FACTOR DISTRIBUTIONS IMPLIED BY QUOTED CDO SPREADS AND TRANCHE PRICING

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ABSTRACT. The rapid pace of innovation in the market for credit risk has given rise to a liquid market in synthetic collateralised debt obligation (CDO) tranches on standardised portfolios. To the extent that tranche spreads depend on default dependence between different obligors in the reference portfolio, quoted spreads can be seen as aggregating the market views on this dependence. In a manner reminiscent of the volatility smiles found in liquid option markets, practitioners speak of implied correlation "smiles" and "skews" . We explore how this analogy can be taken a step further to extract implied factor distributions from the market quotes for synthetic CDO tranches.

1. INTRODUCTION

"Implied correlation" has become a buzzword in portfolio credit risk modelling. In a development similar to implied volatility for vanilla options, a model parameter affecting derivative financial instruments, but not directly observable in the market, is backed out from derivatives prices as those derivatives become competitively quoted. The derivatives in question are synthetic CDO tranches and the parameter is correlation in a Gaussian single factor model of default, essentially along the lines of Vasicek (1987) and Li (2000). This model has become the point of reference when pricing portfolio credit derivatives, in this sense much like the Black and Scholes (1973) model in option pricing. Even more so than in the case of Black/Scholes, however, the severe limitations of this model are recognised by practitioners. Given these limitations, it is unsurprising that it cannot consistently fit the market, i.e. for different tranches on the same portfolio, different values of the correlation parameter are required in order to fit observed tranche spreads. This has led market practitioners to speak of implied correlation "smiles" and "skews", evoking an analogy to the volatility smiles found in vanilla option markets.

Several alternative approaches to modelling portfolio credit risk, as required for pricing CDO tranches, have been proposed. These include lifting the structural (asset–based) models pioneered by Black and Scholes (1973) and Merton (1974) to the portfolio level, for example as illustrated by Hull, Predescu and White (2005). The intensity–based modelling approach initially proposed by Jarrow and Turnbull (1995) can also be applied at the portfolio level, by modelling dependent default intensities as in Duffie and Gârleanu (2001) or using a more general copula–based framework as introduced by Schönbucher and Schubert (2001). Most recently, a new methodology for the pricing of portfolio credit derivatives has been proposed by Schönbucher (2006), who takes a "top–down" approach (as opposed to the "bottom–up" approach of the aforementioned papers) to directly model

Date. This version: April 23, 2008. First version: May 17, 2004.
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the stochastic dynamics of portfolio losses (and the associated loss transition rates) in an arbitrage–free manner.[1]

The choice of approach to construct a model fitting observed tranche quotes primarily depends on the application envisioned for the calibrated model. The point of view that we take in the present paper is that we wish to price related (but illiquid) instruments in a manner consistent with the market quotes for standard synthetic tranches on standard reference portfolios (such as CDX or iTraxx). We want to be reasonably satisfied that the calibrated model subsumes all the market information relevant to the pricing of the illiquid instruments. In this, the problem is somewhat more complicated than calibrating, say, a single–name equity option model to observed standard option prices.

Since we are concerned with relative pricing of similar instruments, we abstract from the fundamental asset values and use credit spreads (for single names as well as for competitively quoted synthetic CDO tranches) directly as inputs. Furthermore, the results of Burtschell, Gregory and Laurent (2005) suggest that credit spread *dynamics* are of minor importance in pricing CDO tranches. Consequently, the simplest solution would be to modify the Vasicek/Li static factor model to fit observed tranche spreads.

The normal distribution of the common factor in the Vasicek model implies a Gaussian dependence structure (copula) between the latent variables driving the defaults of the various obligors. Numerous authors have suggested replacing this by a Student–t, a Marshall/Olkin or various types of Archimedean copulae.[2] Burtschell, Gregory and Laurent (2005) compare a selection of these models; in their study, the best fit to market data seems to be achieved by the double t one–factor model of Hull and White (2004).

The basic model can be extended in various ways, thus introducing additional parameters, which facilitate an improved fit. One obvious way to introduce further degrees of freedom into the model is to allow the systematic factor loadings (corresponding to the constant correlation parameter in the reference Vasicek model) to vary across obligors. However, doing so without any structural assumptions results in more than 100 free parameters in the typical case of a CDX or iTraxx portfolio, making any meaningful calibration impossible. Mashal, Naldi and Tejwani (2004) suggest bringing the number of free parameters back down to one taking historical correlations as an input and scaling all correlations by a constant chosen to fit the market as well as possible. Between these two extremes, intermediate solutions could be achieved by perturbing one or more eigenvectors of the historical variance/covariance matrix.[3]

Andersen and Sidenius (2005) pursue two possible extensions, one which allows for random recovery rates and another, in which the factor loadings are random. They motivate this by the stylised empirical observations that recovery rates are correlated with the business cycle and that default correlation appears stronger in a bear market. In particular for the latter case, examples are given where the model produces implied correlation skews qualitatively similar to those observed in the market.

We pursue a third path, seeking to imply the underlying factor distribution (and thereby the distribution of conditional default probabilities) directly from the market quotes for

[1]Schönbucher (2006) demonstrates how his model can be disaggregated from the portfolio level all the way down to the level of multivariate, intensity–driven dynamics of the defaults of individual obligors, thus providing a general framework for the modelling of credit derivatives. The aim of the present paper is far less ambitious.

[2]See Burtschell, Gregory and Laurent (2005) and references therein.

[3]Mashal, Naldi and Tejwani (2004) mention only scaling all correlations by a single constant because of the attraction of being able to quote a single number, the "implied correlation bump," as representing the default dependence implied from market quotes.

synthetic CDO tranches. In this, we build on the well–known methods of implying risk–neutral distributions from option prices, a strand of the literature initiated by the seminal paper of Breeden and Litzenberger (1978).[4]

As the normal factor distribution used by Vasicek (1987) and Li (2000) remains the benchmark for pricing CDO tranches, this seems a natural starting point for a factor distribution calibrated to market data. The Edgeworth and Gram/Charlier Type A series[5] expand a distribution around the normal in terms of higher order moments. In the case of risk–neutral distributions implied by standard option prices, this is an approach that is well known in the literature,[6] where typically the series is truncated after the fourth moment (representing kurtosis). Jondeau and Rockinger (2001) show how one can ensure that the truncated series yield a valid density. In the sections that follow, we derive the theoretical results required to implement CDO tranche pricing where the common factor follows a Gram/Charlier density, fit this density to market data and apply the model to the pricing of general tranches on standard portfolios.

2. Modelling

1. Assumption. *Along the lines of Vasicek (1987), assume that the latent variable ζ_i driving the default (or survival) of the i–th obligor can be written as*

$$\zeta_i = \beta_i Y + \sqrt{1-\beta_i^2}\epsilon_i \tag{1}$$

where Y, ϵ_1, ..., ϵ_M are independent, $\epsilon_i \sim \mathcal{N}(0,1)$ and (departing from Vasicek's normality assumption) the distribution of Y is given by a Gram/Charlier Type A series expansion in the standard measure, i.e. the density f of Y is given by

$$\begin{aligned} f(x) &= \sum_{j=0}^{\infty} c_j \mathrm{He}_j(x)\phi(x) \\ c_r &= \frac{1}{r!}\int_{-\infty}^{\infty} f(x)\mathrm{He}_r(x)dx \\ \phi(x) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \end{aligned} \tag{2}$$

where $\mathrm{He}_j(x)$ denotes the Hermite polynomial[7] of order j.

Default of obligor i is considered to have occurred before time t if the latent variable ζ_i lies below the threshold $D_i(t)$.

Note that in this context, a large homogeneous portfolio (LHP) approximation is easy to derive. Follow Vasicek (1987) and consider a large homogeneous portfolio of M issuers. Homogeneity of the the portfolio means that, in addition to the ζ_i being identically distributed, the exposures to each obligor in the portfolio are the same, as are the recovery rates R and the correlation (β_i) with the common factor.[8] In this case, the randomness

[4]See Bahra (1997) for an overview.

[5]See e.g. Kendall and Stuart (1969).

[6]See e.g. Jackwerth and Rubinstein (1996), Corrado and Su (1996) and Jurczenko, Maillet and Negrea (2002).

[7]See Definition 3 in the appendix.

[8]Note that some authors write (1) as $\zeta_i = \sqrt{\rho_i}Y + \sqrt{1-\rho_i}\epsilon_i$, in which case $\sqrt{\rho_i}\sqrt{\rho_j}$ is the correlation between the latent variables for obligors i and j. Then in the homogeneous case ($\rho_i = \rho_j = \rho$), this correlation between latent variables is simply ρ.

due to the idiosyncratic risk factors ϵ_i diversifies out as the size M of the portfolio grows large. In the limit, given the value of the systematic risk factor Y, the loss fraction L on the portfolio notional is[9]

$$L \approx (1-R)\Phi\left(\frac{D-\beta Y}{\sqrt{1-\beta^2}}\right) \tag{3}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution. Setting

$$h(x) = (1-R)\Phi\left(\frac{x}{\sqrt{1-\beta^2}}\right) \tag{4}$$

the CDF of the portfolio loss fraction can be expressed as

$$P[L \le \theta] = 1 - F\left(\frac{D-h^{-1}(\theta)}{\beta}\right) \tag{5}$$

where $F(\cdot)$ is the CDF corresponding to the density $f(\cdot)$ given by (2).

The key result needed in order to implement the factor model of Assumption 1 is an explicit relationship between the default thresholds $D_i(t)$ and the (risk–neutral) probability of default of obligor i:

2. Proposition. *Under Assumption 1,*

$$P[\zeta_i \le D_i(t)] = \Phi(D_i(t)) - \sum_{j=1}^{\infty} \beta_i^j c_j \phi(D_i(t))\mathrm{He}_{j-1}(D_i(t)) \tag{6}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and density, respectively, of the standard normal distribution.

Proof: We first derive the density $g(\cdot)$ of ζ_i.

$$\begin{aligned}
g(x) &= \frac{\partial}{\partial x}P[\zeta_i \le x] \\
&= \frac{\partial}{\partial x}\int_{-\infty}^{\infty}\int_{-\infty}^{\frac{x-\beta_i y}{\sqrt{1-\beta_i^2}}}\phi(t)dtf(y)dy \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x-\beta_i y)^2}{2(1-\beta_i^2)}\right\}\frac{1}{\sqrt{1-\beta_i^2}}\sum_{j=0}^{\infty}c_j\mathrm{He}_j(y)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\
&= \exp\left\{-\frac{1}{2(1-\beta_i^2)}(x^2 - x^2\beta_i^2)\right\} \\
&\qquad \int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-\beta_i^2}}\exp\left\{-\frac{1}{2(1-\beta_i^2)}(y^2-2x\beta_i y + x^2\beta_i^2)\right\}\sum_{j=0}^{\infty}c_j\mathrm{He}_j(y)dy \\
&= e^{-\frac{x^2}{2}}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-\beta_i^2}}\exp\left\{-\frac{(y-\beta_i x)^2}{2(1-\beta_i^2)}\right\}\sum_{j=0}^{\infty}c_j\mathrm{He}_j(y)dy
\end{aligned}$$

[9] cf. O'Kane and Schlögl (2005)

Setting $z := (y - x\beta_i)/\sqrt{1-\beta_i^2}$, this is

$$= e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{z^2}{2}} \sum_{j=0}^{\infty} c_j \mathrm{He}_j(\sqrt{1-\beta_i^2}z + x\beta_i)dz$$

and applying Lemma 4,[10]

$$= \phi(x) \int_{-\infty}^{\infty} \phi(z) \sum_{k=0}^{\infty} c_k$$

$$\sum_{j=0}^{k} \binom{k}{j} (\sqrt{1-\beta_i^2})^{k-j} \mathrm{He}_{k-j}(z) j! \sum_{m=0}^{[\frac{j}{2}]} \frac{1}{m! 2^m (j-2m)!} (1-\beta_i^2)^m \mathrm{He}_{j-2m}(x\beta_i) dz$$

which by the orthogonality property of Hermite polynomials[11] simplifies to

$$= \phi(x) \sum_{k=0}^{\infty} c_k k! \sum_{m=0}^{[\frac{k}{2}]} \frac{1}{m! 2^m (k-2m)!} (1-\beta_i^2)^m \mathrm{He}_{k-2m}(x\beta_i)$$

Reordering terms, this becomes

$$= \phi(x) \sum_{j=0}^{\infty} \mathrm{He}_j(x\beta_i) \underbrace{\sum_{m=0}^{\infty} c_{j+2m}(j+2m)! \frac{1}{m! 2^m j!} (1-\beta_i^2)^m}_{=:d_j}$$

Applying Corollary 5 (see appendix),

$$= \phi(x) \sum_{j=0}^{\infty} d_j j! \sum_{m=0}^{[\frac{j}{2}]} \beta_i^{j-2m} \mathrm{He}_{j-2m}(x) \frac{(\beta_i^2 - 1)^m}{(j-2m)! 2^m m!}$$

$$= \phi(x) \sum_{k=0}^{\infty} \mathrm{He}_k(x) \frac{\beta_i^k}{k!} \sum_{m=0}^{\infty} d_{k+2m}(k+2m)! \frac{(\beta_i^2-1)^m}{2^m m!} \tag{7}$$

Consider the term

$$\sum_{m=0}^{\infty} d_{k+2m}(k+2m)! \frac{(\beta_i^2-1)^m}{2^m m!}$$

$$= \sum_{m=0}^{\infty} (k+2m)! \frac{(\beta_i^2-1)^m}{2^m m!} \sum_{n=0}^{\infty} c_{k+2m+2n}(k+2m+2n)! \frac{1}{n! 2^n (k+2m)!} (1-\beta_i^2)^n$$

[10]Reproduced from Schlögl (2008) for the reader's convenience in the appendix. Note that this Lemma is a special case of scaling and translation results well-known in white noise theory, see e.g. Kuo (1996) or Hida, Kuo, Potthoff and Streit (1993). We thank John van der Hoek for pointing this out.

These results essentially afford densities given in terms of Edgeworth or Gram/Charlier expansions the same amount of tractability as the Gaussian. Calculations can be performed directly at the level of the infinite series expansion, without the need to truncate the series before deriving the desired results (as has been the practice in the option pricing literature using these expansions). See Schlögl (2008) for an application to traditional option pricing.

[11]see for example Kendall and Stuart (1969)

	Market				**Vasicek Model**		**Gram/Charlier**	
	Upfront (pts)		**Spread (bp)**					
Subordination	**Bid**	**Ask**	**Bid**	**Ask**	**Upfront**	**Spread**	**Upfront**	**Spread**
0%	37	42	500	500	41.79	500.00	39.88	500.00
3%	0	0	280	330	0.00	365.93	0.00	306.55
7%	0	0	102	110	0.00	93.71	0.00	104.84
10%	0	0	39	59	0.00	24.27	0.00	53.82
15%	0	0	6	16	0.00	1.60	0.00	9.48
30%								
Latent variable correlation coefficient β^2:					16.98%		17.58%	
Skewness:							−0.2825	
Excess kurtosis:							1.8986	

TABLE 1. Calibration to CDX NA I tranche quotes on 21 April 2004

and change indices to $j := m + n$, so that this

$$= \sum_{j=0}^{\infty} c_{k+2j}(k+2j)!(1-\beta_i^2)^j \frac{1}{2^j j!} \sum_{m=0}^{j} \binom{j}{m} (-1)^m$$

The inner sum is zero for all $j > 0$, so that we have

$$= c_k k!$$

Substituting this into (7) yields

$$g(x) = \phi(x) \sum_{k=0}^{\infty} \mathrm{He}_k(x) \beta_i^k c_k \tag{8}$$

It follows from Lemma 6 (see appendix) that

$$\begin{aligned} P[\zeta_i \leq D_i(t)] &= \int_{-\infty}^{D_i(t)} g(x)dx \\ &= \Phi(D_i(t)) - \sum_{j=1}^{\infty} \beta_i^j c_j \phi(D_i(t)) \mathrm{He}_{j-1}(D_i(t)) \end{aligned}$$

□

Lemma 2 permits the term structures of default thresholds to be fitted to the risk–neutral probabilities of default backed out of the single–name credit default swap spreads.[12] CDO tranche spreads can then be calculated by semi-analytical methods — we use the method described by Andersen, Sidenius and Basu (2003), the only modification required being that the calculation of the unconditional loss probabilities from the conditional loss probabilities by numerical integration is carried out with respect to a factor distribution given by a Gram/Charlier density.

[12]For the relationship between CDS spreads and risk–neutral probabilities of default/survival, see Schönbucher (2003).

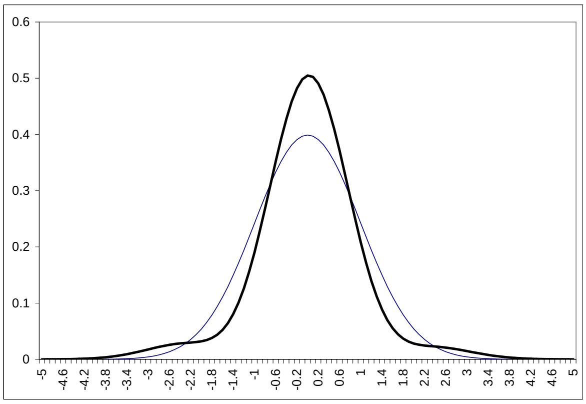

Figure 1. CDX data example

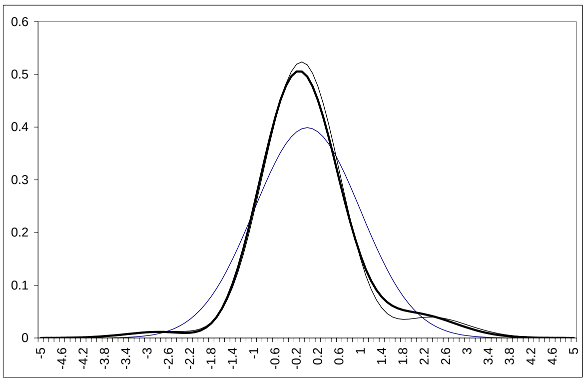

Figure 2. iTraxx data example

Fitted (thick line) versus normal densities

3. Examples

3.1. **Implied factor distributions.** To extract implied factor distributions from competitively quoted tranche spreads, we assume a flat correlation structure (i.e. $\beta_i \equiv \beta$ for all i) and truncate the Gram/Charlier series expansion after some even–numbered moment,[13] thus essentially generalising the Vasicek/Li factor model by allowing for non-zero skewness, excess kurtosis and possibly higher order terms. We implement a non-linear optimisation[14] to find β, skewness and excess kurtosis and any desired higher moments such that the squared relative error in the model tranche spreads versus the mid-market quoted spreads is minimised.

Two market data examples are given in Tables 1 and 2. For the CDX NA I tranche quotes on 21 April 2004, the Gram/Charlier calibration produces very good results, with the model spreads very close to the mid-market quotes, especially when compared to the spreads representing the best fit of a Vasicek flat correlation model calibrated using the same objective function. This is also reflected in the shape of the calibrated density itself, which is nicely unimodal, as Figure 1 shows. For the ITRX.EUR 2 tranche quotes on 21 March 2005, the fit is not as good. In fact, an unconstrained calibration of skewness and kurtosis to the market quotes in this case would not result in a valid density — the density shown here is the result of a constrained optimisation as suggested by Schlögl (2008). Difficulties are encountered in particular in fitting the senior tranche spread — this appears to be a problem common to most (possibly all) variations of the Vasicek factor approach.[16] Adding a fifth and sixth moment to the expansion improves the situation

[13]Truncating the expansion after an odd–numbered moment would unavoidably result in an invalid density. For truncation after an arbitrary even–numbered moment, Schlögl (2008) describes an algorithm, which ensures that the coefficients of the truncated expansion are calibrated in a way that ensures that the density is positive everywhere.

[14]E.g. Powell's method (see Press, Teukolsky, Vetterling and Flannery (1992)) modified as in Schlögl (2008).

[15]"Excess" is to be interpreted in a manner analogous to excess kurtosis, i.e. the number quoted is the excess of the sixth moment about the mean above the corresponding moment of the standard normal distribution, which in this case is 15.

[16]We obtained similar results using other distributional assumptions on the common factor, including the normal inverse Gaussian along similar lines as Guegan and Houdain (2005) and Kalemanova, Schmid and Werner (2007).

	Market		Gram/Charlier 4		Gram/Charlier 6	
	Upfront (pts)	**Spread (bp)**				
Subordination	**Mid**	**Mid**	**Upfront**	**Spread**	**Upfront**	**Spread**
0%	17.5	500.00	17.36	500.00	17.39	500.00
3%	0.0	112.50	0.00	112.38	0.00	112.63
6%	0.0	36.13	0.00	35.69	0.00	34.95
9%	0.0	18.00	0.00	19.28	0.00	20.03
12%	0.0	10.00	0.00	6.27	0.00	7.20
22%						
Latent variable correlation coefficient β^2**:**			18.19%		18.22%	
Skewness:			0.5438		0.3929	
Excess kurtosis:			2.3856		1.9772	
Fifth moment about the mean:					2.6704	
Excess[15] sixth moment about the mean:					29.8882	

Table 2. Calibration to ITRX.EUR 2 tranche quotes on 21 March 2005

	Market		Gram/Charlier 4		Gram/Charlier 6	
	Upfront (pts)	**Spread (bp)**				
Subordination	**Mid**	**Mid**	**Upfront**	**Spread**	**Upfront**	**Spread**
0%	13.01	500.00	12.74	500.00	12.89	500.00
3%	0.00	67.10	0.00	62.87	0.00	72.68
6%	0.00	17.93	0.00	16.86	0.00	15.17
9%	0.00	8.34	0.00	8.26	0.00	10.70
12%	0.00	3.28	0.00	0.99	0.00	1.33
22%						
Latent variable correlation coefficient β^2**:**			14.60%		14.49%	
Skewness:			0.6886		−0.0047	
Excess kurtosis:			2.3650		1.2638	
Fifth moment about the mean:					−0.7185	
Excess sixth moment about the mean:					30.5882	

Table 3. Calibration to ITRX.EUR 2 tranche quotes on 2 July 2007

somewhat and also smoothes the resulting density more toward the unimodal, as can be seen in Figure 2 (the thick line represents the higher order fit).

A similar result is obtained for more recent data, as reported for 2 July 2007 in Table 3. Again, the fit via an implied factor distribution is difficult in particular for the most senior tranche, where calibrated spreads are too low. Adding a fifth and sixth moment to the expansion allows us to increase the model spread for the most senior tranche, but at the cost of radically changing the factor distribution: As Figure 3 shows, the Gram/Charlier four–moment fit (the non-normal density plotted with a thin line) is very similar in shape

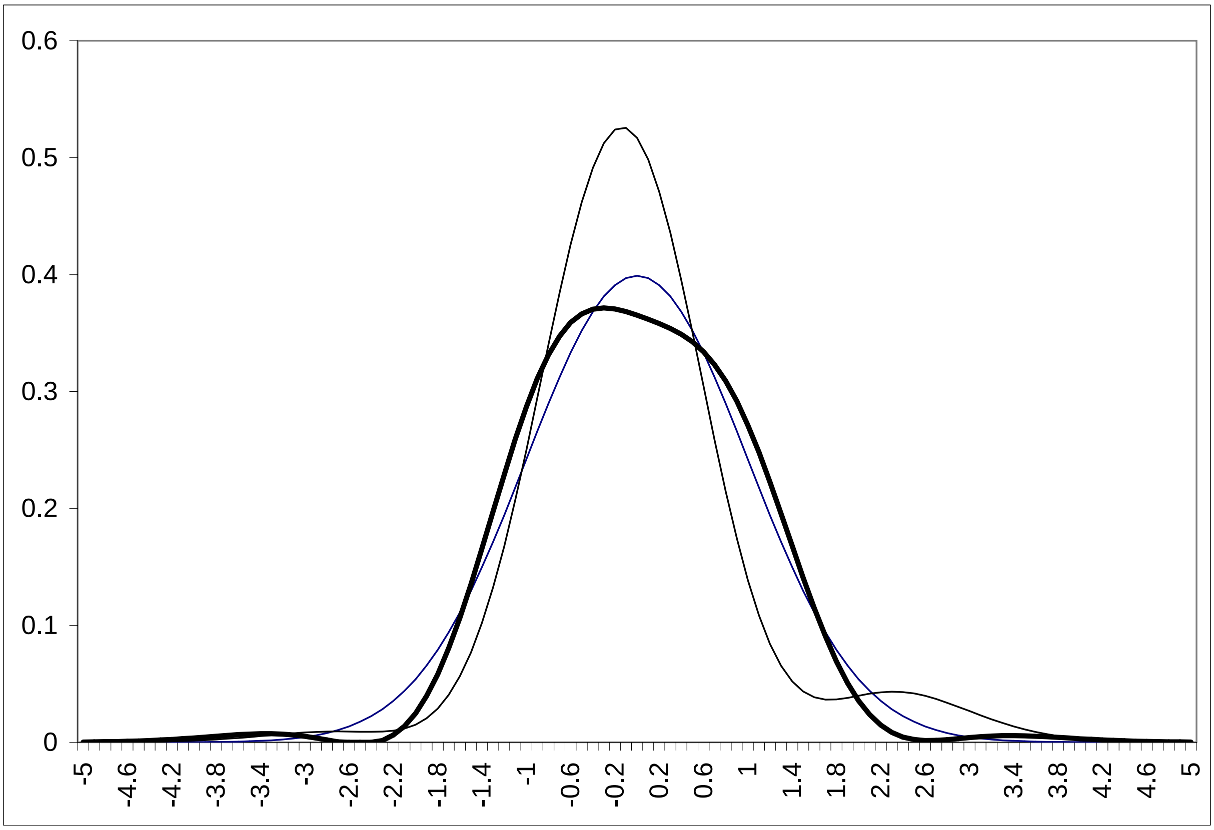

FIGURE 3. iTraxx implied factor distribution for 2 July 2007

to the 21 March 2005 result, while the six–moment fit (the thick line) is substantially different.[17]

One should also note that during the period of considerable turmoil in the credit markets (from August 2007) the quality of the fit deteriorated markedly (results not reported here), and it is a matter of further research whether this market situation can be accurately mirrored by any of the existing models, in particular those based on the one–factor approach.

3.2. **Interpolation of implied and base correlation.** By fitting the factor distribution to the tranche spreads quoted in the market, we are essentially subsuming all departures from the flat correlation single factor Gaussian model in the implied factor distribution. As such, the implied factor distribution is specific to the underlying portfolio, which limits the applicability — in the same way a risk–neutral distribution extracted from, say, S&P 500 index option prices is applicable only to that particular index. In practical terms, such implied distributions are useful for interpolating prices (or implied volatility/correlation) in a manner consistent with the absence of arbitrage. This is what is involved when pricing non-standard tranches on the index portfolios.

The question that one might therefore ask is: Does the implied factor distribution also deal well with situations where the "implied correlation smile" is caused by influences

[17]One can argue that is might be due to overfitting, as this can be avoided by changing the weighting of the individual tranches in the objective function for the minimisation. For the results reproduced here, the *relative* pricing error for each tranche was weighted equally.

Compound and base correlation as defined in O'Kane and Livesey (2004)

		Correlation		Gram/Charlier fit		
Subordination	**Actual spread**	**compound**	**base**	**Spread**	**Compound corr.**	**Base corr.**
0%	8170.34	13.72%	13.72%	8170.45	13.72%	13.72%
3%	2391.05	13.42%	13.54%	2390.92	13.42%	13.54%
7%	1172.06	12.67%	13.36%	1172.65	12.64%	13.37%
10%	568.41	17.72%	13.01%	567.6	17.26%	13.10%
15%	97.46	13.10%	11.71%	98.07	13.21%	12.41%
30%						

Table 4. Standard tranches, heterogeneous portfolio

		Correlation		Gram/Charlier fit		
Subordination	**Actual spread**	**compound**	**base**	**Spread**	**Compound corr.**	**Base corr.**
0%	5081.53	13.64%	13.64%	5082.33	13.64%	13.64%
5%	1428.59	13.00%	13.36%	1428.94	12.98%	13.37%
10%	739.64	7.27 %	13.23%	739.58	7.30 %	13.26%
12%	463.83	15.08%	13.01%	462.59	14.81%	13.10%
15%	58.5	13.01%	10.56%	59.08	13.16%	—
40%						

Table 5. Non-standard tranches, heterogeneous portfolio

other than non-normality of the common factor? Mashal, Naldi and Tejwani (2004) identify heterogeneity in correlation and spreads as one of the potential causes of an "implied correlation smile." This motivates the following experiment: On a portfolio of 100 names, vary the CDS spreads between 30 and 300 basis points, and vary the β between 20% and 47.5%, (higher spread names have higher correlations). Calculate the "correct" spreads using a Gaussian model with the heterogeneous correlations, and then fit the (flat correlation) Gram/Charlier model to the standard tranches. The result of this calibration is given in Table 4. Then, calculate the spreads for non-standard tranches using the previously fitted Gram/Charlier model and compare these with the correct spreads. As Table 5 shows, the spreads calculated using the fitted model agree very closely with those given by the postulated "correct" model, demonstrating that the portfolio heterogeneity has been absorbed well into the modified distribution of the common factor.[18]

When applied to market data, the model interpolates (and extrapolates) base correlation in a manner in line with the accuracy of the calibration. Compared to direct interpolation/extrapolation of the base correlation obtained from market quotes, base correlation calculated from a calibrated model has the advantage that it is guaranteed to be consistent with the absence of arbitrage.[19] Figure 4 shows the interpolation/extrapolation of

[18]Incidentally, the numbers in Tables 4 and 5 also demonstrate a fact well–known to practitioners: The non-monotonicity and relative instability of implied correlation for mezzanine tranches implies that if one does chose to interpolate the "correlation smile" directly, one should do this at the level of base correlation, rather than the correlation implied by tranche spreads. Base correlation for a subordination level of $x\%$ is the implied correlation of an equity tranche covering losses from zero to $x\%$ of the CDO notional.

[19]This is not guaranteed in the case of direct interpolation, see O'Kane and Livesey (2004).

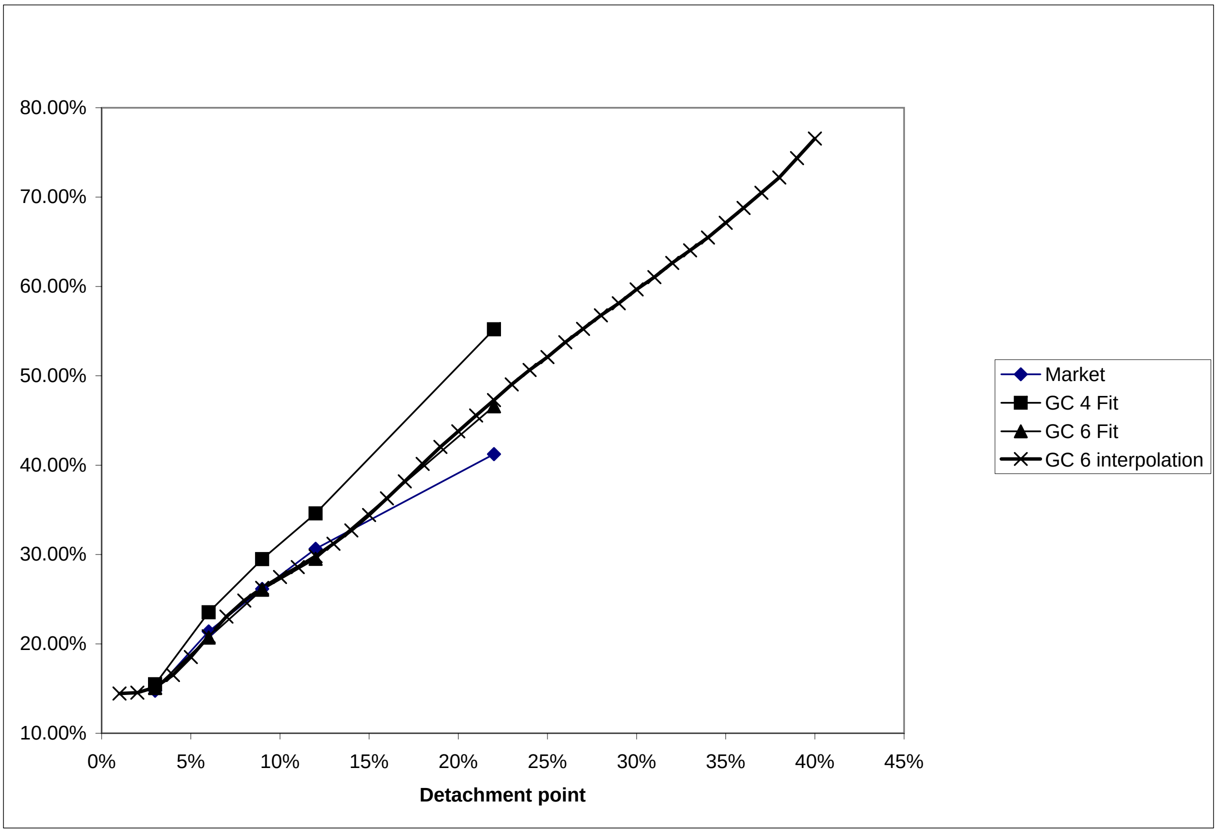

FIGURE 4. Interpolated base correlation for 2 July 2007

base correlation the implied by the model fitted to iTraxx market data on 2 July 2007 (i.e. the calibration reported in Table 3). At the senior end, the model extrapolates based on the calibration of the most senior tranche (and thus departs substantially from what one would obtain by directly extrapolating the base correlations). At the equity end, directly extrapolating the market base correlations would result in much lower correlations (and thus much lower spreads or upfront payments) than the essentially flat extrapolation implied by the model.

4. Conclusion

In a way similar to volatility for standard options, (default) correlation is the key parameter for the pricing of CDO tranches, which is not directly observable in the market. As prices for these derivative financial instruments become competitively quoted in the market, values for these parameters can be implied. We have demonstrated how, in a way similar to how one can extract risk-neutral distributions from standard option prices, an implied factor distribution for a CDO pricing model can be constructed in a semi-parametric way. Essentially, in the sense that the factor distribution determines the copula of the joint distribution of default times, a default dependence structure has thus been extracted from market quoted tranche spreads.

Appendix: Some useful results on Hermite polynomials under linear coordinate transforms

We use the definition of Hermite polynomials customary in statistics, as given for example in Kendall and Stuart (1969), where they are called "Chebyshev–Hermite polynomials." In the literature these polynomials are usually denoted $\mathrm{He}_i(\cdot)$, as opposed to a slightly different version of Hermite polynomials, which are usually denoted $H_i(x)$ (see e.g. Abramowitz and Stegun (1964)).

3. Definition. The Hermite polynomials $\mathrm{He}_i(\cdot)$ are defined by the identity

$$(-D)^i \phi(x) = \mathrm{He}_i(x)\phi(x) \tag{9}$$

where

$$D = \frac{d}{dx}$$

is the differential operator and

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

The following results are reproduced for the reader's convenience. See Schlögl (2008) for proofs.

4. Lemma. *The Hermite polynomials* $\mathrm{He}_i(\cdot)$ *satisfy*

$$\mathrm{He}_i(ax+b) = \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \mathrm{He}_{i-j}(x) j! \sum_{m=0}^{\left[\frac{j}{2}\right]} \frac{1}{m!2^m(j-2m)!} a^{2m} \mathrm{He}_{j-2m}(b) \tag{10}$$

5. Corollary. *The Hermite polynomials* $\mathrm{He}_i(\cdot)$ *satisfy*

$$\mathrm{He}_i(y+a) = \sum_{j=0}^{i} \binom{i}{j} \mathrm{He}_{i-j}(y) a^j \tag{11}$$

$$\mathrm{He}_i(ax) = i! \sum_{m=0}^{\left[\frac{i}{2}\right]} a^{i-2m} \mathrm{He}_{i-2m}(x) \frac{(a^2-1)^m}{(i-2m)!2^m m!} \tag{12}$$

6. Lemma. *We have for* $i \geq 1$

$$\begin{aligned} \int_a^b \mathrm{He}_i(y) \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2}} dy = & \\ \sum_{j=0}^{i-1} \binom{i}{j} \mu^j \left(\phi(a-\mu)\mathrm{He}_{i-j-1}(a-\mu) - \phi(b-\mu)\mathrm{He}_{i-j-1}(b-\mu)\right) & \\ + \mu^i \left(\Phi(b-\mu) - \Phi(a-\mu)\right) & \end{aligned} \tag{13}$$

where $\Phi(\cdot)$ *is the cumulative distribution function of the standard normal distribution. If* $\mu = 0$ *and* $i \geq 1$

$$\int_a^b \mathrm{He}_i(y) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \phi(a)\mathrm{He}_{i-1}(a) - \phi(b)\mathrm{He}_{i-1}(b) \tag{14}$$

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