Sphere-Packing Bound for Symmetric Classical-Quantum Channels

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ABSTRACT. We provide a sphere-packing lower bound for the optimal error probability in finite blocklengths when coding over a symmetric classical-quantum channel. Our result shows that the pre-factor can be significantly improved from the order of the subexponential to the polynomial. This established pre-factor is essentially optimal because it matches the best known random coding upper bound in the classical case. Our approaches rely on a sharp concentration inequality in strong large deviation theory and crucial properties of the error-exponent function.

1. INTRODUCTION

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The probability of decoding error is one of the fundamental criteria for evaluating the performance of a communication system. In Shannon's seminal work [1], he pioneered the study of the noisy coding theorem, which states that the error probability can be made arbitrarily small as the coding blocklength grows when the coding rate R is below the channel capacity C. Later, Shannon [2] made a further step in exploring the exponential dependency of the optimal error probability $\epsilon^*(n, R)$ on the blocklength nand rate R, and defined the *reliability function* as follows: given a fixed coding rate R < C, E(R) := $\limsup_{n \to +\infty} -\frac{1}{n} \log \epsilon^*(n, R)$. The quantity E(R) then provides a measure of how rapidly the error probability approaches zero with an increase in blocklength. This asymptotic characterization of the optimal error probability under a fixed rate is hence called the *error exponent analysis*. For a classical channel, the upper bounds of the optimal error can be established using a random coding argument [3]. On the other hand, the lower bound was first developed by Shannon, Gallager, and Berlekamp [4] and was called the *sphere-packing bound*. Alternative approaches by Haroutunian [5] and Blahut [6] were subsequently proposed.

In recent years, much attention has been paid to the finite blocklength regime [7, 8]. Altuğ and Wagner employed strong large deviation techniques [9] to prove a sphere-packing bound with a finite blocklength n. Moreover, the pre-factor of the bound was significantly refined from the order of the subexponential $\exp\{-O(\sqrt{n})\}$ [4] to the polynomial [10, 11]. This refinement is substantial especially at rates near capacity, where the error-exponent function is close to zero; hence, the pre-factor dominants the bound [12, 13].

Error exponent analysis in classical-quantum (c-q) channels is much more difficult because of the noncommutative nature of quantum mechanics. Burnashev and Holevo [15, 16] investigated reliability functions in c-q channels and proved the random coding upper bound for pure-state channels. Winter [17] adopted Haroutunian's method to derive a sphere-packing bound for c-q channels in the form of relative entropy functions [5]. Dalai [18] employed Shannon-Gallager-Berlekamp's approach to establish a sphere-packing bound with Gallager's expression [4]. It was later pointed out that these two sphere-packing exponents are not equal for general c-q channels [19]. In this work, we initiate the study of the refined sphere-packing bound in the quantum scenario. In particular, we consider a "symmetric c-q channel" (see Section 2 for a detailed definition), which is an important class of *covariant channels* (e.g. [20]), and establish a sphere-packing bound with the pre-factor improved from the order of the subexponential

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in Dalai's result [18] to the polynomial. Our result recovers Altuğ and Wagner's work [10] for classical symmetric channels including the binary symmetric channel and binary erasure channel. Furthermore, the proved pre-factor matches that of the best known random coding upper bound [21] in the classical case. Hence, our result yields the exact asymptotics for the sphere-packing bound in symmetric c-q channels. The main ingredients in our proof are a tight concentration inequality from Bahadur and Ranga Rao [9], [13] (see Appendix A) and the major properties of the sphere-packing exponent [22]. We remark that the result obtained in this paper might enable analysis in the medium error probability regime of a classical-quantum channel [12, 13, 14]. We leave the case for general c-q channels as future work [23].

This paper is organized as follows. We introduce the necessary notation and state our main result in Section 2. Section 3 includes the crucial properties of the error-exponent function. We provide the proof of the main result in Section 4. Section 5 concludes this paper.

2. NOTATION AND MAIN RESULT

2.1. Notation. Throughout this paper, we consider a finite-dimensional Hilbert space \mathcal{H} . The set of density operators (i.e. positive semi-definite operators with unit trace) on \mathcal{H} are defined as $\mathcal{S}(\mathcal{H})$. For $\rho, \sigma \in$ $\mathcal{S}(\mathcal{H})$, we write $\rho \ll \sigma$ if $\operatorname{supp}(\rho) \subset \operatorname{supp}(\sigma)$, where $\operatorname{supp}(\rho)$ denotes the support of ρ . The identity operator on \mathcal{H} is denoted by $\mathbb{1}_{\mathcal{H}}$. When there is no possibility of confusion, we skip the subscript \mathcal{H} . We use Tr $[\cdot]$ as the trace function. Let \mathbb{N} , \mathbb{R} , and $\mathbb{R}_{>0}$ denote the set of integers, real numbers, and positive real numbers, respectively. Define $[n] := \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$. Given a pair of positive semi-definite operators $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, we define the (quantum) relative entropy as $\mathbb{D}(\rho \| \sigma) := \operatorname{Tr} [\rho (\log \rho - \log \sigma)]$, when $\rho \ll \sigma$, and $+\infty$ otherwise. For every $\alpha \in [0,1)$, we define the (Petz) quantum Rényi divergences $D_{\alpha}(\rho \| \sigma) :=$ $\frac{1}{\alpha-1}\log \operatorname{Tr}\left[\rho^{\alpha}\sigma^{1-\alpha}\right]. \text{ For } \alpha = 1, \ D_1(\rho\|\sigma) := \lim_{\alpha \to 1} D_\alpha(\rho\|\sigma) = \mathbb{D}(\rho\|\sigma). \text{ Let } \mathcal{X} = \{1, 2, \dots, |\mathcal{X}|\} \text{ be}$ a finite alphabet, and let $\mathcal{P}(\mathcal{X})$ be the set of probability distributions on \mathcal{X} . In particular, we denote by $U_{\mathcal{X}}$ the uniform distribution on \mathcal{X} . A classical-quantum (c-q) channel W maps elements of the finite set \mathcal{X} to the density operators in $\mathcal{S}(\mathcal{H})$, i.e., $W: \mathcal{X} \to \mathcal{S}(\mathcal{H})$. Let \mathcal{M} be a finite alphabetical set with size $M = |\mathcal{M}|$. An (n-block) encoder is a map $f_n : \mathcal{M} \to \mathcal{X}^n$ that encodes each message $m \in \mathcal{M}$ to a codeword $\mathbf{x}^n(m) := x_1(m) \dots x_n(m) \in \mathcal{X}^n$. The codeword $\mathbf{x}^n(m)$ is then mapped to a state $W_{\mathbf{x}^n(m)}^{\otimes n} = W_{x_1(m)} \otimes \cdots \otimes W_{x_n(m)} \in \mathcal{S}(\mathcal{H}^{\otimes n}).$ The *decoder* is described by a positive operator-valued measurement (POVM) $\Pi_n = {\Pi_{n,1}, \ldots, \Pi_{n,M}}$ on $\mathcal{H}^{\otimes n}$, where $\Pi_{n,i} \ge 0$ and $\sum_{i=1}^M \Pi_{n,i} = \mathbb{1}$. The pair $(f_n, \Pi_n) =: \mathcal{C}_n$ is called a *code* with *rate* $R = \frac{1}{n} \log |\mathcal{M}|$. The error probability of sending a message *m* with the code \mathcal{C}_n is $\epsilon_m(W, \mathcal{C}_n) := 1 - \text{Tr}(\prod_{n,m} W_{\mathbf{x}^n(m)})$. We use $\epsilon_{\max}(W, \mathcal{C}_n) = \max_{m \in \mathcal{M}} \epsilon_m(W, \mathcal{C}_n)$ and $\bar{\epsilon}(W, \mathcal{C}_n) = \frac{1}{M} \sum_{m \in \mathcal{M}} \epsilon_m(W, \mathcal{C}_n)$ to denote the maximal error probability and the average error probability, respectively. Given a sequence $\mathbf{x}^n \in \mathcal{X}^n$, we denote by $P_{\mathbf{x}^n}(x) := \frac{1}{n} \sum_{i=1}^n \mathbf{1} \{x = x_i\}$ the empirical distribution of \mathbf{x}^n .

Throughout this paper, we consider a symmetric c-q channel defined as

$$W_x := V^{x-1} W_1 (V^{\dagger})^{x-1}, \quad \forall x \in \mathcal{X},$$

$$\tag{1}$$

where $W_1 \in \mathcal{S}(\mathcal{H})$ is an arbitrary density operator, and V satisfies $V^{\dagger}V = VV^{\dagger} = V^{|\mathcal{X}|} = \mathbb{1}_{\mathcal{H}}$. We define the following conditional entropic quantities for the channel W with $P \in \mathcal{P}(\mathcal{X})$: $D_{\alpha}(W \| \sigma | P) := \sum_{x \in \mathcal{X}} P(x)D_{\alpha}(W_x \| \sigma)$. The mutual information of the c-q channel $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$ with prior distribution $P \in \mathcal{P}(\mathcal{X})$ is defined as $I(P, W) := \mathbb{D}(W \| PW | P)$, where $PW^{\alpha} := \sum_{x \in \mathcal{X}} P(x)W_x^{\alpha}$, $\alpha \in (0, 1]$. The (classical) capacity of the channel $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$ is denoted by $C := \max_{P \in \mathcal{P}(\mathcal{X})} I(P, W)$. Let

$$E_{\rm sp}^{(1)}(R,P) := \sup_{s \ge 0} \left\{ E_0(s,P) - sR \right\}$$
$$E_{\rm sp}^{(2)}(R,P) := \sup_{0 < \alpha \le 1} \min_{\sigma \in \mathcal{S}(\mathcal{H})} \frac{\alpha - 1}{\alpha} \left(R - D_\alpha \left(W \| \sigma | P \right) \right),$$

where we denote by $E_0(s, P) := -\log \operatorname{Tr}\left[\left(PW^{1/(1+s)}\right)^{1+s}\right]$ an auxiliary function [16, 22]. The spherepacking exponent is defined by

$$E_{\rm sp}(R) := \max_{P \in \mathcal{P}(\mathcal{X})} E_{\rm sp}^{(1)}(R, P) = \max_{P \in \mathcal{P}(\mathcal{X})} E_{\rm sp}^{(2)}(R, P),$$
(2)

where the last equality follows from [24, Proposition IV.2]. Further, we define a rate [25, p. 152], [18]:

$$R_{\infty} := \lim_{s \to +\infty} \max_{P \in \mathcal{P}(\mathcal{X})} \min_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) D_{\frac{1}{1+s}} (W_x \| \sigma)$$
$$= \max_{P \in \mathcal{P}(\mathcal{X})} \min_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) \operatorname{Tr} \left[W_x^0 \sigma \right].$$
(3)

It follows that $E_{\rm sp}(R) = +\infty$ for any $R \leq R_{\infty}$ (see also [4, p. 69] and [3, Eq. (5.8.5)]).

Consider a binary hypothesis whose null and alternative hypotheses are $\rho \in \mathcal{S}(\mathcal{H})$ and $\sigma \in \mathcal{S}(\mathcal{H})$, respectively. The *type-I error* and *type-II error* of the hypothesis testing, for an operator $0 \leq Q \leq 1$, are defined as $\alpha(Q;\rho) := \text{Tr}[(1-Q)\rho]$, and $\beta(Q;\sigma) := \text{Tr}[Q\sigma]$. There is a trade-off between these two errors. Thus, we can define the minimum type-I error, when the type-II error is below $\mu \in (0, 1)$, as

$$\widehat{\alpha}_{\mu}\left(\rho\|\sigma\right) := \min_{0 \le Q \le 1} \left\{ \alpha\left(Q;\rho\right) : \beta\left(Q;\sigma\right) \le \mu \right\}.$$

$$\tag{4}$$

2.2. Main Result. Let us now consider any symmetric c-q channel with capacity C.

Theorem 1 (Exact Sphere-Packing Bound). For any rate $R \in [0, C)$, there exist an $N_0 \in \mathbb{N}$ such that for all codes \mathcal{C}_n of length $n \geq N_0$, we have

$$\epsilon_{\max}\left(\mathcal{C}_n\right) \ge \frac{1 - o(1)}{n^{\frac{1}{2}\left(1 + \left|E'_{\rm sp}\left(R\right)\right|\right)}} \exp\left\{-nE_{\rm sp}\left(R\right)\right\},\tag{5}$$

where $E'_{\rm sp}(R) := \partial \max_{P \in \mathcal{P}(\mathcal{X})} E^{(1)}_{\rm sp}(r, P) / \partial r|_{r=R}.$

3. PROPERTIES OF THE SPHERE-PACKING EXPONENT

Lemma 2 (Optimal Input Distribution). For any $R > R_{\infty}$, the distribution $U_{\mathcal{X}}$ is a maximizer of $E_{sp}^{(1)}(R,\cdot)$ and $E_{sp}^{(2)}(R,\cdot)$.

Proof. We first prove that $U_{\mathcal{X}}$ attains $\max_{P \in \mathcal{P}(\mathcal{X})} E_0(s, P)$. From Eq. (1), it is not hard to verify that $U_{\mathcal{X}} W^{\alpha} = V U_{\mathcal{X}} W^{\alpha} V^{\dagger}$ for all $\alpha \in (0, 1]$. Hence, it follows that

$$\operatorname{Tr}[W_{x}^{\alpha}(U_{\mathcal{X}}W^{\alpha})^{\frac{1-\alpha}{\alpha}}] = \operatorname{Tr}[V^{x-1}W_{1}^{\alpha}V^{\dagger x-1}(U_{\mathcal{X}}W^{\alpha})^{\frac{1-\alpha}{\alpha}}]$$
(6)

$$= \operatorname{Tr}[W_1^{\alpha} V^{\dagger x-1} (U_{\mathcal{X}} W^{\alpha})^{\frac{1-\alpha}{\alpha}} V^{x-1}]$$
(7)

$$= \operatorname{Tr}[W_1^{\alpha}(U_{\mathcal{X}}W^{\alpha})^{\frac{1-\alpha}{\alpha}}]$$
(8)

$$= \operatorname{Tr}[(U_{\mathcal{X}}W^{\alpha})^{\frac{1}{\alpha}}] \tag{9}$$

for all $\alpha \in (0, 1]$. The above equation shows that the distribution $U_{\mathcal{X}}$ that maximizes $E_0(s, P), \forall s \ge 0$ [16, Eq. (38)]. Then we have

$$E_{\rm sp}^{(1)}(R, U_{\mathcal{X}}) = \sup_{s \ge 0} \left\{ \max_{P \in \mathcal{P}(\mathcal{X})} E_0(s, P) - sR \right\} = E_{\rm sp}(R).$$

Further, Jensen's inequality implies that $E_{\rm sp}^{(2)}(R, U_{\mathcal{X}}) \geq E_{\rm sp}^{(1)}(R, U_{\mathcal{X}}) = E_{\rm sp}(R)$, which completes the proof.

Lemma 3 (Saddle-Point Property). Consider any $R \in (R_{\infty}, C)$ and $P \in \mathcal{P}(\mathcal{X})$. Let $\mathcal{S}_{P,W}(\mathcal{H}) := \{\sigma \in \mathcal{S}(\mathcal{H}) : \forall x \in \operatorname{supp}(P), \operatorname{supp}(W_x) \cap \operatorname{supp}(\sigma) \neq \emptyset\}$. We define

$$F_{R,P}(\alpha,\sigma) := \frac{\alpha - 1}{\alpha} \left(R - D_{\alpha} \left(W \| \sigma | P \right) \right), \tag{10}$$

on $(0,1] \times S_{P,W}(\mathcal{H})$, and let $\mathcal{P}_R(\mathcal{X}) := \{P \in \mathcal{P}(\mathcal{X}) : \min_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{0 < \alpha \leq 1} F_{R,P}(\alpha,\sigma) \in \mathbb{R}_{>0}\}$. The following holds

(i) For any $P \in \mathcal{P}(\mathcal{X})$, $F_{R,P}(\cdot, \cdot)$ has a saddle-point with the saddle-value:

$$\min_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{0 < \alpha \le 1} F_{R,P}(\alpha, \sigma) = \sup_{0 < \alpha \le 1} \min_{\sigma \in \mathcal{S}(\mathcal{H})} F_{R,P}(\alpha, \sigma) = E_{\rm sp}^{(2)}(R, W, P).$$
(11)

- (ii) The saddle-point is unique for $P \in \mathfrak{P}_R(\mathcal{X})$.
- (iii) Let $P \in \mathfrak{P}_R(\mathcal{X})$. The unique saddle-point (α, σ) of $F_{R,P}(\cdot, \cdot)$ satisfies $\alpha \in (0, 1)$ and

$$\sigma = \frac{\left(\sum_{x \in \mathcal{X}} P(x) W_x^{\alpha} \mathrm{e}^{(1-\alpha)D_{\alpha}(W_x \| \sigma)}\right)^{1/\alpha}}{\mathrm{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) W_x^{\alpha} \mathrm{e}^{(1-\alpha)D_{\alpha}(W_x \| \sigma)}\right)^{1/\alpha}\right]} \gg W_x, \quad \forall x \in \mathrm{supp}(P).$$
(12)

The proof is provided in Appendix B.1.

Lemma 4 (Representation). For any $R \in (R_{\infty}, C)$, let $(\alpha_R^{\star}, \sigma_R^{\star})$ be the saddle-point of $F_{R,U_{\mathcal{X}}}(\cdot, \cdot)$. It follows that

$$(\alpha_R^{\star}, \sigma_R^{\star}) = \left(-E_{\rm sp}'(R), \frac{\left(U_{\mathcal{X}} W^{\alpha_R^{\star}} \right)^{1/\alpha_R^{\star}}}{\operatorname{Tr} \left[\left(U_{\mathcal{X}} W^{\alpha_R^{\star}} \right)^{1/\alpha_R^{\star}} \right]} \right).$$
(13)

Proof. Since Lemma 2 implies that $U_{\mathcal{X}}$ attains $E_{sp}^{(2)}(R, \cdot)$, one observes from the definition of $E_{sp}^{(2)}$ that all the quantities $D_{\alpha_R^{\star}}(W_x \| \sigma_R^{\star}), x \in \mathcal{X}$ are equal. By item (iii) of Lemma 3, we obtain a representation of σ_R^* in Eq. (13). The optimal $\alpha_R^* = -\partial E_{\rm sp}(r, U_{\mathcal{X}})/\partial r|_{r=R}$ follows from [22, Eq. (42)].

Lemma 5 (Invariance). For any $R \in (R_{\infty}, C)$, we have

$$F_{R,P}(\alpha_R^{\star}, \sigma_R^{\star}) = E_{\rm sp}(R) > 0, \quad \forall P \in \mathcal{P}(\mathcal{X}), \tag{14}$$

where α_{R}^{\star} and σ_{R}^{\star} are defined in Eq. (13).

Proof. Following the argument in Lemma 2 and recalling Eq. (13) in Lemma 4, one can verify that $\sup_{\alpha \in (0,1]} F_{R,P}(\alpha, \sigma_R^{\star}) = \sup_{s \ge 0} \{ E_0(s, U_{\mathcal{X}}) - sR \} = E_{sp}(R) \text{ for all } P \in \mathcal{P}(\mathcal{X}).$ Further, we obtain $E_{\rm sp}(R) > 0$ for $R \in (R_{\infty}, C)$ from the result in [22, Proposition 10].

4. Proof of the Main Result

For rates in the range $R \leq R_{\infty}$, we have $E_{\rm sp}(R) = +\infty$. The bound in Eq. (5) obviously holds. Hence, we consider the case of $R \in (R_{\infty}, C)$ and fix the rate throughout the proof.

We first pose the channel coding problem into a binary hypothesis testing through Lemma 6, which originates from Blahut [6] for the classical case.

Lemma 6 (Hypothesis Testing Reduction). For any code C_n with message size e^{nr} , there exists an $\mathbf{x}^n \in C_n$ such that

$$\epsilon_{\max}\left(\mathcal{C}_{n}\right) \geq \max_{\sigma \in \mathcal{S}(H)} \widehat{\alpha}_{\exp\{-nr\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma^{\otimes n}\right).$$
(15)

The proof is provided in Appendix B.2.

Let us now commence with the proof of Theorem 1. Fix arbitrary $\gamma, \xi > 0$. Let $\gamma_n := \left(\frac{1}{2} + \gamma\right) \frac{\log n}{n}$ and $R_n := R - \gamma_n$. The choice of the rate back-off term γ_n will become evident later. Choose $N_1 \in \mathbb{N}$ such that $R_n \ge R - \xi > R_\infty$. Let σ_R^* be defined in Eq. (13), and from Lemma 6, we have

$$\epsilon_{\max}\left(\mathcal{C}_{n}\right) \geq \widehat{\alpha}_{\exp\{-nR_{n}\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma_{R}^{\star \otimes n}\right).$$

$$(16)$$

In the following, we provide a lower bound for the type-I error $\widehat{\alpha}_{\exp\{-nR_n\}} \left(W_{\mathbf{x}^n}^{\otimes n} \| \sigma_R^{\star \otimes n} \right)$. Let $p^n :=$ $\bigotimes_{i=1}^{n} p_{x_i}$ and $q^n := \bigotimes_{i=1}^{n} q_{x_i}$, where (p_{x_i}, q_{x_i}) are Nussbaum-Szkoła distributions [26] of $(W_{x_i}, \sigma_R^{\star})$ for every $i \in [n]$. Since $D_{\alpha}(W_{x_i} \| \sigma_R^{\star}) = D_{\alpha}(p_{x_i} \| q_{x_i})$, for all $\alpha \in (0, 1]$, we shorthand $\phi_n(R_n) := \sup_{\alpha \in (0, 1]} F_{R_n, P_{\mathbf{x}^n}}(\alpha, \sigma_R^{\star})$, where $P_{\mathbf{x}^n}$ is the empirical distribution of \mathbf{x}^n . Moreover, item (iii) in Lemma 3 implies that the state σ_R^{\star} dominates all the channel outputs: $\sigma_R^* \gg W_x$, for all $x \in \operatorname{supp}(P_{\mathbf{x}^n})$, Hence, we have $p^n \ll q^n$. Subsequently, for every $i \in [n]$, we let $q_{x_i}(\omega) = 0$, for all $\omega \notin \operatorname{supp}(p_{x_i})$. We apply Nagaoka's argument [27] by choosing $\delta = \exp\{nR_n - n\phi_n(R_n)\}\$ to yield, for any $0 \le Q_n \le \mathbb{1}$,

$$\alpha \left(Q_n; W_{\mathbf{x}^n}^{\otimes n} \right) + \delta \beta \left(Q_n; \sigma_R^{\star \otimes n} \right) \ge \frac{\alpha \left(\mathcal{U}; p^n \right) + \delta \beta \left(\mathcal{U}; q^n \right)}{2}, \tag{17}$$

where $\alpha(\mathfrak{U};p^n) := \sum_{\omega \in \mathfrak{U}^c} p^n(\omega), \ \beta(\mathfrak{U};q^n) := \sum_{\omega \in \mathfrak{U}} q^n(\omega), \ \text{and} \ \mathfrak{U} := \left\{ \omega : p^n(\omega) e^{n\phi_n(R_n)} > q^n(\omega) e^{nR_n} \right\}.$

Next, we employ Bahadur-Ranga Rao's concentration inequality, Theorem 9 in Appendix A, to further lower bound $\alpha(\mathcal{U}; p^n)$ and $\beta(\mathcal{U}; q^n)$. Before proceeding, we need to introduce some notation. We define the *tilted distributions*, for every $i \in [n], \omega \in \operatorname{supp}(p_{x_i})$, and $t \in [0, 1]$ by

$$\hat{q}_{x_i,t}(\omega) := \frac{p_{x_i}(\omega)^{1-t} q_{x_i}(\omega)^t}{\sum_{\omega \in \operatorname{supp}(p_{x_i})} p_{x_i}(\omega)^{1-t} q_{x_i}(\omega)^t}.$$
(18)

Let

$$\Lambda_{0,x_i}(t) := \log \mathbb{E}_{p_{x_i}} \left[e^{t \log \frac{q_{x_i}}{p_{x_i}}} \right];$$

$$\Lambda_{1,x_i}(t) := \log \mathbb{E}_{q_{x_i}} \left[e^{t \log \frac{p_{x_i}}{q_{x_i}}} \right].$$
(19)

Since p^n and q^n are mutually absolutely continuous, the maps $t \mapsto \Lambda_{j,x_i}(t)$, $j \in \{0,1\}$ are differentiable for all $t \in [0,1]$. One can immediately verify the following partial derivatives with respect to t:

$$\Lambda_{0,x_i}'(t) = \mathbb{E}_{\hat{q}_{x_i,t}} \left[\log \frac{q_{x_i}}{p_{x_i}} \right], \ \Lambda_{0,x_i}'(t) = \operatorname{Var}_{\hat{q}_{x_i,t}} \left[\log \frac{q_{x_i}}{p_{x_i}} \right],$$

$$\Lambda_{0,x_i}'(t) = \operatorname{Var}_{\hat{q}_{x_i,t}} \left[\log \frac{q_{x_i}}{p_{x_i}} \right], \ \Lambda_{1,x_i}'(t) = \mathbb{E}_{\hat{q}_{x_i,1-t}} \left[\log \frac{p_{x_i}}{q_{x_i}} \right].$$
(20)

With $\Lambda_{j,x_i}(t)$ in Eq. (19), we can define

$$\Lambda_{j,P_{\mathbf{x}^n}}(t) := \sum_{x \in \mathcal{X}} P_{\mathbf{x}^n}(x) \Lambda_{j,x}(t), \qquad j \in \{0,1\};$$

$$(21)$$

$$\Lambda_{j,P_{\mathbf{x}^n}}^*(z) := \sup_{t \in \mathbb{R}} \left\{ tz - \Lambda_{j,P_{\mathbf{x}^n}}(t) \right\}, \quad j \in \{0,1\},$$
(22)

where $\Lambda_{j,P_{\mathbf{x}^n}}^*(z)$ in Eq. (22) is the *Fenchel-Legendre transform* of $\Lambda_{j,P_{\mathbf{x}^n}}(t)$. The quantities $\Lambda_{j,P_{\mathbf{x}^n}}^*(z)$ would appear in the lower bounds of $\alpha(\mathcal{U};p^n)$ and $\beta(\mathcal{U};q^n)$ obtained by Bahadur-Randga Rao's inequality as shown later.

In the following, we relate the Fenchel-Legendre transform $\Lambda_{j,P_{\mathbf{x}^n}}^*(z)$ to the desired error-exponent function $\phi_n(R_n)$. Such a relationship is stated in Lemma 7; the proof is provided in Appendix B.3.

Lemma 7. Under the prevailing assumptions and for all $R_n \in (R_\infty, C)$, the following holds:

- (i) $\Lambda_{0,P_{\mathbf{x}^n}}^* (\phi_n(R_n) R_n) = \phi_n(R_n);$
- (ii) $\Lambda_{1,P_{rn}}^{*}(R_n \phi_n(R_n)) = R_n;$
- (iii) There exists a unique $t^* = \frac{s^*}{1+s^*} \in (0,1)$, such that $\Lambda'_{0,P_{\mathbf{x}^n}}(t^*) = \phi_n(R_n) R_n$, where $s^* := \frac{\partial \phi_n(r)}{\partial r}|_{r=R_n}$.

Item (iii) in Lemma 7 shows that the optimizer t in Eq. (22) always lies in the compact set [0, 1]. Further, Eqs. (19) and (20) ensure that $\Lambda_{0,x_i}(t) = \Lambda_{1,x_i}(1-t)$, $\Lambda'_{0,x_i}(t) = -\Lambda'_{1,x_i}(1-t)$, $\Lambda''_{0,x_i}(t) = \Lambda''_{1,x_i}(1-t)$. We define the following quantities:

$$V_{\max} := \max_{t \in [0,1], x \in \mathcal{X}} \Lambda_{0,x}''(t);$$
(23)

$$V_{\min} := \min_{t \in [0,1], x \in \mathcal{X}} \Lambda_{0,x}''(t);$$

$$(24)$$

$$T_{\max} := \max_{t \in [0,1], x \in \mathcal{X}} T_{0,x}(t);$$
(25)

$$T_{0,x}(t) := \mathbb{E}_{\hat{q}_{x,t}} \left[\left| \log \frac{q_x}{p_x} - \Lambda'_{0,x}(t) \right|^3 \right];$$

$$(26)$$

 $T_{1,x}(t) := T_{0,x}(1-t)$; and $K_{\max} := 15\sqrt{2\pi}T_{\max}/V_{\min}$. Note that for every $x \in \mathcal{X}$, $\Lambda_{0,x}'(\cdot)$ and $T_{0,x}(\cdot)$ are continuous functions on [0, 1] from the definitions in Eqs. (20), (26) (see also [10, Lemma 9]). The maximization and minimization in the above definitions are well-defined and finite. Moreover, Lemma 8 guarantees that V_{\min} is bounded away from zero.

Lemma 8 (Positivity). For any $R_n \in (R_{\infty}, C)$ and $P_{\mathbf{x}^n} \in \mathcal{P}(\mathcal{X}), \Lambda_{0, P_{\mathbf{x}}^n}'(t) > 0$, for all $t \in [0, 1]$.

Proof. Assume $\Lambda_{0,P_{\star}^n}^{\prime\prime}(t)$ is zero for some $t \in [0,1]$. This is equivalent to

$$p_{x_i}(\omega) = q_{x_i}(\omega) \cdot e^{-\Lambda'_{0,x_i}(t)}, \quad \forall \omega \in p_{x_i}, \quad \forall i \in [n].$$
(27)

Summing the right-hand side of Eq. (27) over $\omega \in p_{x_i}$ gives $1 = \text{Tr}\left[p_{x_i}^0 q_{x_i}\right] e^{-\Lambda'_{0,x_i}(t)}, \quad \forall i \in [n].$ Then, Eqs. (27) and the above equation imply that

$$\phi_n(R_n) = \sup_{0 < \alpha \le 1} \frac{\alpha - 1}{\alpha} \left(R_n + \sum_{x \in \mathcal{X}} P_{\mathbf{x}^n}(x) \log \operatorname{Tr} \left[p_x^0 q_x \right] \right)$$
$$= 0,$$

where we use the fact that $R_n > R_{\infty} = -\sum_{x \in \mathcal{X}} P_{\mathbf{x}^n}(x) \log \operatorname{Tr} \left[p_x^0 q_x \right]$; see Eq. (3)). However, Lemma 5 implies that $\phi_n(R_n) = E_{\operatorname{sp}}(R_n) > 0$, which leads to a contradiction.

Now, we are ready to derive the lower bounds to $\alpha(\mathcal{U}; p^n)$ and $\beta(\mathcal{U}; q^n)$. Let $N_2 \in \mathbb{N}$ be sufficiently large such that for all $n \geq N_2$,

$$\sqrt{n} \ge \frac{1 + (1 + K_{\max})^2}{\sqrt{V_{\min}}}.$$
(28)

Applying Bahadur-Randga Rao's inequality (Theorem 9) to $Z_i = \log q_i - \log p_i$ with the probability measure $\lambda_i = p_i$, and $z = R_n - \phi_n(R_n)$ gives

$$\alpha\left(\mathfrak{U};p^{n}\right) = \Pr\left\{\frac{1}{n}\sum_{i=1}^{n}Z_{i} \ge R_{n} - \phi_{n}(R_{n})\right\}$$
(29)

$$\geq \frac{2A}{\sqrt{n}} \exp\left\{-n\Lambda_{0,P_{\mathbf{x}^n}}^*\left(\phi_n(R_n) - R_n\right)\right\}$$
(30)

where $A := \frac{e^{-K_{\max}}}{\sqrt{4\pi V_{\max}}}$. Similarly, applying Theorem 9 to $Z_i = \log p_i - \log q_i$ with the probability measure $\lambda_i = q_i$, and $z = \phi_n(R_n) - R_n$ yields

$$\beta\left(\mathcal{U};q^n\right) = \Pr\left\{\frac{1}{n}\sum_{i=1}^n Z_i \ge \phi_n(R_n) - R_n\right\}$$
(31)

$$\geq \frac{2A}{\sqrt{n}} \exp\left\{-n\Lambda_{1,P_{\mathbf{x}^n}}^* \left(R_n - \phi_n(R_n)\right)\right\}.$$
(32)

Continuing from Eq. (30) and item (i) in Lemma 7 gives

$$\alpha\left(\mathfrak{U};p^{n}\right) \geq \frac{2A}{\sqrt{n}}\exp\{-n\phi_{n}\left(R_{n}\right)\}.$$
(33)

Eq. (32) together with item (iii) in Lemma 7 yields

$$\beta\left(\mathfrak{U};q^{n}\right) \geq \frac{2A}{\sqrt{n}}\exp\{-nR_{n}\} = 2An^{\gamma}\exp\{-nR\}.$$
(34)

Let $N_3 \in \mathbb{N}$ such that $An^{\gamma} > 1$, for all $n \geq N_3$. Then Eq. (34) implies that $\beta(\mathcal{U}; q^n) > 2 \exp\{-nR\}$. Thus, we can bound the left-hand side of Eq. (17) from below by $\frac{A}{\sqrt{n}} e^{-n\phi_n(R_n)}$. For any test $0 \leq Q_n \leq \mathbb{1}$ such that $\beta(Q_n; \sigma_R^{*\otimes n}) \leq \exp\{-nR\}$, we have

$$\widehat{\alpha}_{\exp\{-nR_n\}} \left(W_{\mathbf{x}^n}^{\otimes n} \| \sigma_R^{\star \otimes n} \right) = \alpha(Q_n; \rho^n)$$

$$\geq \frac{A}{\sqrt{n}} \exp\{-n\phi_n(R_n)\} = \frac{A}{\sqrt{n}} \exp\{-nE_{\operatorname{sp}}(R_n)\}, \qquad (35)$$

where the last equality follows from Lemma 5.

Finally, it remains to remove the back-off term $R_n = R - \gamma_n$ in Eq. (35). By Taylor's theorem, we have

$$E_{\rm sp}(R - \gamma_n) = E_{\rm sp}(R) - \gamma_n E_{\rm sp}'(R) + \frac{\gamma_n^2}{2} E_{\rm sp}''(\bar{R}),$$
(36)

for some $\bar{R} \in (R - \xi, R)$ and $E_{\rm sp}''(\bar{R}) := \left. \frac{\partial^2 E_{\rm sp}^{(1)}(r, U_{\chi})}{\partial r^2} \right|_{r=\bar{R}}$. Further, one can calculate that

$$E_{\rm sp}^{\prime\prime}(\bar{R}) = -\left. \left(\frac{\partial^2 E_0(s, U_{\mathcal{X}})}{\partial s^2} \right|_{s=\bar{s}} \right)^{-1} \tag{37}$$

$$=\frac{(1+\bar{s})^3}{\Lambda_{0,U_{\mathcal{X}}}''\left(\frac{\bar{s}}{1+\bar{s}}\right)} \le \frac{(1+\bar{s})^3}{V_{\min}} =: \Upsilon,$$
(38)

where $\bar{s} = \frac{1-\alpha_{\bar{R}}^*}{\alpha_{\bar{R}}^*}$. From item (iii) in Lemma 3, it follows that both \bar{s} and $|E'_{\rm sp}(R)| = s^*$ are both positive and finite for $\bar{R} \in (R_{\infty}, C)$ and $R \in (R_{\infty}, C)$. Together with the fact that $V_{\rm min} > 0$, we have $\Upsilon \in \mathbb{R}_{>0}$. We apply Taylor's expansion on the function $n^{-(\cdot)}$ again to yield

$$n^{-\frac{1}{2}\left(1+\left|E_{\rm sp}'(R)\right|\right)-\gamma_{n}\Upsilon} = n^{-\frac{1}{2}\left(1+\left|E_{\rm sp}'(R)\right|\right)} \cdot \left(1 - \frac{\log n}{n^{\bar{x}\Gamma}}\gamma_{n}\Upsilon\right)$$
$$= n^{-\frac{1}{2}\left(1+\left|E_{\rm sp}'(R)\right|\right)} \cdot \left(1 - o(1)\right), \tag{39}$$

where the first equality holds for some $\bar{x} \in (0, \gamma_n)$, and the last line follows from the definition $\gamma_n = (\frac{1}{2} + \gamma) \frac{\log}{n}$. Finally, by combining Eqs. (16), (35), and (39), we obtain the desired Eq. (5) for sufficiently large $n \ge N_0 := \max \{N_1, N_2, N_3\}$.

5. DISCUSSION

In this work, we establish a sphere-packing bound with a refined polynomial pre-factor that coincides with the best classical results [10, Theorem 1] to date. As discussed by Altuğ and Wagner [10, Sec. VII], the pre-factor is correct for binary symmetric channels but slightly worse for binary erasure channels (in the order of $1/\sqrt{n}$). On the other hand, our pre-factor matches the recent result of the random coding upper bound [21, Theorem 2], where the pre-factor has been shown to be exact. Hence, we conjecture that the established result is optimal for general symmetric c-q channels.

This work admits variety of potential extensions. First, the symmetric c-q channel studied in this paper is a covariant channel with a cyclic group:

$$W_{\mathcal{U}_{\rm in}(g)x\mathcal{U}_{\rm in}(g)^{\dagger}} = \mathcal{U}_{\rm out}(g)W_x\mathcal{U}_{\rm out}(g)^{\dagger}, \quad \forall g, x \in \mathcal{X},$$

$$\tag{40}$$

where \mathcal{U}_{in} and \mathcal{U}_{out} are the unitary representations on \mathcal{X} and $\mathcal{S}(\mathcal{H})$ such that $\mathcal{U}_{in}(g) \, x \, \mathcal{U}_{in}(g)^{\dagger} = (x + g) \mod |\mathcal{X}|$ and $\mathcal{U}_{out}(g) = V^g$. It would be interesting to investigate whether the refined sphere-packing bound can be extended to covariant quantum channels $\mathcal{N} : \mathcal{S}(\mathcal{H}_{in}) \to \mathcal{S}(\mathcal{H}_{out})$ with arbitrary compact groups. Second, the random coding bound in the quantum case has been proved only for pure-state channels [16]. It is promising to prove the bound for this class of c-q channels by employing the symmetry property. Finally, the refinement provides a new possibility for moderate deviation analysis in c-q channels [13], which is left as future work.

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APPENDIX A. A TIGHT CONCENTRATION INEQUALITY

Let $(Z_i)_{i=1}^n$ be a sequence of independent, real-valued random variables whose probability measures are λ_i . Let $\Lambda_i(t) := \log \mathbb{E}\left[e^{tZ_i}\right]$ and define the Fenchel-Legendre transform of $\frac{1}{n}\sum_{i=1}^n \Lambda_i(\cdot)$ to be: $\Lambda_n^*(z) := \sup_{t \in \mathbb{R}} \left\{ zt - \frac{1}{n} \sum_{i=1}^n \Lambda_i(t) \right\}$, $\forall z \in \mathbb{R}$. Thenb there exists a real number $t^* \in (0,1]$ for every $z \in \mathbb{R}$ such that $z = \frac{1}{n} \sum_{i=1}^n \Lambda_i'(t^*)$ and $\Lambda_n^*(z) = zt^* - \frac{1}{n} \sum_{i=1}^n \Lambda_i(t^*)$. Define the probability measure $\tilde{\lambda}_i$ via $\frac{d\tilde{\lambda}_i}{d\lambda_i}(z_i) := e^{t^* z_i - \Lambda_i(t^*)}$, and let $\bar{Z}_i := Z_i - \mathbb{E}_{\tilde{\lambda}_i}[Z_i]$. Furthermore, define $m_{2,n} := \sum_{i=1}^n \operatorname{Var}_{\tilde{\lambda}_i}[\bar{Z}_i]$, $m_{3,n} := \sum_{i=1}^n \mathbb{E}_{\tilde{\lambda}_i} \left[\left| \bar{Z}_i \right|^3 \right]$, and $K_n(t^*) := \frac{15\sqrt{2\pi}m_{3,n}}{m_{2,n}}$. With these definitions, we can now state the following sharp concentration inequality for $\frac{1}{n} \sum_{i=1}^n Z_i$:

Theorem 9 (Bahadur-Ranga Rao's Concentration Inequality [11, Proposition 5], [28]). Given

$$\sqrt{m_{2,n}} \ge 1 + (1 + K_n(t^*))^2,$$
(41)

it follows that

$$\Pr\left\{\frac{1}{n}\sum_{i=1}^{n} Z_{i} \ge z\right\} \ge e^{-n\Lambda_{n}^{*}(z)} \frac{e^{-K_{n}(t^{*})}}{2\sqrt{2\pi m_{2,n}}}.$$
(42)

Appendix B. Proofs of Miscellaneous Lemmas

B.1. **Proof of Lemma 3.** Let $R > R_{\infty}$ and $P \in \mathcal{P}(\mathcal{X})$ be arbitrary. It is convenient to reparameterize the function $F_{R,P}$ by the substitution $\alpha = \frac{1}{1+s}$:

$$F_{R,P}(\alpha,\sigma)|_{\alpha=\frac{1}{1+s}} = -sR + sD_{\frac{1}{1+s}}(W||\sigma|P) =: K_{R,P}(s,\sigma).$$
(43)

In the following, we prove the existence of a saddle-point of $K_{R,P}(\cdot, \cdot)$ on $\mathbb{R}_{\geq 0} \times \mathcal{S}_{P,W}(\mathcal{H})$, where $\mathbb{R}_{\geq 0} := [0, \infty)$. By Ref. [29, Lemma 36.2], (s^*, σ^*) is a saddle point of $K_{R,P}(\cdot, \cdot)$ if and only if the supremum in

$$\sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in \mathcal{S}_{P,W}(\mathcal{H})} K_{R,P}(s,\sigma)$$
(44)

is attained at s^* , the infimum in

$$\inf_{\sigma \in \mathcal{S}_{P,W}(\mathcal{H})} \sup_{s \in \mathbb{R}_{\ge 0}} K_{R,P}(s,\sigma) \tag{45}$$

is attained at σ^* , and the two extrema in Eqs. (44), (45) are equal and finite. We first claim that

$$\forall s \in \mathbb{R}_{\geq 0}, \quad \inf_{\sigma \in \mathcal{S}_{P,W}(\mathcal{H})} K_{R,P}(s,\sigma) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} K_{R,P}(s,\sigma).$$
(46)

To see that, observe that for any $s \in \mathbb{R}_{\geq 0}$, the definition of the α -Rényi divergence yields

$$\forall \sigma \in \mathcal{S}(\mathcal{H}) \setminus \mathcal{S}_{P,W}(\mathcal{H}), \quad D_{\frac{1}{1+s}}(W \| \sigma | P) = +\infty,$$
(47)

which, in turn, implies

$$\forall \sigma \in \mathcal{S}(\mathcal{H}) \backslash \mathcal{S}_{P,W}(\mathcal{H}), \quad K_{R,P}(s,\sigma) = +\infty.$$
(48)

Hence, Eq. (46) yields

$$\sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in \mathcal{S}_{P,W}(\mathcal{H})} K_{R,P}(s,\sigma) = \sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} K_{R,P}(s,\sigma) = \sup_{s \in \mathbb{R}_{\geq 0}} \min_{\sigma \in \mathcal{S}(\mathcal{H})} K_{R,P}(s,\sigma),$$
(49)

where the last equality in Eq. (49) follows from the lower semi-continuity of the map $\sigma \mapsto D_{1/(1+s)}(W \| \sigma | P)$ [24, Corollary III.25] and the compactness of $\mathcal{S}(\mathcal{H})$. Further, by the fact $R > R_{\infty}$ and the definition of $E_{sp}^{(2)}$, we have

$$E_{\rm sp}^{(2)}(R,P) = \sup_{s \in \mathbb{R}_{\ge 0}} \min_{\sigma \in \mathcal{S}(\mathcal{H})} K_{R,P}(s,\sigma) < +\infty, \tag{50}$$

which guarantees the supremum in the right-hand side of Eq. (49) is attained at some $s \in \mathbb{R}_{\geq 0}$, i.e.,

$$\sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in \mathcal{S}_{P,W}(\mathcal{H})} K_{R,P}(s,\sigma) = \max_{s \in \mathbb{R}_{\geq 0}} \min_{\sigma \in \mathcal{S}(\mathcal{H})} K_{R,P}(s,\sigma) < +\infty.$$
(51)

Thus, we complete our claim in Eq. (44). It remains to show that the infimum in Eq.(45) is attained at some $\sigma^* \in \mathcal{S}_{P,W}(\mathcal{H})$ and the supremum and infimum are exchangeable. To achieve this, we will show that $(\mathbb{R}_{\geq 0}, \mathcal{S}_{P,W}(\mathcal{H}), K_{R,P})$ is a closed saddle-element (see Definition 10 below) and apply Rockafellar's saddle-point result, Theorem 11, to conclude our claim.

Definition 10 (Closed Saddle-Element [29]). The triple $(\mathcal{A}, \mathcal{B}, F)$ is called a closed saddle-element if for any¹ $x \in ri(\mathcal{A})$ (resp. $y \in ri(\mathcal{B})$):

- (a) \mathcal{B} (resp. \mathcal{A}) is convex;
- (b) $F(x, \cdot)$ (resp. $F(\cdot, y)$) is convex (resp. concave) and lower (resp. upper) semi-continuous; and
- (c) any accumulation point of \mathcal{B} (resp. \mathcal{A}) that does not belong to \mathcal{B} (resp. \mathcal{A}), say y_o (resp. x_o) satisfies $\lim_{y\to y_o} F(x,y) = +\infty$ (resp. $\lim_{x\to x_o} F(x,y) = -\infty$).

Theorem 11 (The Existence of Saddle-Points [29, Theorem 8], [30, Theorem 37.3]). Let $(\mathcal{A}, \mathcal{B}, F)$ be any closed saddle-element on $\mathbb{R}^m \times \mathbb{R}^n$.

- (I) No non-zero x_0 has the property that, for all $x \in ri(\mathcal{A})$ and $y \in ri(\mathcal{B})$, the half-line $\{x + tx_0 : t \ge 0\}$ is contained in \mathcal{A} and $F(x + tx_0, y)$ is a non-zero and non-decreasing function for $t \ge 0$.
- (II) No non-zero y_0 has the property that, for all $x \in ri(\mathcal{A})$ and $y \in ri(\mathcal{B})$, the half-line $\{y + ty_0 : t \ge 0\}$ is contained in \mathcal{B} and $F(x, y + ty_0)$ is a non-increasing function for $t \ge 0$.

If condition (I) is satisfied, then

$$\max_{x \in \mathcal{A}} \inf_{y \in \mathcal{B}} F(x, y) = \inf_{y \in \mathcal{B}} \sup_{x \in \mathcal{A}} F(x, y) < +\infty.$$
(52)

If condition (II) is satisfied, then

$$-\infty < \sup_{x \in \mathcal{A}} \inf_{y \in \mathcal{B}} F(x, y) = \min_{y \in \mathcal{B}} \sup_{x \in \mathcal{A}} F(x, y).$$
(53)

If (I) and (II) are both satisfied, then F has a saddle-point on $\mathcal{A} \times \mathcal{B}$.

Fix an arbitrary $s \in \operatorname{ri}(\mathbb{R}_{\geq 0}) = \mathbb{R}_{>0}$. We check that $(\mathcal{S}_{P,W}(\mathcal{H}), K_{R,P}(s, \cdot))$ fulfills the three items in Definition 10. (a) The set $\mathcal{S}_{P,W}(\mathcal{H})$ is clearly convex. (b) Since the map $\sigma \mapsto D_{1/(1+s)}(W \| \sigma | P)$ is convex (owing to Lieb's concavity theorem [31]) and lower semi-continuous on $\mathcal{L}(\mathcal{H})_+$ [24, Corollary III.25], by Eq. (43), $\sigma \mapsto K_{R,P}(\alpha, \sigma)$ is also convex and lower semi-continuous on $\mathcal{S}_{P,W}(\mathcal{H})$. (c) Due to the compactness of $\mathcal{S}(\mathcal{H})$, any accumulation point of $\mathcal{S}_{P,W}(\mathcal{H})$ that does not belong to $\mathcal{S}_{P,W}(\mathcal{H})$, say σ_o , satisfies $\sigma_o \in \mathcal{S}(\mathcal{H}) \setminus \mathcal{S}_{P,W}(\mathcal{H})$. By Eqs. (47), (48), one finds $K_{R,P}(\alpha, \sigma_o) = +\infty$.

Next, fix an arbitrary $\sigma \in ri(\mathcal{S}_{P,W}(\mathcal{H}))$. Owing to the convexity of $\mathcal{S}_{P,W}(\mathcal{H})$, it follows that $ri(\mathcal{S}_{P,W}(\mathcal{H})) = ri(cl(\mathcal{S}_{P,W}(\mathcal{H})))$ (see e.g. [30, Theorem 6.3]). We first claim $cl(\mathcal{S}_{P,W}(\mathcal{H})) = \mathcal{S}(\mathcal{H})$. To see this, observe that $\mathcal{S}(\mathcal{H})_{++} \subseteq \mathcal{S}_{P,W}(\mathcal{H})$ since a full-rank density operator is not orthogonal with every $W_x, x \in \mathcal{X}$. Hence,

$$\mathcal{S}(\mathcal{H}) = \mathsf{cl}\left(\mathcal{S}(\mathcal{H})_{++}\right) \subseteq \mathsf{cl}\left(\mathcal{S}_{P,W}(\mathcal{H})\right).$$
(54)

On the other hand, the fact $\mathcal{S}_{P,W}(\mathcal{H}) \subseteq \mathcal{S}(\mathcal{H})$ leads to

$$cl(\mathcal{S}_{P,W}(\mathcal{H})) \subseteq cl(\mathcal{S}(\mathcal{H})) = \mathcal{S}(\mathcal{H}).$$
(55)

By Eqs. (54) and (55), we deduce that

$$\operatorname{ri}(\mathcal{S}_{P,W}(\mathcal{H})) = \operatorname{ri}(\operatorname{cl}(\mathcal{S}_{P,W}(\mathcal{H}))) = \operatorname{ri}(\mathcal{S}(\mathcal{H})) = \mathcal{S}(\mathcal{H})_{++},$$
(56)

where the last equality in Eq. (56) follows from [32, Proposition 2.9]. Hence, we obtain

$$\forall \sigma \in \operatorname{ri}\left(\mathcal{S}_{P,W}(\mathcal{H})\right) \quad \text{and} \quad \forall x \in \mathcal{X}, \quad \sigma \gg W_x.$$
(57)

Now, we verify that $(\mathbb{R}_{\geq 0}, K_{R,P}(\cdot, \sigma))$ satisfies the three items in Definition 10. (a) The set $\mathbb{R}_{\geq 0}$ is obviously convex. (b) From Eqs. (57) and the definition of the Rényi divergence, the map $s \mapsto D_{1/(1+s)}(W \| \sigma | P)$ is continuous on $\mathbb{R}_{\geq 0}$. Further, $s \mapsto sD_{1/(1+s)}(W \| \sigma | P)$ is concave on $\mathbb{R}_{\geq 0}$ [24, Appendix B]. By Eq. (43), the map $s \mapsto K_{R,P}(s,\sigma)$ is concave and continuous on $\mathbb{R}_{\geq 0}$. (c) Since $\mathbb{R}_{\geq 0}$ is closed, there is no accumulation point of $\mathbb{R}_{\geq 0}$ that does not belong to $\mathbb{R}_{\geq 0}$.

¹We denote by ri and cl the relative interior and the closure of a set, respectively.

We are now in a position to prove item (i) of this Proposition. Since the set $S_{P,W}(\mathcal{H})$ is bounded, condition (II) is satisfied. Equation (53) in Theorem 11 implies that

$$-\infty < \sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in \mathcal{S}_{P,W}(\mathcal{H})} K_{R,P}(s,\sigma) = \min_{\sigma \in \mathcal{S}_{P,W}(\mathcal{H})} \sup_{s \in \mathbb{R}_{\geq 0}} K_{R,P}(s,\sigma).$$
(58)

Then Eqs. (51) and (58) lead to the existence of a saddle-point of $K_{R,P}(\cdot, \cdot)$ on $\mathbb{R}_{\geq 0} \times S_{P,W}(\mathcal{H})$. Note that $K_{R,P}(s,\sigma) = F_{R,P}(1/(1+s),\sigma)$. We conclude the existence of a saddle-point of $F_{R,P}(\cdot, \cdot)$ on $(0,1] \times S_{P,W}(\mathcal{H})$. Hence, item (i) is proved.

We postpone the proof of the uniqueness of the optimizer to later and now show item (iii). Given any $R \in (R_{\infty}, C)$ and $P \in \mathcal{P}_R(\mathcal{X})$, one finds

$$\min_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{0 < \alpha \le 1} F_{R,P}(\alpha, \sigma) \in (0, +\infty).$$
(59)

If $\alpha^* = 1$ and σ^* is a saddle point of $F_{R,P}(\cdot, \cdot)$, by Eq. (10) we deduce that $F_{R,P}(1, \sigma^*) = 0$ for every possible σ^* , which contradicts Eq. (59). Hence, $\alpha^* = 1$ is not a saddle point of $F_{R,P}(\cdot, \sigma^*)$.

For any saddle-point $(\alpha^{\star}, \sigma^{\star})$ of $F_{R,P}(\cdot, \cdot)$, it holds that

$$F_{R,P}(\alpha^{\star},\sigma^{\star}) = \min_{\sigma \in \mathcal{S}(\mathcal{H})} F_{R,P}(\alpha^{\star},\sigma) = \frac{\alpha^{\star}-1}{\alpha^{\star}}R + \frac{1-\alpha^{\star}}{\alpha^{\star}}\min_{\sigma \in \mathcal{S}(\mathcal{H})} D_{\alpha^{\star}}(W||\sigma|P).$$
(60)

We claim the minimizer of Eq. (60) must satisfy

$$\sigma^{\star} = \frac{\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha^{\star}}}{\operatorname{Tr}[W_x^{\alpha^{\star}}(\sigma^{\star})^{1-\alpha^{\star}}]}\right)^{\frac{1}{\alpha^{\star}}}}{\operatorname{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha^{\star}}}{\operatorname{Tr}[W_x^{\alpha^{\star}}(\sigma^{\star})^{1-\alpha^{\star}}]}\right)^{\frac{1}{\alpha^{\star}}}\right]} = \frac{\left(\sum_{x \in \mathcal{X}} P(x) W_x^{\alpha^{\star}} \mathrm{e}^{(1-\alpha^{\star})D_{\alpha^{\star}}(W_x \| \sigma)}\right)^{\frac{1}{\alpha^{\star}}}}{\operatorname{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) W_x^{\alpha^{\star}} \mathrm{e}^{(1-\alpha^{\star})D_{\alpha^{\star}}(W_x \| \sigma)}\right)^{\frac{1}{\alpha^{\star}}}\right]}$$
(61)

for every $\alpha^* \in (0, 1)$. Our approach closely follows from Hayashi and Tomamichel [33, Lemma 5]. For two density operators $\sigma, \omega \in \mathcal{S}(\mathcal{H})$ and a map $G : \mathcal{S}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})_{sa}$ (where $\mathcal{L}(\mathcal{H})_{sa}$ denotes the self-adjoint operators on \mathcal{H}), define the Fréchet derivative (see e.g. [33, Appendix C], [34]²)

$$\partial_{\omega} G(\sigma) := \mathsf{D} G(\sigma)[\omega - \sigma]. \tag{62}$$

By letting

$$g_{\alpha}(\sigma) := \sum_{x \in \mathcal{X}} P(x) \log \operatorname{Tr} \left[W_x^{\alpha} \sigma^{1-\alpha} \right],$$
(63)

it follows that

$$\sigma^{\star} = \underset{\sigma \in \mathcal{S}(\mathcal{H})}{\arg\min} D_{\alpha} \left(W \| \sigma | P \right) = \underset{\sigma \in \mathcal{S}(\mathcal{H})}{\arg\max} g_{\alpha}(\sigma), \quad \forall \alpha \in (0, 1).$$
(64)

Since the map $\sigma \mapsto g_{\alpha}(\sigma)$ is strictly concave for every $\alpha \in (0,1)$ [31], a sufficient and necessary condition for σ to be an optimizer of Eq. (64) is $\partial_{\omega}g_{\alpha}(\sigma) = 0$ for all $\omega \in \mathcal{S}(\mathcal{H})$. Direct calculation shows that

$$\partial_{\omega}g_{\alpha}(\sigma) = \operatorname{Tr}\left[\sum_{x\in\mathcal{X}} P(x)\frac{W_{x}^{\alpha}}{\operatorname{Tr}\left[W_{x}^{\alpha}\sigma^{1-\alpha}\right]}\partial_{\omega}\sigma^{1-\alpha}\right]$$
(65)

Next, we check that the fixed-points of the following map achieves the optimum:

$$\sigma \mapsto \frac{\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha}}{\operatorname{Tr}[W_x^{\alpha} \sigma^{1-\alpha}]}\right)^{\frac{1}{\alpha}}}{\operatorname{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha}}{\operatorname{Tr}[W_x^{\alpha} \sigma^{1-\alpha}]}\right)^{\frac{1}{\alpha}}\right]}.$$
(66)

 $^{^{2}}$ We note that the Fréchet derivative of functions involving matrices has other applications in quantum information theory; see e.g. [35, 36, 37].

Let

$$\chi_{\alpha}(\sigma) := \operatorname{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha}}{\operatorname{Tr}\left[W_x^{\alpha} \sigma^{1-\alpha}\right]}\right)^{\frac{1}{\alpha}}\right] > 0, \quad \forall \alpha \in (0, 1),$$
(67)

and let $\bar{\sigma}$ be a fix-point of the map in Eq. (66). Then, by Eqs. (66), (67), we have

$$\chi_{\alpha}(\bar{\sigma}) \cdot \bar{\sigma} = \left(\sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha}}{\operatorname{Tr}\left[W_x^{\alpha} \bar{\sigma}^{1-\alpha}\right]}\right)^{\frac{1}{\alpha}}.$$
(68)

Substituting Eq. (68) into Eq. (65) yields

$$\partial_{\omega}g_{\alpha}(\bar{\sigma}) = \operatorname{Tr}\left[\chi_{\alpha}(\bar{\sigma})^{\alpha}\bar{\sigma}^{\alpha}\partial_{\omega}\bar{\sigma}^{1-\alpha}\right] = \operatorname{Tr}\left[\chi_{\alpha}(\bar{\sigma})^{\alpha}\bar{\sigma}^{\alpha}(1-\alpha)\bar{\sigma}^{-\alpha}(\omega-\bar{\sigma})\right]$$

= $(1-\alpha)\chi_{\alpha}(\bar{\sigma})^{\alpha}\operatorname{Tr}\left[\omega-\bar{\sigma}\right] = 0.$ (69)

By Brouwer's fixed-point theorem, the map in Eq. (66) is indeed the optimizer for Eq. (64). Further, from Eq. (61), it is clear that

$$\sigma^* \gg W_x, \quad \forall x \in \operatorname{supp}(P), \tag{70}$$

and thus item (iii) is proved.

Lastly, we show the uniqueness of the saddle-point. Since the map $\sigma \mapsto D_{\alpha}(W || \sigma | P)$ is strictly concave [31], the minimizer of Eq. (59) is unique for any $\alpha \in (0, 1)$. Then, it remains to prove the uniqueness of the maximizer. Let σ^* attain the minimum in Eq. (59). By using the reparameterization again, we have

$$K_{R,P}(s,\sigma^{\star}) = -sR + sD_{\frac{1}{1+s}}(W \| \sigma^{\star} | P)$$
(71)

$$= -sR + s \sum_{x \in \mathcal{X}} P(x) D_{\frac{1}{1+s}}(p_x || q_x), \qquad (72)$$

where p_x, q_x are the Nussbaum-Szkoła distributions of W_x and σ^* . The second-order partial derivative can be calculated as

$$\frac{\partial^2 K_{R,P}\left(s,\sigma^\star\right)}{\partial s^2} = -\frac{1}{(1+s)^3} \sum_{x \in \mathcal{X}} P(x) \operatorname{Var}_{\hat{q}_{\frac{1}{1+s},x}} \left[\log \frac{q_x}{p_x}\right],\tag{73}$$

where

$$\hat{q}_{t,x}(\omega) := \frac{p_x(\omega)^{1-t}q_x(\omega)^t}{\sum_{\omega \in \operatorname{supp}(p_x) \cap \operatorname{supp}(q_x)} p_x(\omega)^{1-t}q_x(\omega)^t}, \quad \forall \omega \in \operatorname{supp}(p_x) \cap \operatorname{supp}(q_x), \ t \in [0,1].$$
(74)

Now, we assume the right-hand side of Eq. (73) is zero, which is equivalent to

 $p_x(\omega) = c_x \cdot q_x(\omega), \quad \forall \omega \in \operatorname{supp}(p_x) \cap \operatorname{supp}(q_x)$ (75)

for some constant $c_x > 0$ and $x \in \text{supp}(P)$. From Eq. (70), one finds $p_x \ll q_x$. Summing the right-hand side of Eq. (75) over $\omega \in p_x^0$ yields

$$1 = c_x \cdot \operatorname{Tr}\left[p_x^0 q_x\right], \quad \forall x \in \operatorname{supp}(P).$$
(76)

By combining Eqs. (75) and (76), one can verify

$$\sup_{s \in \mathbb{R}_{>0}} \left\{ -sR + s \sum_{x \in \mathcal{X}} P(x) D_{\frac{1}{1+s}}(p_x \| q_x) \right\} = \sup_{s \in \mathbb{R}_{>0}} \left\{ -sR - s \sum_{x \in \mathcal{X}} P(x) \log \operatorname{Tr}\left[p_x^0 q_x\right] \right\} = 0, \quad (77)$$

where we rely the fact $R > R_{\infty}(W) \ge -\sum_{x \in \mathcal{X}} P(x) \log \operatorname{Tr} \left[p_x^0 q_x \right]$ from Eq. (3). However, Eq. (77) contradicts the assumption $P \in \mathcal{P}_R(\mathcal{X})$, which in turn implies that the right-hand side of Eq. (73) is strictly negative. Therefore, the map $s \to K_{R,P}(s, \sigma^*)$ is strictly concave for all $s \in \mathbb{R}_{>0}$ and thus the maximizer of Eq. (59) is unique.

B.2. **Proof of Lemma 6.** Let $\mathbf{x}^n(m)$ be the codeword encoding the message $m \in \{1, \ldots, \exp\{nr\}\}$. We define a binary hypothesis testing problem as:

$$\mathsf{H}_{0}: W_{\mathbf{x}^{n}(m)}^{\otimes n}; \tag{78}$$

$$\mathsf{H}_1: \sigma^n := \bigotimes_{i=1}^n \sigma_i,\tag{79}$$

where $\sigma^n \in \mathcal{S}(\mathcal{H}^{\otimes n})$ can be viewed as a dummy channel output. Since $\sum_{m=1}^{M} \beta(\Pi_{n,m}; \sigma^n) = 1$ for any POVM $\Pi_n = {\Pi_{n,1}, \ldots, \Pi_{n,\exp\{nr\}}}$, and $\beta(\Pi_{n,m}; \sigma^{\otimes n}) \ge 0$ for every $m \in \mathcal{M}$, there must exist a message $m \in \mathcal{M}$ for any code \mathcal{C}_n such that $\beta(\Pi_{n,m}; \sigma^n) \le \exp\{-nr\}$. Let $\mathbf{x}^n := \mathbf{x}^n(m)$ be the codeword for that message m. Then

$$\epsilon_{\max}\left(\mathcal{C}_{n}\right) \geq \epsilon_{m}\left(\mathcal{C}_{n}\right) = \alpha\left(\Pi_{n,m}; W_{\mathbf{x}^{n}}^{\otimes n}\right) \geq \widehat{\alpha}_{\exp\{-nr\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma^{n}\right).$$

$$(80)$$

Since the above inequality (80) holds for every $\sigma^n \in \mathcal{S}(\mathcal{H}^{\otimes n})$, it follows that

$$\epsilon_{\max}\left(\mathcal{C}_{n}\right) \geq \max_{\sigma \in \mathcal{S}(H)} \widehat{\alpha}_{\exp\{-nr\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma^{\otimes n}\right).$$
(81)

B.3. Proof of Lemma 7. This lemma closely follows from Altuğ and Wagner's [11, Lemma 9]. However, the major difference is that we prove the claim using the expression ϕ_n as the error-exponent instead of the discrimination function: $\min \{ \mathbb{D}(\tau || \rho) : \mathbb{D}(\tau || \sigma) \leq R_n \}$. This expression is crucial to obtaining the sphere-packing bound in Theorem 1 in the strong form of Gallager's expression

For convenience, we shorthand $r = R_n$. From Lemma 5, it can be verified that

$$E_0(s) := -\frac{1+s}{n} \log \operatorname{Tr}\left[(p^n)^{\frac{1}{1+s}} (q^n)^{\frac{s}{1+s}} \right]$$
(82)

$$= -(1+s)\Lambda_{0,P_{\mathbf{x}^n}}\left(\frac{s}{1+s}\right),\tag{83}$$

where Eq. (83) follows from the definition of $\Lambda_{0,P_{\mathbf{x}^n}}$ in Eq. (21). Then, we rewrite the error-exponent function $\phi_n(r)$ by the Legendre-Fenchel transform of $E_0(s)$, i.e.,

$$\phi_n(r) = \sup_{\alpha \in (0,1]} \left\{ \frac{\alpha - 1}{\alpha} \left(r - \sum_{x \in \mathcal{X}} P_{\mathbf{x}^n}(x) D_\alpha\left(p_x \| q_x\right) \right) \right\}$$
(84)

$$= \sup_{s \ge 0} \left\{ -sr + E_0(s) \right\}.$$
(85)

Direct calculation shows that

$$\frac{\partial E_0(s)}{\partial s} = -\Lambda_{0,P_{\mathbf{x}^n}} \left(\frac{s}{1+s}\right) - \frac{1}{1+s}\Lambda_{0,P_{\mathbf{x}^n}}' \left(\frac{s}{1+s}\right),\tag{86}$$

$$\frac{\partial^2 E_0(s)}{\partial s^2} = -\frac{1}{(1+s)^3} \Lambda_{0,P_{\mathbf{x}^n}}'' \left(\frac{s}{1+s}\right).$$
(87)

Now assume the second-order derivative $\Lambda_{0,P_{\mathbf{x}^n}}'(t)$ in right-hand side of Eq. (87) is zero for some $t \in [0, 1]$. This is equivalent to

$$p_x(\omega) = q_x(\omega) \cdot e^{-\Lambda'_{0,x}(t)}, \quad \forall \omega \in p_x, \quad \forall x \in \operatorname{supp}(P_{\mathbf{x}^n}).$$
 (88)

Summing the right-hand side of Eq. (88) over $\omega \in p_x$ gives

$$1 = \operatorname{Tr}\left[p_x^0 q_x\right] e^{-\Lambda'_{0,x}(t)}.$$
(89)

Then, Eqs. (88) and (89) imply that

$$\phi_n(r) = \sup_{0 < \alpha \le 1} \frac{\alpha - 1}{\alpha} \left(r - \sum_{x \in \mathcal{X}} P_{\mathbf{x}^n}(x) D_\alpha\left(p_x \| q_x\right) \right)$$
(90)

$$= \sup_{0 < \alpha \le 1} \frac{\alpha - 1}{\alpha} \left(r + \sum_{x \in \mathcal{X}} P_{\mathbf{x}^n}(x) \log \operatorname{Tr} \left[p_x^0 q_x \right] \right) = 0,$$
(91)

where in Eq. (91) we use the fact that $r > -\sum_{x \in \mathcal{X}} P_{\mathbf{x}^n}(x) \log \operatorname{Tr} [p_x^0 q_x]$; see Eq. (3). However, from Lemma 5 we know that $\phi_n(r) = E_{\mathrm{sp}}(R) > 0$, which leads to a contradiction. Hence, we obtain

$$\Lambda_{0,P_{\mathbf{x}^n}}''(t) > 0, \quad \forall t \in [0,1],$$
(92)

and prove item (i).

From Eqs. (87) and (92), the objective function $-sr + E_0(s)$ in Eq. (85) is strictly concave in s for $s \in \mathbb{R}_+$. Further, by recalling that $\phi_n(r) = E_{sp}(R) > 0$, s = 0 will not be an optimum in Eq. (85). We deduce that there exists a unique maximizer $s^* \in \mathbb{R}_{>0}$ such that

$$r = \left. \frac{\partial E_0(s)}{\partial s} \right|_{s=s^\star},\tag{93}$$

$$\phi_n(r) = E_0(s^*) - s^* \left. \frac{\partial E_0(s)}{\partial s} \right|_{s=s^*},\tag{94}$$

if r lies in the range:

$$-\frac{1}{n}\log\operatorname{Tr}\left[(p^{n})^{0}q^{n}\right] = \lim_{s \to +\infty} \frac{\partial E_{0}(s)}{\partial s} \leq r \leq \left.\frac{\partial E_{0}(s)}{\partial s}\right|_{s=0} = \frac{1}{n}\mathbb{D}\left(p^{n}\|q^{n}\right),\tag{95}$$

where the boundary values $-\frac{1}{n}\log \operatorname{Tr}\left[(p^n)^0q^n\right]$ and $\frac{1}{n}\mathbb{D}\left(p^n\|q^n\right)$ can be obtained from Eqs. (86), (19) and (20). Substituting Eq. (93) into (86) gives

$$r = -\Lambda_{0,P_{\mathbf{x}^n}} \left(\frac{s^{\star}}{1+s^{\star}}\right) - \frac{1}{1+s^{\star}} \Lambda_{0,P_{\mathbf{x}^n}}' \left(\frac{s^{\star}}{1+s^{\star}}\right).$$
(96)

Further, Eqs. (85), (83), (96) imply that

$$\phi_n(r) = -s^* r + E_0(s^*) \tag{97}$$

$$= \frac{s^{\star}}{1+s^{\star}} \Lambda_{0,P_{\mathbf{x}^n}}^{\prime} \left(\frac{s^{\star}}{1+s^{\star}}\right) - \Lambda_{0,P_{\mathbf{x}^n}} \left(\frac{s^{\star}}{1+s^{\star}}\right).$$
(98)

By comparing Eqs. (96) and (98), we obtain

$$\Lambda_{0,P_{\mathbf{x}^n}}^{\prime}\left(\frac{s^{\star}}{1+s^{\star}}\right) = \phi_n(r) - r \tag{99}$$

which is exactly the optimum solution to the Fenchel-Legendre transform $\Lambda^*_{0,P_{\mathbf{x}^n}}(z)$ in Eq. (22) with

$$t^{\star} = \frac{s^{\star}}{1+s^{\star}} \in (0,1), \tag{100}$$

$$z = \phi_n(r) - r. \tag{101}$$

From Eqs. (22), (99) and (98), we conclude the item (i) of Lemma 7:

$$\Lambda_{0,P_{\mathbf{x}^n}}^*(\phi_n(r) - r) = t^* z - \Lambda_{0,P_{\mathbf{x}^n}}(t^*)$$
(102)

$$= \frac{s^{\star}}{1+s^{\star}} \left(\phi_n(r) - r\right) - \Lambda_{0,P_{\mathbf{x}^n}} \left(\frac{s^{\star}}{1+s^{\star}}\right)$$
(103)

$$= \frac{s^{\star}}{1+s^{\star}} \Lambda_{0,P_{\mathbf{x}^n}}' \left(\frac{s^{\star}}{1+s^{\star}}\right) - \Lambda_{0,P_{\mathbf{x}^n}} \left(\frac{s^{\star}}{1+s^{\star}}\right)$$
(104)
$$= \phi_n(r).$$
(105)

$$=\phi_n(r).\tag{105}$$

Item (ii) follows from item (i), the symmetry $\Lambda_{0,x_i}(t) = \Lambda_{1,x_i}(1-t)$ and $\Lambda'_{0,x_i}(t) = -\Lambda'_{1,x_i}(1-t)$, and Eq. (22). $\Lambda^*_{1,P_{\mathbf{x}^n}}(r-\phi(r)) = r$.

For the item (iii), the positivity of $\Lambda_{0,P_{\mathbf{x}^n}}'(t)$, for $t \in [0,1]$, implies that the objective function $tz - \Lambda_{0,P_{\mathbf{x}^n}}(t)$ in Eq. (22) is strictly concave in t for $t \in [0,1]$. Hence, by Eq. (100), the optimizer $t^* \in (0,1)$ exists uniquely. By recalling Eq. (99), we complete the claim in item (iii).

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