prime q. Nodes A and B modulate $w_{A,l}$ and $w_{B,l}$ into $x_{A,l}$ and $x_{B,l}$, respectively, where $x_{A,l}, x_{B,l} \in \mathcal{A}$. Suppose that each user considers q-level pulse amplitude modulation (q-PAM), i.e., $\mathcal{A}$ is the constellation of q-PAM. Thus, $x_{i,l} = \frac{1}{\mu}(w_{i,l} - \frac{q-1}{2})$, where μ is a power normalization factor such that $E\{x_{i,l}^2\} = 1$ for $i \in \{A, B\}$. Then, we consider a full-rate STBC to achieve full-rate transmission at each node.

Definition 1 (Full-rate STBC): The code rate of an STBC $\mathbf{S}$ with size of $T \times M$ is defined as $R = \frac{L}{T}$ symbols per channel use, where L denotes the independent symbols embedded in $\mathbf{S}$. If $L = TM$, the STBC is said to have full rate, i.e., $\mathcal{R} = M$ symbols per channel use [12].

In particular, nodes A and B first encode $x_{A,l}, x_{B,l}, \forall l \in \{1, 2, \ldots, L\}$ to linear dispersion STBC $\mathbf{S}_A \in \mathbf{R}^{\frac{L}{2}\times 2}$ and $\mathbf{S}_B \in \mathbf{R}^{\frac{L}{2}\times 2}$ across $\frac{L}{2}$ time slots and two antennas, respectively, and then transmit them to relay R simultaneously. We refer to this system as *space-time coded TWRN*. From *Definition 1*, each node transmits at full rate. At relay R, nodes A and B are perfectly synchronized. Then, the received signal at relay R is given by

$$\mathbf{Y}_R \triangleq \sqrt{P}\mathbf{S}_A\mathbf{H}_A + \sqrt{P}\mathbf{S}_B\mathbf{H}_B + \mathbf{Z}, \tag{1}$$

where $\mathbf{H}_i \in \mathbb{R}^{2\times 2}$ denotes the real-valued MIMO channel matrix from node $i, i \in \{A, B\}$ to relay R. The (m, n)th element of $\mathbf{H}_i$, h^i_{mn}, denotes the channel coefficient between the mth transmit antenna at node i and the nth receive antenna at relay R, and all elements are assumed to be i.i.d. Gaussian distributed with zero mean and unit variance. Moreover, P is the transmit power at each node, and $\mathbf{Z} \in \mathbb{R}^{T\times 2}$ is the AWGN matrix whose entries are i.i.d. with zero mean and variance σ_z^2.

Following the equivalent channel transformation of linear dispersion STBC [14], an equivalent channel model of $\mathbf{Y}_R$ in (1) is given by

$$\mathbf{y}_R \triangleq vec\{\mathbf{Y}_R\} \triangleq \sqrt{P}\boldsymbol{\mathcal{H}}_A\mathbf{x}_A + \sqrt{P}\boldsymbol{\mathcal{H}}_B\mathbf{x}_B + \mathbf{z}, \tag{2}$$

where $vec\{\}$ arranges a matrix in one column by putting its columns one after the other. Thus, vectorized versions of $\mathbf{Y}_R, \mathbf{Z} \in \mathbb{R}^{\frac{L}{2}\times 2}$ are $\mathbf{y}_R, \mathbf{z} \in \mathbb{R}^{L\times 1}$, respectively. Moreover, we define $\mathbf{x}_A \triangleq (x_{A,1}, x_{A,2}, \ldots, x_{A,L})^{\mathcal{T}}$ and $\mathbf{x}_B \triangleq (x_{B,1}, x_{B,2}, \ldots, x_{B,L})^{\mathcal{T}}$ as *modulated symbol vectors* of nodes A and B, and $\boldsymbol{\mathcal{H}}_A, \boldsymbol{\mathcal{H}}_B \in \mathbb{R}^{L\times L}$ as *equivalent channel* matrices of nodes A and B, respectively.

In our proposed scheme, nodes A and B use the same ST code. To achieve a full-rate transmission, node A encodes $\mathbf{x}_A \triangleq (x_{A,1}, x_{A,2}, x_{A,2}, x_{A,3})^{\mathcal{T}}$ to a universal STBC $\mathbf{S}_A$ [12]:

$$\mathbf{S}_A \triangleq \frac{1}{\sqrt{1+|\phi|}}\begin{pmatrix} s^1_{A,1} & \sqrt{\phi}s^2_{A,1} \\ \sqrt{\phi}s^2_{A,2} & s^1_{A,2}\end{pmatrix}, \tag{3}$$

where $\mathbf{s}^p_A \triangleq (s^p_{A,1}, s^p_{A,2})^{\mathcal{T}} = \mathbf{M}(x_{A,2p-1}, x_{A,2p})^{\mathcal{T}}$ for $p = 1, 2$. Moreover, $\mathbf{M}$ is an algebraic rotation matrix that maximizes the associated minimum product distance, and the optimal 2×2 rotation matrix $\mathbf{M}$ is given in [12]. The STBC in (3) can achieve the full diversity order in the 2×2 MIMO system under maximum likelihood decoding.

Since q-PAM establishes a bijection mapping from w_i to x_i, we rewrite $\mathbf{y}_R$ in (2) as follows:

$$\begin{aligned}\mathbf{y}_R &= \tfrac{\sqrt{P}}{\mu}\boldsymbol{\mathcal{H}}_A(\mathbf{w}_A - \tfrac{q-1}{2}) + \tfrac{\sqrt{P}}{\mu}\boldsymbol{\mathcal{H}}_B(\mathbf{w}_B - \tfrac{q-1}{2}) + \mathbf{z} \\ &= \tfrac{\sqrt{P}}{\mu}(\boldsymbol{\mathcal{H}}_A\mathbf{w}_A + \boldsymbol{\mathcal{H}}_B\mathbf{w}_B) + \mathbf{z} - \tfrac{\sqrt{P}(q-1)}{2\mu}(\boldsymbol{\mathcal{H}}_A + \boldsymbol{\mathcal{H}}_B),\end{aligned}$$

where $\mathbf{w}_i \triangleq (w_{i,1}, w_{i,2}, \ldots, w_{i,L})^{\mathcal{T}}$ for $i \in \{A, B\}$. Since $w_{i,l} \in \mathbb{F}_q, \forall l \in \{1, 2, \ldots, L\}$, we have $\mathbf{w}_A, \mathbf{w}_B \in \mathbb{F}_q^L$.

Define a *joint symbol vector* as $(\mathbf{w}_A^{\mathcal{T}}, \mathbf{w}_B^{\mathcal{T}})^{\mathcal{T}}$ and a set of $(\mathbf{w}_A^{\mathcal{T}}, \mathbf{w}_B^{\mathcal{T}})^{\mathcal{T}}$ as $\mathcal{W}_{(A,B)}$. Since $\mathbf{w}_A, \mathbf{w}_B \in \mathbb{F}_q^L$, $\mathcal{W}_{(A,B)} = \mathbb{F}_q^{2L}$ and $|\mathcal{W}_{(A,B)}| = q^{2L}$. Given a joint symbol vector, we define a *superimposed symbol vector* as $\mathbf{w}_S \triangleq \boldsymbol{\mathcal{H}}_A\mathbf{w}_A + \boldsymbol{\mathcal{H}}_B\mathbf{w}_B$ and a set of $\mathbf{w}_S$ as $\mathcal{W}_S$. Note that the mapping from $\mathcal{W}_{(A,B)}$ to $\mathcal{W}_S$ may not be bijective, depending on the *equivalent channel-gain pair* $(\boldsymbol{\mathcal{H}}_A, \boldsymbol{\mathcal{H}}_B)$. For simplicity, we use the term "channel-gain pair". At a particular channel-gain pair, the relay can determine a constellation of $\mathcal{W}_S$.

A straightforward approach for decoding at relay R is to decode $(\mathbf{w}_A, \mathbf{w}_B)$ individually as $(\hat{\mathbf{w}}_A, \hat{\mathbf{w}}_B)$ using the ML rule, which is referred to as the complete decoding (CD). Then, the relay broadcasts $\hat{\mathbf{w}}_A$ and $\hat{\mathbf{w}}_B$ to nodes A and B respectively. For CD, a decoding error at the relay occurs if $(\mathbf{w}_A, \mathbf{w}_B) \neq (\hat{\mathbf{w}}_A, \hat{\mathbf{w}}_B)$ (i.e., if $\mathbf{w}_A \neq \hat{\mathbf{w}}_A$ or $\mathbf{w}_B \neq \hat{\mathbf{w}}_B$). At the high SNR regime, symbol error ratio (SER) of CD is dominated by the minimum distance between any two distinct superimposed symbol vectors $\mathbf{w}_S$ and $\mathbf{w}'_S$ in the constellation of $\mathcal{W}_S$, defined as follows [4], [5]:

$$l_{\min} \triangleq \underset{(\mathbf{w}_A,\mathbf{w}_B)\neq(\mathbf{w}'_A,\mathbf{w}'_B)}{\arg\min} \|\mathbf{w}_S - \mathbf{w}'_S\|, \tag{4}$$

[1]For the complex modulation (such as QAM) that introduces phase difference at the relay, we need a different mathematical tool in complex field (such as Gaussian integer field), which is out of the scope of the present work.

where $\|\cdot\|$ denotes the Frobenius norm.

An advanced approach is to use PNC, where the role of relay R in TWRN is not to decode the packets of nodes A and B separately, but to forward the useful information to help a combined information exchange between nodes A and B. Recently, linear PNC (LPNC) was first proposed for single-antenna TWRN in [4] and was then generalized to MIMO TWRN [10], allowing the relay to decode a linear combination of packets from nodes A and B. The optimal LPNC design can bring the SER gain over CD in the high SNR regime.

As we pointed out in *Introduction*, the existing works have not considered optimal LPNC for full-rate and full-diversity MIMO TWRN, and this is the key problem to be solved in this paper. Before delving into the details, let us review general formulation of conventional MIMO LPNC in [10].

B. General Formulation of MIMO LPNC

Generally, under the MIMO LPNC mapping, a joint symbol vector $(\mathbf{w}_A^{\mathcal{T}}, \mathbf{w}_B^{\mathcal{T}})$ is mapped to a *network-coded (NC) symbol vector* $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ as follows:

$$\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \triangleq \boldsymbol{\alpha}\mathbf{w}_A + \boldsymbol{\beta}\mathbf{w}_B \pmod q, \tag{5}$$

where $(\boldsymbol{\alpha},\boldsymbol{\beta})$ denotes a pair of *network-coded (NC) coefficient matrices*. Each NC coefficient matrix has the size of $L \times L$ and its elements belong to $\mathbb{F}_q$. We see that $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ is a linear combination of $\mathbf{w}_A$ and $\mathbf{w}_B$ in $\mathbb{F}_q$. Let $\mathcal{M}_L(\mathbb{F}_q)$ denote the set of $L \times L$ matrices with entries in $\mathbb{F}_q$. Therefore, $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathcal{M}_L(\mathbb{F}_q)$. Note that a vector modulo q equals to every element modulo q in the vector. Furthermore, $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \in \mathbb{F}_q^L$. For a pair $(\boldsymbol{\alpha},\boldsymbol{\beta})$, we define $\mathcal{W}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \triangleq \{\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} | \mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \boldsymbol{\alpha}\mathbf{w}_A + \boldsymbol{\beta}\mathbf{w}_B \pmod q, \forall (\mathbf{w}_A,\mathbf{w}_B) \in \mathbb{F}_q^{2L}\}$.

NC Partition of $\mathcal{W}_{(A,B)}$: For a fixed pair $(\boldsymbol{\alpha},\boldsymbol{\beta})$ with one of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ invertible, $\mathcal{W}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \mathbb{F}_q^L$ and $|\mathcal{W}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}| = q^L$. For every pair $(\boldsymbol{\alpha},\boldsymbol{\beta})$, the LPNC mapping defined in (5) partitions $\mathcal{W}_{(A,B)}$ into q^L subsets, and each subset corresponds to a particular $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ as follows

$$\mathcal{W}_{(A,B)}(\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \triangleq \big\{(\mathbf{w}_A,\mathbf{w}_B) \in \mathbb{F}_q^{2L} \big| \; \boldsymbol{\alpha}\mathbf{w}_A + \boldsymbol{\beta}\mathbf{w}_B = \mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \pmod q\big\}. \tag{6}$$

That is, the MIMO LPNC mapping from $\mathcal{W}_{(A,B)}$ to $\mathcal{W}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ is a q^L to 1 mapping.

NC Grouping of $\mathcal{W}_S$: Since each $(\mathbf{w}_A^{\mathcal{T}}, \mathbf{w}_B^{\mathcal{T}})$ corresponds to a superimposed symbol vector $\mathbf{w}_S$, the MIMO LPNC mapping also groups $\mathcal{W}_S$ into q^L subsets for $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ as follows

$$\mathcal{W}_S(\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \triangleq \big\{\mathbf{w}_S \in \mathcal{W}_S \big| \exists (\mathbf{w}_A^{\mathcal{T}}, \mathbf{w}_B^{\mathcal{T}})^{\mathcal{T}} \in \mathcal{W}_{(A,B)} : \mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \boldsymbol{\alpha}\mathbf{w}_A + \boldsymbol{\beta}\mathbf{w}_B \pmod q, \mathbf{w}_S = \boldsymbol{\mathcal{H}}_A\mathbf{w}_A + \boldsymbol{\mathcal{H}}_B\mathbf{w}_B\big\}. \tag{7}$$

Note that two different groups above may not be necessarily disjoint.

Decoding Process at the Relay: Given $\mathbf{y}_R$ in (4), relay R aims to decode $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ by choosing a proper pair of NC coefficient matrices $(\boldsymbol{\alpha},\boldsymbol{\beta})$. The optimal selection of $(\boldsymbol{\alpha},\boldsymbol{\beta})$ will be studied in Section III-B. First, the relay estimates the joint symbol vector $(\mathbf{w}_A,\mathbf{w}_B)$ as $(\hat{\mathbf{w}}_A,\hat{\mathbf{w}}_B)$ by CD. Second, the relay maps this $(\hat{\mathbf{w}}_A,\hat{\mathbf{w}}_B)$ to an NC symbol vector in (5). Note that even though $(\mathbf{w}_A,\mathbf{w}_B) \neq (\hat{\mathbf{w}}_A,\hat{\mathbf{w}}_B)$, the decoding of NC symbol vector at relay R is still correct as long as $\boldsymbol{\alpha}\mathbf{w}_A + \boldsymbol{\beta}\mathbf{w}_B = \boldsymbol{\alpha}\hat{\mathbf{w}}_A + \boldsymbol{\beta}\hat{\mathbf{w}}_B \pmod q$. Here, we denote $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \neq \hat{\mathbf{w}}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ as a decoding error of NC symbol vector at the relay. Therefore, the SER of $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ in MIMO LPNC systems is generally smaller than that of $(\mathbf{w}_A,\mathbf{w}_B)$ in CD.

Broadcast Phase: Suppose that the decoding of $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ at relay R is perfect. The relay broadcasts this $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ to nodes A and B. Then, from its received signal, each node estimates $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ as $\hat{\mathbf{w}}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ by the maximum likelihood detection for $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \in \mathbb{F}_q^L$ in (5). Assuming that $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \hat{\mathbf{w}}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ (an error-free broadcasting), node A can recover $\mathbf{w}_B$ from node B with the knowledge of $(\boldsymbol{\alpha},\boldsymbol{\beta})$, as follows:

$$\boldsymbol{\beta}^{-1}(\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} - \boldsymbol{\alpha}\mathbf{w}_A) = \boldsymbol{\beta}^{-1}\boldsymbol{\beta}\mathbf{w}_B = \mathbf{w}_B \pmod q, \tag{8}$$

where $\boldsymbol{\beta}^{-1}$ denotes the inverse matrix of $\boldsymbol{\beta}$ in $\mathcal{M}_L(\mathbb{F}_q)$. The recovery of $\mathbf{w}_A$ at node B follows similarly. According to (8), we require that both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are invertible in $\mathbb{F}_q$ to guarantee the successful information exchange in (8). Otherwise, there will be information loss of $\mathbf{w}_B$ and node A cannot recover $\mathbf{w}_B$ in (8) due to the rank deficiency of $\boldsymbol{\beta}$. Denote the set of invertible matrices in $\mathcal{M}_L(\mathbb{F}_q)$ by $GL_L(\mathbb{F}_q)$. Then, we define the valid MIMO LPNC mapping as follows:

Definition 2: The MIMO LPNC mapping under $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is *valid* if $\boldsymbol{\alpha},\boldsymbol{\beta} \in GL_L(\mathbb{F}_q)$.

From the LPNC mapping in the MAC phase and recovery in the BC phase, the core of MIMO LPNC systems is to minimize the SER of NC symbol vector by choosing the optimal $(\boldsymbol{\alpha},\boldsymbol{\beta})$.

III. On the Optimal Design of Full-Rate Full-Diversity LPNC

In this section, we put forth the optimal full-rate full-diversity LPNC (FRFD LPNC) mapping in MIMO TWRN, by taking into account the equivalent channel induced by STBC at each node.

A. Optimal FRFD LPNC Mapping

To derive the optimal $(\boldsymbol{\alpha},\boldsymbol{\beta})$ in FRFD LPNC systems, we focus on distance metrics of superimposed constellation $\mathcal{W}_S$.

First, we introduce the general Euclidean distance in the superimposed constellation. At a particular $(\boldsymbol{\mathcal{H}}_A, \boldsymbol{\mathcal{H}}_B)$, the Euclidean distance between any two superimposed symbol vectors $\mathbf{w}_S$ and $\mathbf{w}'_S$ associated with two distinct $(\mathbf{w}_A^{\mathcal{T}}, \mathbf{w}_B^{\mathcal{T}})^{\mathcal{T}}$ and $((\mathbf{w}'_A)^{\mathcal{T}}, (\mathbf{w}'_B)^{\mathcal{T}})^{\mathcal{T}}$ is given by

$$l_e \triangleq \|\mathbf{w}_S - \mathbf{w}'_S\| = \|\boldsymbol{\mathcal{H}}_A\boldsymbol{\delta}_A + \boldsymbol{\mathcal{H}}_B\boldsymbol{\delta}_B\|, \tag{9}$$

where we define $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}}$ as the *difference vector* between the two distinct joint symbol vectors, i.e., $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}} \triangleq (\mathbf{w}_A^{\mathcal{T}}, \mathbf{w}_B^{\mathcal{T}})^{\mathcal{T}} - ((\mathbf{w}'_A)^{\mathcal{T}}, (\mathbf{w}'_B)^{\mathcal{T}})^{\mathcal{T}}$. Therefore, $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}} \neq \mathbf{0}$. Let $\delta_{A,l}$ and $\delta_{B,l}$ denote the lth element in $\boldsymbol{\delta}_A$ and $\boldsymbol{\delta}_B$, respectively. Since every element in $\mathbf{w}_A, \mathbf{w}_B$ belongs to $\mathbb{F}_q$, $\delta_{A,l}, \delta_{B,l} \in \Delta\mathcal{A} \triangleq \{1-q, \ldots, -1, 0, 1, \ldots q-1\}, \forall l \in \{1, 2, \ldots, L\}$, where $\mathcal{A}$ has been defined as the constellation of q-PAM. Then, $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}} \in \Delta\mathcal{A}^{2L}$ with $\boldsymbol{\delta}_A, \boldsymbol{\delta}_B \in \Delta\mathcal{A}^L$.

Given a particular $(\boldsymbol{\mathcal{H}}_A, \boldsymbol{\mathcal{H}}_B)$, we further define a collection of Euclidean distances induced by $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}} \in \Delta\mathcal{A}^{2L}$ as

$$\ell = \left\{ l_e \in \mathbb{R} | l_e = \|\boldsymbol{\mathcal{H}}_A \boldsymbol{\delta}_A + \boldsymbol{\mathcal{H}}_B \boldsymbol{\delta}_B\|, \forall (\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}} \in \Delta\mathcal{A}^{2L} \right\}.$$

We sort the elements in l_e in ascending order. Let $\tilde{\ell} \triangleq \{l_{e,1}, l_{e,2}, \ldots\}$ denote the ordered sequence, where $l_{e,i}$ denotes the ith smallest distance. Furthermore, $\tilde{\ell}$ contains only distinct elements in ℓ. As we defined in (4), $l_{e,1} = l_{\min}$. Note that at some $(\boldsymbol{\mathcal{H}}_A, \boldsymbol{\mathcal{H}}_B)$, it is possible that multiple difference vectors $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}}$ correspond to the same $l_{e,i}$.

Recall that the valid MIMO LPNC mapping can map any two joint symbol vectors to the same NC symbol vector $\mathbf{w}_N$ if and only if $\boldsymbol{\alpha}\mathbf{w}_A + \boldsymbol{\beta}\mathbf{w}_B = \boldsymbol{\alpha}\mathbf{w}'_A + \boldsymbol{\beta}\mathbf{w}'_B \pmod q$. In other words, a valid MIMO LPNC mapping under $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is said to *cluster* $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}}$ if and only if $\boldsymbol{\alpha}\boldsymbol{\delta}_A + \boldsymbol{\beta}\boldsymbol{\delta}_B = \mathbf{0} \pmod q$, where $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}} \triangleq (\mathbf{w}_A^{\mathcal{T}}, \mathbf{w}_B^{\mathcal{T}})^{\mathcal{T}} - ((\mathbf{w}'_A)^{\mathcal{T}}, (\mathbf{w}'_B)^{\mathcal{T}})^{\mathcal{T}}$.

Remark 1: The MIMO LPNC mapping under $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is *equivalent* to that under $(\boldsymbol{\alpha}', \mathbf{I}_{L\times L})$ where $\boldsymbol{\alpha}' \triangleq \boldsymbol{\beta}^{-1}\boldsymbol{\alpha}$. To see this, we revisit $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \triangleq \boldsymbol{\alpha}\mathbf{w}_A + \boldsymbol{\beta}\mathbf{w}_B \pmod q$. Since $\boldsymbol{\beta} \in GL_L(\mathbb{F}_q)$, $\boldsymbol{\beta}^{-1}$ exists. Therefore, $\boldsymbol{\beta}^{-1}\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \triangleq \boldsymbol{\beta}^{-1}(\boldsymbol{\alpha}\mathbf{w}_A + \boldsymbol{\beta}\mathbf{w}_B) = \boldsymbol{\beta}^{-1}\boldsymbol{\alpha}\mathbf{w}_A + \mathbf{w}_B \pmod q$. Note that $\boldsymbol{\beta}^{-1} \in GL_L(\mathbb{F}_q)$. For any two distinct $(\mathbf{w}_A^{\mathcal{T}}, \mathbf{w}_B^{\mathcal{T}})^{\mathcal{T}}$ and $((\mathbf{w}'_A)^{\mathcal{T}}, (\mathbf{w}'_B)^{\mathcal{T}})^{\mathcal{T}}$, $\boldsymbol{\alpha}\mathbf{w}_A + \boldsymbol{\beta}\mathbf{w}_B = \boldsymbol{\alpha}\mathbf{w}'_A + \boldsymbol{\beta}\mathbf{w}'_B \pmod q$ if and only if $\boldsymbol{\beta}^{-1}\boldsymbol{\alpha}\mathbf{w}_A + \mathbf{w}_B = \boldsymbol{\beta}^{-1}\boldsymbol{\alpha}\mathbf{w}'_A + \mathbf{w}'_B \pmod q$.

Remark 1 implies that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ has the same NC partition of $\mathcal{W}_{(A,B)}$ as $(\boldsymbol{\beta}^{-1}\boldsymbol{\alpha}, \mathbf{I})$.

In FRFD LPNC systems, the relay only needs to distinguish the superimposed symbol vectors that correspond to distinct NC symbol vectors. Therefore, the distance induced by the difference vectors that are clustered together are not of interest. Then, we define a distance metric relevant to the SER of NC symbol vectors in the FRFD LPNC mapping as follows:

$$d_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \triangleq \mathop{\arg\min}_{\substack{(\mathbf{w}_A,\mathbf{w}_B)\neq(\mathbf{w}'_A,\mathbf{w}'_B),\\ \boldsymbol{\alpha}\mathbf{w}_A+\boldsymbol{\beta}\mathbf{w}_B\neq\boldsymbol{\alpha}\mathbf{w}'_A+\boldsymbol{\beta}\mathbf{w}'_B \pmod q}} \|\mathbf{w}_S - \mathbf{w}'_S\|. \quad (10)$$

Note that $d_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ is the minimum distance among all pairs of $\mathbf{w}_S$ and $\mathbf{w}'_S$ that belong to different groups in (7). Recall that $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \neq \hat{\mathbf{w}}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ represents a decoding error of NC symbol vector. Then, the SER of $\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ is given by [10]

$$P_e^{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\mathbf{w}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \to \hat{\mathbf{w}}_N^{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \overset{\rho\to\infty}{\lessapprox} \frac{\mathcal{A}_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}}{q^L} Q\Big(\sqrt{\frac{\rho}{2\mu^2} d_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}}\Big), \quad (11)$$

where $\mathcal{A}_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ denotes a total multiplicity with respect to $d_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, $Q(x)$ denotes the Q-function, and $\rho = P/\sigma_z^2$. From (11), $d_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ dominates the SER of NC symbol vectors in the high SNR regime. The larger $d_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, the smaller SER. Furthermore, since $d_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ depends on $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, we focus on the optimal $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ such that

$$(\boldsymbol{\alpha}_{opt}, \boldsymbol{\beta}_{opt}) = \mathop{\arg\max}_{\boldsymbol{\alpha},\boldsymbol{\beta}\in GL_L(\mathbb{F}_q)} d_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}. \quad (12)$$

If we can find $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to cluster all $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}}$ that correspond to $l_{e,1}, l_{e,2}, \ldots l_{e,i}$ in (10), then $d_{\min}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} = l_{e,i+1}$. Therefore, for arbitrary $(\boldsymbol{\mathcal{H}}_A, \boldsymbol{\mathcal{H}}_B)$, we choose $(\boldsymbol{\alpha}_{opt}, \boldsymbol{\beta}_{opt})$ to maximizes i.

B. Solution of $(\boldsymbol{\alpha}_{opt}, \boldsymbol{\beta}_{opt})$ and Distance Analysis

This part specifies the solution of $(\boldsymbol{\alpha}_{opt}, \boldsymbol{\beta}_{opt})$ for arbitrary channel-gain pair. First, we investigate the optimal FRFD LPNC mapping for some channel-gain pairs at which $l_{\min} = 0$.

Definition 3: Consider a particular valid $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}}$. We have $l_{\min} = 0$ when $\boldsymbol{\mathcal{H}}_A^o \boldsymbol{\delta}_A + \boldsymbol{\mathcal{H}}_B^o \boldsymbol{\delta}_B = \mathbf{0}$. We refer to such $(\boldsymbol{\mathcal{H}}_A^o, \boldsymbol{\mathcal{H}}_B^o)$ as a *zero-$l_{\min}$ channel-gain pair.*

With respect to STBC at each node, *Proposition 1* below introduces a property of equivalent channel if the STBC can achieve full diversity in point-to-point MIMO systems.

Proposition 1: [15] Consider a point-to-point MIMO system. Let $\mathbf{S}$ be a linear dispersion STBC associated with an modulated symbol vector $\mathbf{x} \in \mathcal{A}^L$. Let $\boldsymbol{\mathcal{H}}$ be the equivalent channel of $\mathbf{S}$. Considering any two distinct STBC matrices $\mathbf{S}$ and $\mathbf{S}'$ associated with $\mathbf{x}$ and $\mathbf{x}'$ where $\mathbf{x} \neq \mathbf{x}'$, $\Delta\mathbf{S} \triangleq \mathbf{S} - \mathbf{S}'$ associated with $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}' \in \Delta\mathcal{A}^L$ has the full rank property for all $\Delta\mathbf{s} \in \Delta\mathcal{A}^L$ *if and only if* the column vectors of $\boldsymbol{\mathcal{H}}$ are linear independent over $\Delta\mathcal{A}^L$.

Given any *rational* zero-$l_{\min}$ channel-gain pair $(\boldsymbol{\mathcal{H}}_A^o, \boldsymbol{\mathcal{H}}_B^o)$,[2] we can find at least two superimposed symbol vectors overlapping with each other. Note that we may have multiple $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}}$ yielding $l_{\min} = 0$ at the same $(\boldsymbol{\mathcal{H}}_A^o, \boldsymbol{\mathcal{H}}_B^o)$, where all elements in $\boldsymbol{\mathcal{H}}_A^o$ and $\boldsymbol{\mathcal{H}}_B^o$ are rational. For example, we have $\boldsymbol{\mathcal{H}}_A^o \boldsymbol{\delta}_A + \boldsymbol{\mathcal{H}}_B^o \boldsymbol{\delta}_B = \boldsymbol{\mathcal{H}}_A^o \boldsymbol{\delta}'_A + \boldsymbol{\mathcal{H}}_B^o \boldsymbol{\delta}'_B = \mathbf{0}$, if $(\boldsymbol{\delta}'_A, \boldsymbol{\delta}'_B) = v(\boldsymbol{\delta}_A, \boldsymbol{\delta}_B)$ and $(\boldsymbol{\delta}'_A, \boldsymbol{\delta}'_B) \in \Delta\mathcal{A}^{2L}$ for some $v \in \mathbb{Z}$. Then, we define a set that collects all $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}}$ corresponding to $l_{\min} = 0$ as follows

$$\Delta^o \triangleq \{(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}} | \begin{pmatrix} \boldsymbol{\mathcal{H}}_A^o & \boldsymbol{\mathcal{H}}_B^o \end{pmatrix} \begin{pmatrix} \boldsymbol{\delta}_A \\ \boldsymbol{\delta}_B \end{pmatrix} = \mathbf{0}\}. \quad (13)$$

From (13), we remark that $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}} \in \Delta\mathcal{A}^{2L}$ lies in the null space of $(\boldsymbol{\mathcal{H}}_A^o \ \boldsymbol{\mathcal{H}}_B^o) \in \mathbb{R}^{L\times 2L}$. According to *Proposition 1*, all column vectors of $\boldsymbol{\mathcal{H}}_A^o$ or $\boldsymbol{\mathcal{H}}_B^o$ are linearly independent in $\Delta\mathcal{A}^L$, since $\|\boldsymbol{\mathcal{H}}_A^o \boldsymbol{\delta}_A\| \neq 0, \forall \boldsymbol{\delta}_A \in \Delta\mathcal{A}^L$ or $\|\boldsymbol{\mathcal{H}}_B^o \boldsymbol{\delta}_B\| \neq 0, \forall \boldsymbol{\delta}_B \in \Delta\mathcal{A}^L$. Therefore, by rank-nullity theorem, the nullity of $(\boldsymbol{\mathcal{H}}_A^o \ \boldsymbol{\mathcal{H}}_B^o)$ in $\Delta\mathcal{A}^{2L}$ is $2L - L = L$. Since the elements of $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}}$ can reach any values in $\Delta\mathcal{A}$, we can find exactly L linearly independent vectors of $(\boldsymbol{\delta}_A^{\mathcal{T}}, \boldsymbol{\delta}_B^{\mathcal{T}})^{\mathcal{T}} \in \Delta^o$ in the null space of $(\boldsymbol{\mathcal{H}}_1^o \ \boldsymbol{\mathcal{H}}_2^o)$, denoted by $\{(\bar{\boldsymbol{\delta}}_{A,1}^{\mathcal{T}}, \bar{\boldsymbol{\delta}}_{B,1}^{\mathcal{T}})^{\mathcal{T}}, (\bar{\boldsymbol{\delta}}_{A,2}^{\mathcal{T}}, \bar{\boldsymbol{\delta}}_{B,2}^{\mathcal{T}})^{\mathcal{T}}, \ldots, (\bar{\boldsymbol{\delta}}_{A,L}^{\mathcal{T}}, \bar{\boldsymbol{\delta}}_{B,L}^{\mathcal{T}})^{\mathcal{T}}\}$.

Proposition 2: At an arbitrary zero-$l_{\min}$ channel-gain pair $(\boldsymbol{\mathcal{H}}_A^o \ \boldsymbol{\mathcal{H}}_B^o)$, the associated $\boldsymbol{\Phi}_A$ and $\boldsymbol{\Phi}_B$ are both full rank, where $\boldsymbol{\Phi}_A \triangleq (\bar{\boldsymbol{\delta}}_{A,1}, \ldots \bar{\boldsymbol{\delta}}_{A,L})$ and $\boldsymbol{\Phi}_B \triangleq (\bar{\boldsymbol{\delta}}_{B,1}, \ldots \bar{\boldsymbol{\delta}}_{B,L})$.

Proof of Proposition 2: W.l.o.g, suppose that $\boldsymbol{\Phi}_A$ is full rank and $\boldsymbol{\Phi}_B$ is rank deficient. From *Proposition 1*, we can deduce that the column vectors of $\boldsymbol{\mathcal{H}}_A^o \boldsymbol{\Phi}_A \in \mathbf{R}^{L\times L}$ are linearly independent in $\Delta\mathcal{A}^L$, since $\boldsymbol{\Phi}_A$ is full rank. Furthermore, the column vectors of $\boldsymbol{\mathcal{H}}_B^o \boldsymbol{\Phi}_B \in \mathbb{R}^{L\times L}$ are linearly dependent in $\Delta\mathcal{A}^L$ because of a rank-deficient $\boldsymbol{\Phi}_B$. It contradicts with $\boldsymbol{\mathcal{H}}_A^o \boldsymbol{\Phi}_A = -\boldsymbol{\mathcal{H}}_B^o \boldsymbol{\Phi}_B$. Therefore, $\boldsymbol{\Phi}_A$ and $\boldsymbol{\Phi}_B$ are both full rank. ■

[2] Mathematically, if one of the elements in $\boldsymbol{\mathcal{H}}_A^o$ and $\boldsymbol{\mathcal{H}}_B^o$ are irrational, it is not possible to find a $(\boldsymbol{\delta}_A, \boldsymbol{\delta}_B) \in \Delta\mathcal{A}^{2L}$ to satisfy $\boldsymbol{\mathcal{H}}_A^o \boldsymbol{\delta}_A + \boldsymbol{\mathcal{H}}_B^o \boldsymbol{\delta}_B = 0$, since $\Delta\mathcal{A} \in \mathbb{Z}$. In real communications systems, channel gains are represented by rational values due to finite resolution in processors.

Theorem 1: At a particular zero-$l_{\min}$ channel-gain pair, the optimal FRFD LPNC mapping is given by

$$(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt}) = (-\boldsymbol{\Phi}_B^{-1}\boldsymbol{\Phi}_A, \mathbf{I}_{L\times L}) \pmod q, \tag{14}$$

where $\boldsymbol{\Phi}_A \triangleq (\bar{\boldsymbol{\delta}}_{A,1},\ldots\bar{\boldsymbol{\delta}}_{A,L})$ and $\boldsymbol{\Phi}_B \triangleq (\bar{\boldsymbol{\delta}}_{B,1},\ldots\bar{\boldsymbol{\delta}}_{B,L})$. Then, $d_{\min}^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})} = l_{e,2}$.

Proof of Theorem 1: Generally, we need to cluster all difference pairs in Δ_o at $(\boldsymbol{\mathcal{H}}_1^o\ \boldsymbol{\mathcal{H}}_2^o)$. Otherwise, $d_{\min}^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})} = l_{\min} = 0$. First, we choose $(\boldsymbol{\alpha},\boldsymbol{\beta})$ to cluster the group of L linearly independent difference vectors in $(\boldsymbol{\Phi}_A^{\mathcal{T}}\ \boldsymbol{\Phi}_B^{\mathcal{T}})^{\mathcal{T}}$ that span the null space of $(\boldsymbol{\mathcal{H}}_1^o\ \boldsymbol{\mathcal{H}}_2^o)$ in $\Delta\mathcal{A}^{2L}$. By *Proposition 2*, both $\boldsymbol{\Phi}_A$ and $\boldsymbol{\Phi}_B$ are full rank. Therefore, there exists a solution of $(\boldsymbol{\alpha},\mathbf{I}_{L\times L}) = (-\boldsymbol{\Phi}_B^{-1}\boldsymbol{\Phi}_A,\mathbf{I}_{L\times L}) \pmod q$ such that $\boldsymbol{\alpha}\boldsymbol{\Phi}_A + \boldsymbol{\Phi}_B = \mathbf{0} \pmod q$. By *Remark 1*, other solutions satisfying $\boldsymbol{\alpha}\boldsymbol{\Phi}_A + \boldsymbol{\beta}\boldsymbol{\Phi}_B = \mathbf{0} \pmod q$ are equivalent to the optimal solution in (14). Then, this $(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})$ can also cluster the other difference vectors in Δ^o, since they can be expressed as the linear combinations of $(\boldsymbol{\Phi}_A^{\mathcal{T}}\ \boldsymbol{\Phi}_B^{\mathcal{T}})^{\mathcal{T}}$ in $\Delta\mathcal{A}^{2L}$.

Once $(\boldsymbol{\Phi}_A^{\mathcal{T}}\ \boldsymbol{\Phi}_B^{\mathcal{T}})^{\mathcal{T}}$ are clustered, there is no further freedom for $(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})$ to cluster other difference vectors that are linearly independent of $(\boldsymbol{\Phi}_A^{\mathcal{T}}\ \boldsymbol{\Phi}_B^{\mathcal{T}})^{\mathcal{T}}$, since the nullity of $(\boldsymbol{\Phi}_A^{\mathcal{T}}\ \boldsymbol{\Phi}_B^{\mathcal{T}})^{\mathcal{T}}$ is only L. Furthermore, $d_{\min}^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})} = l_{e,2}$, since $(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})$ can only cluster all difference vectors associated with $l_{\min}$ (i.e., $l_{e,1}$). This completes the proof. ■

Remark 2: In our numerical results, we found that a common scenario is that $l_{\min} = 0$ rarely occurs because of random channel coefficients. However, this does not mean that it is meaningless to investigate the FRFD LPNC mapping for $l_{\min} = 0$. Note that when the channel-gain pair deviates from the zero-$l_{\min}$ channel-gain pairs a little, we may come across the cases at which $l_{\min}$ is very small and close to zero. In such cases, it is possible that the nonzero $l_{\min}$ channel-gain pairs still correspond to the group of linearly independent column vectors of $(\boldsymbol{\Phi}_A^{\mathcal{T}}\ \boldsymbol{\Phi}_B^{\mathcal{T}})^{\mathcal{T}}$ for $l_{\min} = 0$.

In the following, we consider a $(\boldsymbol{\mathcal{H}}_A\ \boldsymbol{\mathcal{H}}_B)$ that corresponds to $l_{\min} \neq 0$. Note that it is possible to have multiple difference vectors yielding the same $l_{\min}$. Suppose that at this $(\boldsymbol{\mathcal{H}}_A\ \boldsymbol{\mathcal{H}}_B)$, $l_{\min} \neq 0$. We can find multiple difference vectors $\{(\bar{\boldsymbol{\delta}}_{A,1}^{\mathcal{T}},\bar{\boldsymbol{\delta}}_{B,1}^{\mathcal{T}})^{\mathcal{T}},\ldots,(\bar{\boldsymbol{\delta}}_{A,r}^{\mathcal{T}},\bar{\boldsymbol{\delta}}_{B,r}^{\mathcal{T}})^{\mathcal{T}}\}$ corresponding to this nonzero $l_{\min}$, i.e., $l_{\min} = \|\boldsymbol{\mathcal{H}}_A\bar{\boldsymbol{\delta}}_{A,i} + \boldsymbol{\mathcal{H}}_B\bar{\boldsymbol{\delta}}_{B,i}\|, \forall i \in \{1,2,\ldots,r\}$. Moreover, these r difference vectors are linearly independent to each other. Let $\boldsymbol{\Phi}_A \triangleq (\bar{\boldsymbol{\delta}}_{A,1},\ldots\bar{\boldsymbol{\delta}}_{A,r})$ and $\boldsymbol{\Phi}_B \triangleq (\bar{\boldsymbol{\delta}}_{B,1},\ldots\bar{\boldsymbol{\delta}}_{B,r})$. First, the FRFD LPNC mapping under $(\boldsymbol{\alpha}_{opt},\mathbf{I}_{L\times L})$ shall cluster all $(\bar{\boldsymbol{\delta}}_{A,i}^{\mathcal{T}},\bar{\boldsymbol{\delta}}_{B,i}^{\mathcal{T}})^{\mathcal{T}}$, i.e., $\boldsymbol{\alpha}_{opt}\bar{\boldsymbol{\delta}}_{A,i} + \bar{\boldsymbol{\delta}}_{B,i} = \mathbf{0} \pmod q, \forall i \in \{1,2,\ldots,r\}$. Otherwise, $d_{\min}^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})} = l_{\min}$. Second, we investigate how the value of r effects the selection of $(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})$ as follows:

- $r > L$: This case is not possible, since the nullity of the valid $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is L.
- $r = L$: In this case, similar to the proof of *Theorem 1*, it is possible to choose $(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt}) = (-\boldsymbol{\Phi}_B^{-1}\boldsymbol{\Phi}_A,\mathbf{I}_{L\times L}) \pmod q$ to cluster the group of difference vectors $(\boldsymbol{\Phi}_A,\boldsymbol{\Phi}_B)$. Thus, $d_{\min}^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})} = l_{e,2}$.
- $r < L$: In this case, the MIMO LPNC mapping has more freedom to cluster not only $(\boldsymbol{\Phi}_A^{\mathcal{T}}\ \boldsymbol{\Phi}_B^{\mathcal{T}})^{\mathcal{T}}$ but also another difference vectors that are linearly independent of $(\boldsymbol{\Phi}_A^{\mathcal{T}}\ \boldsymbol{\Phi}_B^{\mathcal{T}})^{\mathcal{T}}$. Suppose the extra linearly independent difference vectors are denoted by $(\boldsymbol{\delta}_{A,r+1}^{\mathcal{T}},\boldsymbol{\delta}_{B,r+1}^{\mathcal{T}})^{\mathcal{T}},\ \ldots,(\boldsymbol{\delta}_{A,r'}^{\mathcal{T}},\boldsymbol{\delta}_{B,r'}^{\mathcal{T}})^{\mathcal{T}}$. Because of the nullity of $(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})$, we have $r' = L$. In this case, we have $d_{\min}^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})} \geq l_{e,2}$.

IV. Diversity Analysis of FRFD LPNC

In this section, we verify that the optimal FRFD LPNC scheme can achieve full diversity by deriving the lower and upper bounds for $P_e^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})}$ respectively.

Theorem 2: The proposed optimal FDFR LPNC can achieve full diversity in the space-time coded TWRN.

Proof of Theorem 2: Upper bound on $P_e^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})}$: From Section III-B, we can find $(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})$ to ensure that $d_{\min}^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})} > l_{\min}$. Therefore, the average SER of CD is the upper bound on $P_e^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})}$ in the high SNR regime.

Let $P_e^{CD}(\boldsymbol{\delta}_A,\boldsymbol{\delta}_B)$ denote the average SER of CD. To derive $P_e^{CD}(\boldsymbol{\delta}_A,\boldsymbol{\delta}_B)$, we first investigate the conditional SER given $(\boldsymbol{\mathcal{H}}_A,\boldsymbol{\mathcal{H}}_B)$ as follows

$$\begin{aligned} P_e^{CD}(\boldsymbol{\delta}_A,\boldsymbol{\delta}_B|\boldsymbol{\mathcal{H}}_A,\boldsymbol{\mathcal{H}}_B) &= Q\left(\sqrt{\frac{\rho}{2\mu^2}\|\boldsymbol{\mathcal{H}}_A\boldsymbol{\delta}_A + \boldsymbol{\mathcal{H}}_A\boldsymbol{\delta}_B\|^2}\right) \\ &= Q\left(\sqrt{\frac{\rho}{2\mu^2}\|(\Delta\mathbf{S}_A\ \Delta\mathbf{S}_B)(\mathbf{H}_A^{\mathcal{T}}\ \mathbf{H}_B^{\mathcal{T}})^{\mathcal{T}}\|^2}\right), \end{aligned} \tag{15}$$

where $\Delta\mathbf{S}_i$ denotes the STBC difference matrix associated with $\boldsymbol{\delta}_i$, $i \in \{A,B\}$. Let $\Delta\mathbf{S}_{A,B} \triangleq (\Delta\mathbf{S}_A\ \Delta\mathbf{S}_B)$ and $\mathbf{H}_{A,B} \triangleq (\mathbf{H}_A^{\mathcal{T}}\ \mathbf{H}_B^{\mathcal{T}})^{\mathcal{T}}$. Further, we have that

$$\begin{aligned} &\|(\Delta\mathbf{S}_A\ \Delta\mathbf{S}_B)(\mathbf{H}_A^{\mathcal{T}}\ \mathbf{H}_B^{\mathcal{T}})^{\mathcal{T}}\|^2 = \|\Delta\mathbf{S}_{A,B}\mathbf{H}_{A,B}\|^2 \\ &= \mathrm{tr}\{(\mathbf{U}^{\mathcal{H}}\mathbf{H}_{\mathrm{A,B}})^{\mathcal{H}}\Lambda\mathbf{U}^{\mathcal{H}}\mathbf{H}_{\mathrm{A,B}}\}, \end{aligned} \tag{16}$$

where ${}^{\mathcal{H}}$ denotes the the conjugate transpose. The second equality holds since $\Delta\mathbf{S}_{A,B} = \mathbf{U}\Lambda\mathbf{U}^{\mathcal{H}}$, i.e., singular value decomposition of $\Delta\mathbf{S}_{A,B}^{\mathcal{H}}\Delta\mathbf{S}_{A,B} \in \mathbb{R}^{4\times 4}$, where $\mathbf{U}$ is a unitary matrix and the diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ with $\lambda_i > 0, i = 1,2,\ldots,4$. Note that $\Delta\mathbf{S}_{A,B}$ is full rank, since both of $\Delta\mathbf{S}_A$ and $\Delta\mathbf{S}_B$ are full rank of 2, $\forall\boldsymbol{\delta}_A \neq 0$ and $\forall\boldsymbol{\delta}_B \neq 0$. Therefore, $\lambda_1,\lambda_2 > 0$ and $\lambda_1 = \lambda_2 = 0$. Let $\tilde{\mathbf{H}}_{A,B} \triangleq (\tilde{\mathbf{H}}_A^{\mathcal{T}}\ \tilde{\mathbf{H}}_B^{\mathcal{T}})^{\mathcal{T}} \triangleq \mathbf{U}^{\mathcal{H}}\mathbf{H}_{A,B}$. Then, we have

$$\mathrm{tr}\{(\tilde{\mathbf{H}}_{A,B})^{\mathcal{H}}\Lambda\tilde{\mathbf{H}}_{A,B}\} \geq \mathrm{tr}(\lambda_{\min}\tilde{\mathbf{H}}_A^{\mathcal{H}}\tilde{\mathbf{H}}_A) \geq \gamma\sum_{i=1}^{2}\sum_{j=1}^{2}\tilde{h}_{i,j}^2, \tag{17}$$

where $\lambda_{\min} \triangleq \min\{\lambda_1,\lambda_2\} > 0$. Since $\{\Delta\mathbf{S}_{A,B}\}$ is a finite set for all $\boldsymbol{\delta}_A \neq 0$ or $\boldsymbol{\delta}_B \neq 0$, we can find a positive γ such that $\gamma = \min_{\Delta\mathbf{S}_{A,B}}\lambda_{\min}$. Note that γ is independent of ρ. Following [15, *Lemma 3*], we can approximate (15) as $P_e^{CD}(\boldsymbol{\delta}_A,\boldsymbol{\delta}_B) \overset{\rho\to\infty}{\leq} G_c^{CD}\rho^{-2}$, where G_c^{CD} is a constant and independent of ρ. Therefore, TWRN with CD can achieve the full diversity order of 2.

Lower bound on $P_e^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})}$: Consider an *interference-free* model by decoupling TWRN to a pair of one-way relay channels, where nodes A and B transmit their packets to relay R in different time slots. In this model, R can decode the packet from node A without the interference from node B. Obviously, the average SER of interference-free model is the lower bound on $P_e^{(\boldsymbol{\alpha}_{opt},\boldsymbol{\beta}_{opt})}$. Recall that the universal STBC

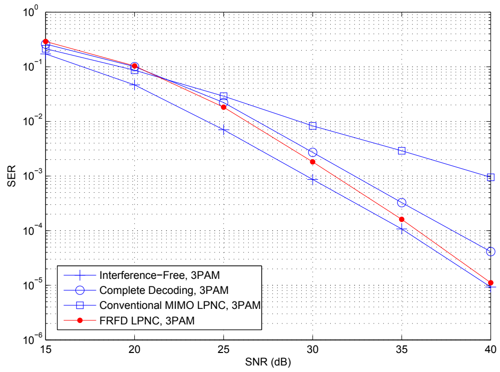

Fig. 1. SER of 3-PAM optimal FRFD LPNC, interference-free model, and TWRN with CD.

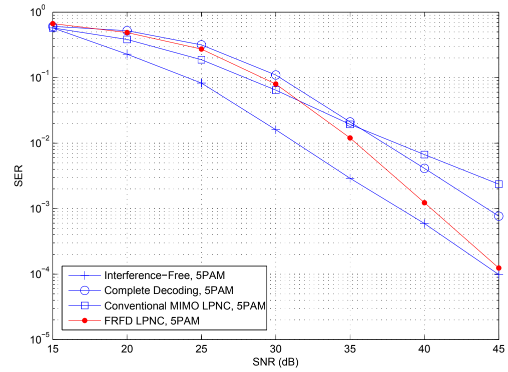

Fig. 2. SER of 5-PAM optimal FRFD LPNC, interference-free model, and TWRN with CD.

of (3) can harvest full diversity in the point-to-point MIMO. Therefore, the average SER of interference-free mode can be expressed as $P_e^{IF}(\boldsymbol{\delta}_A) \overset{\rho\to\infty}{\leq} G_c^{IF}\rho^{-2}$, where G_c^{IF} is the coding gain of universal STBC [12].

Therefore, we conclude that the FRFD LPNC system can achieve full diversity, since its lowerbound and upperbound can achieve diversity order of 2 respectively. ■

V. Simulation Results

This section considers a real-valued MIMO TWRN with two antennas at each node and the relay. The full diversity order of this system is 2.

Figs. 1 and 2 show the SER of optimal FRFD LPNC, interference-free model, TWRN with complete decoding, and conventional MIMO LPNC [10] (without STBC at source nodes), respectively. Figs. 1 and 2 consider 3-PAM and 5-PAM respectively. First, we observe that FRFD LPNC, interference-free model, and TWRN with CD can achieve full diversity order of 2 in the high SNR regime, consistent with our analytical results in *Section IV*, while conventional MIMO LPNC cannot harvest the full diversity. Moreover, we see that the SER of interference-free model and TWRN with CD are lower and upper bounds on that of FRFD LPNC in the high SNR regime, respectively. This also agrees with our analytical results in *Section IV*. As SNR goes high, the SER of FRFD LPNC outperforms that of TWRN with CD by up to 5dB in Fig. 2, and approaches that of interference-free model.

VI. Conclusion

We have put forth a novel MIMO LPNC scheme for full rate and full diversity TWRN. At source nodes, universal STBCs are applied to guarantee the full-rate transmission. At the relay, we formulated the FRFD LPNC mapping under finite field operation, building on top of the conventional MIMO LPNC. In particular, we strictly solved the optimal LPNC mapping for arbitrary equivalent channel-gain pairs. According to our distance analysis, we verified that the optimal FDFR LPNC scheme can harvest full diversity in the high SNR regime. Furthermore, we show that the SER of optimal FDFR LPNC outperforms that of conventional MIMO LPNC and TWRN with complete decoding in the high SNR regime. The spirit of FDFR LPNC is also applicable to other multiuser communication scenarios such as distributed MIMO systems and multiple access networks.

References

[1] S. Zhang, S. C. Liew, and P. P. Lam, "Hot topic: physical-layer network coding," in *Proc. ACM Mobicom*, Sep. 2006.

[2] S. C. Liew, S. Zhang, and L. Lu, "Physical-layer network coding: tutorial, survey, and beyond," *Phys. Commun.*, vol. 6, pp. 4–42, Mar. 2013.

[3] P. Popovski and H. Yomo, "The anti-packets can increase the achievable throughput of a wireless multi-hop network," in *Proc. IEEE ICC 2006*.

[4] T. Yang and I. Collings, "On the optimal design and performance of linear physical-layer network coding for fading two-way relay channels," *IEEE Trans. Wireless Commun.*, vol. 13, no. 7, pp. 956–967, Feb. 2014.

[5] L. Shi, S. C. Liew, and L. Lu, "On the subtleties of q-PAM linear physical-layer network coding," *IEEE Trans. Inf. Theory*, vol. 62, no. 5, pp. 2520–2544, May 2016.

[6] D. Fang, A. Bur, and J. Yuan, "Linear physical-layer network coding over hybrid finite ring for Rayleigh fading two-way relay channels," *IEEE Trans. Commun.*, vol. 62, no. 9, pp. 3249–3261, Sep. 2014.

[7] L. Yang, T. Yang, J. Yuan, and J. An, "Achieving the near-capacity of two-way relay channels with modulation-coded physical-layer network coding," *IEEE Trans. Wireless Commun.*, vol. 14, no. 9, pp. 5225–5239, May 2016.

[8] B. Nazer and M. Gastpar, "Compute-and-forward: harnessing interference through structured codes," *IEEE Trans. Inf. Theory*, vol. 57, no. 10, pp. 6463–6485, Oct. 2011.

[9] T. Yang, Q. T. Sun, J. A. Zhang, and J. Yuan, "A linear network coding approach for uplink distributed MIMO systems: protocol and outage behavior," *IEEE J. Sel. Areas Commun.*, vol. 33, no. 2, pp. 250–263, Feb 2015.

[10] J. Guo, T. Yang, J. Yuan, and J. Zhang, "Linear vector physical-layer network coding for MIMO two-way relay channels design and performance analysis," *IEEE Trans. Commun.*, vol. 63, no. 7, pp. 2591–2604, July 2015.

[11] O. Ordentlich and U. Erez, "Precoded integer-forcing universally achieves the MIMO capacity to within a constant gap," *IEEE Trans. Inf. Theory*, vol. 61, no. 1, pp. 323–340, Jan. 2015.

[12] H. E. Gamal and M. O. Damen, "Universal space-time coding," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1097–1119, May 2003.

[13] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communications: performance criterion and code construction," *IEEE Trans. Inf. Theory*, vol. 44, no. 2, pp. 744–765, Mar. 1998.

[14] B. Hassibi and B. Hochwald, "High-rate codes that are linear in space and time," *IEEE Trans. Inf. Theory*, vol. 48, no. 7, pp. 1804–1824, Jul. 2002.

[15] X. Guo and X.-G. Xia, "On full diversity space-time block dodes with partial interference cancellation group decoding," *IEEE Trans. Inf. Theory*, vol. 55, no. 10, pp. 4366–4385, Oct. 2009.