# Sphere-Packing Bound for Symmetric Classical-Quantum Channels 

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#### Abstract

We provide a sphere-packing lower bound for the optimal error probability in finite blocklengths when coding over a symmetric classical-quantum channel. Our result shows that the pre-factor can be significantly improved from the order of the subexponential to the polynomial. This established pre-factor is essentially optimal because it matches the best known random coding upper bound in the classical case. Our approaches rely on a sharp concentration inequality in strong large deviation theory and crucial properties of the error-exponent function.


## 1. Introduction

The probability of decoding error is one of the fundamental criteria for evaluating the performance of a communication system. In Shannon's seminal work [1], he pioneered the study of the noisy coding theorem, which states that the error probability can be made arbitrarily small as the coding blocklength grows when the coding rate $R$ is below the channel capacity $C$. Later, Shannon [2] made a further step in exploring the exponential dependency of the optimal error probability $\epsilon^{*}(n, R)$ on the blocklength $n$ and rate $R$, and defined the reliability function as follows: given a fixed coding rate $R<C, E(R):=$ $\lim \sup _{n \rightarrow+\infty}-\frac{1}{n} \log \epsilon^{*}(n, R)$. The quantity $E(R)$ then provides a measure of how rapidly the error probability approaches zero with an increase in blocklength. This asymptotic characterization of the optimal error probability under a fixed rate is hence called the error exponent analysis. For a classical channel, the upper bounds of the optimal error can be established using a random coding argument [3]. On the other hand, the lower bound was first developed by Shannon, Gallager, and Berlekamp [4] and was called the sphere-packing bound. Alternative approaches by Haroutunian [5] and Blahut [6] were subsequently proposed.

In recent years, much attention has been paid to the finite blocklength regime [7, 8]. Altuğ and Wagner employed strong large deviation techniques [9] to prove a sphere-packing bound with a finite blocklength $n$. Moreover, the pre-factor of the bound was significantly refined from the order of the subexponential $\exp \{-O(\sqrt{n})\}[4]$ to the polynomial [10, 11]. This refinement is substantial especially at rates near capacity, where the error-exponent function is close to zero; hence, the pre-factor dominants the bound [12, 13].

Error exponent analysis in classical-quantum (c-q) channels is much more difficult because of the noncommutative nature of quantum mechanics. Burnashev and Holevo [15, 16] investigated reliability functions in c-q channels and proved the random coding upper bound for pure-state channels. Winter [17] adopted Haroutunian's method to derive a sphere-packing bound for c-q channels in the form of relative entropy functions [5]. Dalai [18] employed Shannon-Gallager-Berlekamp's approach to establish a spherepacking bound with Gallager's expression [4]. It was later pointed out that these two sphere-packing exponents are not equal for general c-q channels [19]. In this work, we initiate the study of the refined sphere-packing bound in the quantum scenario. In particular, we consider a "symmetric c-q channel" (see Section 2 for a detailed definition), which is an important class of covariant channels (e.g. [20]), and establish a sphere-packing bound with the pre-factor improved from the order of the subexponential

[^0]in Dalai's result [18] to the polynomial. Our result recovers Altuğ and Wagner's work [10] for classical symmetric channels including the binary symmetric channel and binary erasure channel. Furthermore, the proved pre-factor matches that of the best known random coding upper bound [21] in the classical case. Hence, our result yields the exact asymptotics for the sphere-packing bound in symmetric c-q channels. The main ingredients in our proof are a tight concentration inequality from Bahadur and Ranga Rao [9], [13] (see Appendix A) and the major properties of the sphere-packing exponent [22]. We remark that the result obtained in this paper might enable analysis in the medium error probability regime of a classical-quantum channel $[12,13,14]$. We leave the case for general c-q channels as future work [23].

This paper is organized as follows. We introduce the necessary notation and state our main result in Section 2. Section 3 includes the crucial properties of the error-exponent function. We provide the proof of the main result in Section 4. Section 5 concludes this paper.

## 2. Notation and Main Result

2.1. Notation. Throughout this paper, we consider a finite-dimensional Hilbert space $\mathcal{H}$. The set of density operators (i.e. positive semi-definite operators with unit trace) on $\mathcal{H}$ are defined as $\mathcal{S}(\mathcal{H})$. For $\rho, \sigma \in$ $\mathcal{S}(\mathcal{H})$, we write $\rho \ll \sigma$ if $\operatorname{supp}(\rho) \subset \operatorname{supp}(\sigma)$, where $\operatorname{supp}(\rho)$ denotes the support of $\rho$. The identity operator on $\mathcal{H}$ is denoted by $\mathbb{1}_{\mathcal{H}}$. When there is no possibility of confusion, we skip the subscript $\mathcal{H}$. We use $\operatorname{Tr}[\cdot]$ as the trace function. Let $\mathbb{N}, \mathbb{R}$, and $\mathbb{R}_{>0}$ denote the set of integers, real numbers, and positive real numbers,, respectively. Define $[n]:=\{1,2, \ldots, n\}$ for $n \in \mathbb{N}$. Given a pair of positive semi-definite operators $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, we define the (quantum) relative entropy as $\mathbb{D}(\rho \| \sigma):=\operatorname{Tr}[\rho(\log \rho-\log \sigma)]$, when $\rho \ll \sigma$, and $+\infty$ otherwise. For every $\alpha \in[0,1)$, we define the (Petz) quantum Rényi divergences $D_{\alpha}(\rho \| \sigma):=$ $\frac{1}{\alpha-1} \log \operatorname{Tr}\left[\rho^{\alpha} \sigma^{1-\alpha}\right]$. For $\alpha=1, D_{1}(\rho \| \sigma):=\lim _{\alpha \rightarrow 1} D_{\alpha}(\rho \| \sigma)=\mathbb{D}(\rho \| \sigma)$. Let $\mathcal{X}=\{1,2, \ldots,|\mathcal{X}|\}$ be a finite alphabet, and let $\mathcal{P}(\mathcal{X})$ be the set of probability distributions on $\mathcal{X}$. In particular, we denote by $U_{\mathcal{X}}$ the uniform distribution on $\mathcal{X}$. A classical-quantum (c-q) channel $W$ maps elements of the finite set $\mathcal{X}$ to the density operators in $\mathcal{S}(\mathcal{H})$, i.e., $W: \mathcal{X} \rightarrow \mathcal{S}(\mathcal{H})$. Let $\mathcal{M}$ be a finite alphabetical set with size $M=|\mathcal{M}|$. An ( $n$-block) encoder is a map $f_{n}: \mathcal{M} \rightarrow \mathcal{X}^{n}$ that encodes each message $m \in \mathcal{M}$ to a codeword $\mathbf{x}^{n}(m):=x_{1}(m) \ldots x_{n}(m) \in \mathcal{X}^{n}$. The codeword $\mathbf{x}^{n}(m)$ is then mapped to a state $W_{\mathbf{x}^{n}(m)}^{\otimes n}=W_{x_{1}(m)} \otimes \cdots \otimes W_{x_{n}(m)} \in \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)$. The decoder is described by a positive operator-valued measurement (POVM) $\Pi_{n}=\left\{\Pi_{n, 1}, \ldots, \Pi_{n, M}\right\}$ on $\mathcal{H}^{\otimes n}$, where $\Pi_{n, i} \geq 0$ and $\sum_{i=1}^{M} \Pi_{n, i}=\mathbb{1}$. The pair $\left(f_{n}, \Pi_{n}\right)=: \mathcal{C}_{n}$ is called a code with rate $R=\frac{1}{n} \log |\mathcal{M}|$. The error probability of sending a message $m$ with the code $\mathcal{C}_{n}$ is $\epsilon_{m}\left(W, \mathcal{C}_{n}\right):=1-\operatorname{Tr}\left(\Pi_{n, m} W_{\mathbf{x}^{n}(m)}\right)$. We use $\epsilon_{\max }\left(W, \mathcal{C}_{n}\right)=\max _{m \in \mathcal{M}} \epsilon_{m}\left(W, \mathcal{C}_{n}\right)$ and $\bar{\epsilon}\left(W, \mathcal{C}_{n}\right)=\frac{1}{M} \sum_{m \in \mathcal{M}} \epsilon_{m}\left(W, \mathcal{C}_{n}\right)$ to denote the maximal error probability and the average error probability, respectively. Given a sequence $\mathbf{x}^{n} \in \mathcal{X}^{n}$, we denote by $P_{\mathbf{x}^{n}}(x):=\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\left\{x=x_{i}\right\}$ the empirical distribution of $\mathbf{x}^{n}$.

Throughout this paper, we consider a symmetric $c-q$ channel defined as

$$
\begin{equation*}
W_{x}:=V^{x-1} W_{1}\left(V^{\dagger}\right)^{x-1}, \quad \forall x \in \mathcal{X}, \tag{1}
\end{equation*}
$$

where $W_{1} \in \mathcal{S}(\mathcal{H})$ is an arbitrary density operator, and $V$ satisfies $V^{\dagger} V=V V^{\dagger}=V^{|\mathcal{X}|}=\mathbb{1}_{\mathcal{H}}$. We define the following conditional entropic quantities for the channel $W$ with $P \in \mathcal{P}(\mathcal{X}): D_{\alpha}(W \| \sigma \mid P):=$ $\sum_{x \in \mathcal{X}} P(x) D_{\alpha}\left(W_{x} \| \sigma\right)$. The mutual information of the c-q channel $W: \mathcal{X} \rightarrow \mathcal{S}(\mathcal{H})$ with prior distribution $P \in \mathcal{P}(\mathcal{X})$ is defined as $I(P, W):=\mathbb{D}(W \| P W \mid P)$, where $P W^{\alpha}:=\sum_{x \in \mathcal{X}} P(x) W_{x}^{\alpha}, \alpha \in(0,1]$. The (classical) capacity of the channel $W: \mathcal{X} \rightarrow \mathcal{S}(\mathcal{H})$ is denoted by $C:=\max _{P \in \mathcal{P}(\mathcal{X})} I(P, W)$. Let

$$
\begin{aligned}
& E_{\mathrm{sp}}^{(1)}(R, P):=\sup _{s \geq 0}\left\{E_{0}(s, P)-s R\right\} \\
& E_{\mathrm{sp}}^{(2)}(R, P):=\sup _{0<\alpha \leq 1} \min _{\sigma \in \mathcal{S}(\mathcal{H})} \frac{\alpha-1}{\alpha}\left(R-D_{\alpha}(W \| \sigma \mid P)\right),
\end{aligned}
$$

where we denote by $E_{0}(s, P):=-\log \operatorname{Tr}\left[\left(P W^{1 /(1+s)}\right)^{1+s}\right]$ an auxiliary function $[16,22]$. The spherepacking exponent is defined by

$$
\begin{equation*}
E_{\mathrm{sp}}(R):=\max _{P \in \mathcal{P}(\mathcal{X})} E_{\mathrm{sp}}^{(1)}(R, P)=\max _{P \in \mathcal{P}(\mathcal{X})} E_{\mathrm{sp}}^{(2)}(R, P) \tag{2}
\end{equation*}
$$

where the last equality follows from [24, Proposition IV.2]. Further, we define a rate [25, p. 152], [18]:

$$
\begin{align*}
R_{\infty} & :=\lim _{s \rightarrow+\infty} \max _{P \in \mathcal{P}(\mathcal{X})} \min _{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) D_{\frac{1}{1+s}}\left(W_{x} \| \sigma\right) \\
& =\max _{P \in \mathcal{P}(\mathcal{X})} \min _{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) \operatorname{Tr}\left[W_{x}^{0} \sigma\right] \tag{3}
\end{align*}
$$

It follows that $E_{\mathrm{sp}}(R)=+\infty$ for any $R \leq R_{\infty}$ (see also [4, p. 69] and [3, Eq. (5.8.5)]).
Consider a binary hypothesis whose null and alternative hypotheses are $\rho \in \mathcal{S}(\mathcal{H})$ and $\sigma \in \mathcal{S}(\mathcal{H})$, respectively. The type-I error and type-II error of the hypothesis testing, for an operator $0 \leq Q \leq \mathbb{1}$, are defined as $\alpha(Q ; \rho):=\operatorname{Tr}[(\mathbb{1}-Q) \rho]$, and $\beta(Q ; \sigma):=\operatorname{Tr}[Q \sigma]$. There is a trade-off between these two errors. Thus, we can define the minimum type-I error, when the type-II error is below $\mu \in(0,1)$, as

$$
\begin{equation*}
\widehat{\alpha}_{\mu}(\rho \| \sigma):=\min _{0 \leq Q \leq \mathbb{1}}\{\alpha(Q ; \rho): \beta(Q ; \sigma) \leq \mu\} \tag{4}
\end{equation*}
$$

2.2. Main Result. Let us now consider any symmetric c-q channel with capacity $C$.

Theorem 1 (Exact Sphere-Packing Bound). For any rate $R \in[0, C)$, there exist an $N_{0} \in \mathbb{N}$ such that for all codes $\mathcal{C}_{n}$ of length $n \geq N_{0}$, we have

$$
\begin{equation*}
\epsilon_{\max }\left(\mathcal{C}_{n}\right) \geq \frac{1-o(1)}{n^{\frac{1}{2}\left(1+\left|E_{\mathrm{sp}}^{\prime}(R)\right|\right)}} \exp \left\{-n E_{\mathrm{sp}}(R)\right\} \tag{5}
\end{equation*}
$$

where $E_{\mathrm{sp}}^{\prime}(R):=\partial \max _{P \in \mathcal{P}(\mathcal{X})} E_{\mathrm{sp}}^{(1)}(r, P) /\left.\partial r\right|_{r=R}$.

## 3. Properties of the Sphere-Packing Exponent

Lemma 2 (Optimal Input Distribution). For any $R>R_{\infty}$, the distribution $U_{\mathcal{X}}$ is a maximizer of $E_{\mathrm{sp}}^{(1)}(R, \cdot)$ and $E_{\mathrm{sp}}^{(2)}(R, \cdot)$.

Proof. We first prove that $U_{\mathcal{X}}$ attains $\max _{P \in \mathcal{P}(\mathcal{X})} E_{0}(s, P)$. From Eq. (1), it is not hard to verify that $U_{\mathcal{X}} W^{\alpha}=V U_{\mathcal{X}} W^{\alpha} V^{\dagger}$ for all $\alpha \in(0,1]$. Hence, it follows that

$$
\begin{align*}
\operatorname{Tr}\left[W_{x}^{\alpha}\left(U_{\mathcal{X}} W^{\alpha}\right)^{\frac{1-\alpha}{\alpha}}\right] & =\operatorname{Tr}\left[V^{x-1} W_{1}^{\alpha} V^{\dagger x-1}\left(U_{\mathcal{X}} W^{\alpha}\right)^{\frac{1-\alpha}{\alpha}}\right]  \tag{6}\\
& =\operatorname{Tr}\left[W_{1}^{\alpha} V^{\dagger x-1}\left(U_{\mathcal{X}} W^{\alpha}\right)^{\frac{1-\alpha}{\alpha}} V^{x-1}\right]  \tag{7}\\
& =\operatorname{Tr}\left[W_{1}^{\alpha}\left(U_{\mathcal{X}} W^{\alpha}\right)^{\frac{1-\alpha}{\alpha}}\right]  \tag{8}\\
& =\operatorname{Tr}\left[\left(U_{\mathcal{X}} W^{\alpha}\right)^{\frac{1}{\alpha}}\right] \tag{9}
\end{align*}
$$

for all $\alpha \in(0,1]$. The above equation shows that the distribution $U_{\mathcal{X}}$ that maximizes $E_{0}(s, P), \forall s \geq 0$ [16, Eq. (38)]. Then we have

$$
E_{\mathrm{sp}}^{(1)}\left(R, U_{\mathcal{X}}\right)=\sup _{s \geq 0}\left\{\max _{P \in \mathcal{P}(\mathcal{X})} E_{0}(s, P)-s R\right\}=E_{\mathrm{sp}}(R)
$$

Further, Jensen's inequality implies that $E_{\mathrm{sp}}^{(2)}\left(R, U_{\mathcal{X}}\right) \geq E_{\mathrm{sp}}^{(1)}\left(R, U_{\mathcal{X}}\right)=E_{\mathrm{sp}}(R)$, which completes the proof.

Lemma 3 (Saddle-Point Property). Consider any $R \in\left(R_{\infty}, C\right)$ and $P \in \mathcal{P}(\mathcal{X})$. Let $\mathcal{S}_{P, W}(\mathcal{H}):=$ $\left\{\sigma \in \mathcal{S}(\mathcal{H}): \forall x \in \operatorname{supp}(P), \operatorname{supp}\left(W_{x}\right) \cap \operatorname{supp}(\sigma) \neq \emptyset\right\}$. We define

$$
\begin{equation*}
F_{R, P}(\alpha, \sigma):=\frac{\alpha-1}{\alpha}\left(R-D_{\alpha}(W \| \sigma \mid P)\right) \tag{10}
\end{equation*}
$$

on $(0,1] \times \mathcal{S}_{P, W}(\mathcal{H})$, and let $\mathcal{P}_{R}(\mathcal{X}):=\left\{P \in \mathcal{P}(X): \min _{\sigma \in \mathcal{S}(\mathcal{H})} \sup _{0<\alpha \leq 1} F_{R, P}(\alpha, \sigma) \in \mathbb{R}_{>0}\right\}$. The following holds
(i) For any $P \in \mathcal{P}(\mathcal{X}), F_{R, P}(\cdot, \cdot)$ has a saddle-point with the saddle-value:

$$
\begin{equation*}
\min _{\sigma \in \mathcal{S}(\mathcal{H})} \sup _{0<\alpha \leq 1} F_{R, P}(\alpha, \sigma)=\sup _{0<\alpha \leq 1} \min _{\sigma \in \mathcal{S}(\mathcal{H})} F_{R, P}(\alpha, \sigma)=E_{\mathrm{sp}}^{(2)}(R, W, P) \tag{11}
\end{equation*}
$$

(ii) The saddle-point is unique for $P \in \mathcal{P}_{R}(\mathcal{X})$.
(iii) Let $P \in \mathcal{P}_{R}(\mathcal{X})$. The unique saddle-point $(\alpha, \sigma)$ of $F_{R, P}(\cdot, \cdot)$ satisfies $\alpha \in(0,1)$ and

$$
\begin{equation*}
\sigma=\frac{\left(\sum_{x \in \mathcal{X}} P(x) W_{x}^{\alpha} \mathrm{e}^{(1-\alpha) D_{\alpha}\left(W_{x} \| \sigma\right)}\right)^{1 / \alpha}}{\operatorname{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) W_{x}^{\alpha} \mathrm{e}^{(1-\alpha) D_{\alpha}\left(W_{x} \| \sigma\right)}\right)^{1 / \alpha}\right]} \gg W_{x}, \quad \forall x \in \operatorname{supp}(P) \tag{12}
\end{equation*}
$$

The proof is provided in Appendix B.1.
Lemma 4 (Representation). For any $R \in\left(R_{\infty}, C\right)$, let $\left(\alpha_{R}^{\star}, \sigma_{R}^{\star}\right)$ be the saddle-point of $F_{R, U_{\mathcal{X}}}(\cdot, \cdot)$. It follows that

$$
\begin{equation*}
\left(\alpha_{R}^{\star}, \sigma_{R}^{\star}\right)=\left(-E_{\mathrm{sp}}^{\prime}(R), \frac{\left(U_{\mathcal{X}} W^{\alpha_{R}^{\star}}\right)^{1 / \alpha_{R}^{\star}}}{\operatorname{Tr}\left[\left(U_{\mathcal{X}} W^{\alpha_{R}^{\star}}\right)^{1 / \alpha_{R}^{\star}}\right]}\right) . \tag{13}
\end{equation*}
$$

Proof. Since Lemma 2 implies that $U_{\mathcal{X}}$ attains $E_{\mathrm{sp}}^{(2)}(R, \cdot)$, one observes from the definition of $E_{\mathrm{sp}}^{(2)}$ that all the quantities $D_{\alpha_{R}^{\star}}\left(W_{x} \| \sigma_{R}^{\star}\right), x \in \mathcal{X}$ are equal. By item (iii) of Lemma 3, we obtain a representation of $\sigma_{R}^{\star}$ in Eq. (13). The optimal $\alpha_{R}^{\star}=-\partial E_{\mathrm{sp}}\left(r, U_{\mathcal{X}}\right) /\left.\partial r\right|_{r=R}$ follows from [22, Eq. (42)].

Lemma 5 (Invariance). For any $R \in\left(R_{\infty}, C\right)$, we have

$$
\begin{equation*}
F_{R, P}\left(\alpha_{R}^{\star}, \sigma_{R}^{\star}\right)=E_{\mathrm{sp}}(R)>0, \quad \forall P \in \mathcal{P}(\mathcal{X}) \tag{14}
\end{equation*}
$$

where $\alpha_{R}^{\star}$ and $\sigma_{R}^{\star}$ are defined in Eq. (13).
Proof. Following the argument in Lemma 2 and recalling Eq. (13) in Lemma 4, one can verify that $\sup _{\alpha \in(0,1]} F_{R, P}\left(\alpha, \sigma_{R}^{\star}\right)=\sup _{s \geq 0}\left\{E_{0}\left(s, U_{\mathcal{X}}\right)-s R\right\}=E_{\mathrm{sp}}(R)$ for all $P \in \mathcal{P}(\mathcal{X})$. Further, we obtain $E_{\mathrm{sp}}(R)>0$ for $R \in\left(R_{\infty}, C\right)$ from the result in [22, Proposition 10].

## 4. Proof of the Main Result

For rates in the range $R \leq R_{\infty}$, we have $E_{\mathrm{sp}}(R)=+\infty$. The bound in Eq. (5) obviously holds. Hence, we consider the case of $R \in\left(R_{\infty}, C\right)$ and fix the rate throughout the proof.

We first pose the channel coding problem into a binary hypothesis testing through Lemma 6, which originates from Blahut [6] for the classical case.
Lemma 6 (Hypothesis Testing Reduction). For any code $\mathcal{C}_{n}$ with message size $\mathrm{e}^{n r}$, there exists an $\mathbf{x}^{n} \in \mathcal{C}_{n}$ such that

$$
\begin{equation*}
\epsilon_{\max }\left(\mathcal{C}_{n}\right) \geq \max _{\sigma \in \mathcal{S}(H)} \widehat{\alpha}_{\exp \{-n r\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma^{\otimes n}\right) \tag{15}
\end{equation*}
$$

The proof is provided in Appendix B.2.
Let us now commence with the proof of Theorem 1. Fix arbitrary $\gamma, \xi>0$. Let $\gamma_{n}:=\left(\frac{1}{2}+\gamma\right) \frac{\log n}{n}$ and $R_{n}:=R-\gamma_{n}$. The choice of the rate back-off term $\gamma_{n}$ will become evident later. Choose $N_{1} \in \mathbb{N}$ such that $R_{n} \geq R-\xi>R_{\infty}$. Let $\sigma_{R}^{\star}$ be defined in Eq. (13), and from Lemma 6, we have

$$
\begin{equation*}
\epsilon_{\max }\left(\mathcal{C}_{n}\right) \geq \widehat{\alpha}_{\exp \left\{-n R_{n}\right\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma_{R}^{\star \otimes n}\right) \tag{16}
\end{equation*}
$$

In the following, we provide a lower bound for the type-I error $\widehat{\alpha}_{\exp \left\{-n R_{n}\right\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma_{R}^{\star \otimes n}\right)$. Let $p^{n}:=$ $\bigotimes_{i=1}^{n} p_{x_{i}}$ and $q^{n}:=\bigotimes_{i=1}^{n} q_{x_{i}}$, where $\left(p_{x_{i}}, q_{x_{i}}\right)$ are Nussbaum-Szkoła distributions [26] of ( $W_{x_{i}}, \sigma_{R}^{\star}$ ) for every $i \in[n]$. Since $D_{\alpha}\left(W_{x_{i}} \| \sigma_{R}^{\star}\right)=D_{\alpha}\left(p_{x_{i}} \| q_{x_{i}}\right)$, for all $\alpha \in(0,1]$, we shorthand $\phi_{n}\left(R_{n}\right):=\sup _{\alpha \in(0,1]} F_{R_{n}, P_{x^{n}}}\left(\alpha, \sigma_{R}^{\star}\right)$, where $P_{\mathbf{x}^{n}}$ is the empirical distribution of $\mathbf{x}^{n}$. Moreover, item (iii) in Lemma 3 implies that the state $\sigma_{R}^{\star}$ dominates all the channel outputs: $\sigma_{R}^{\star} \gg W_{x}$, for all $x \in \operatorname{supp}\left(P_{\mathbf{x}^{n}}\right)$, Hence, we have $p^{n} \ll q^{n}$. Subsequently, for every $i \in[n]$, we let $q_{x_{i}}(\omega)=0$, for all $\omega \notin \operatorname{supp}\left(p_{x_{i}}\right)$. We apply Nagaoka's argument [27] by choosing $\delta=\exp \left\{n R_{n}-n \phi_{n}\left(R_{n}\right)\right\}$ to yield, for any $0 \leq Q_{n} \leq \mathbb{1}$,

$$
\begin{equation*}
\alpha\left(Q_{n} ; W_{\mathbf{x}^{n}}^{\otimes n}\right)+\delta \beta\left(Q_{n} ; \sigma_{R}^{\star \otimes n}\right) \geq \frac{\alpha\left(\mathcal{U} ; p^{n}\right)+\delta \beta\left(\mathcal{U} ; q^{n}\right)}{2}, \tag{17}
\end{equation*}
$$

where $\alpha\left(\mathcal{U} ; p^{n}\right):=\sum_{\omega \in \mathcal{U c}} p^{n}(\omega), \beta\left(\mathcal{U} ; q^{n}\right):=\sum_{\omega \in \mathcal{U}} q^{n}(\omega)$, and $\mathcal{U}:=\left\{\omega: p^{n}(\omega) \mathrm{e}^{n \phi_{n}\left(R_{n}\right)}>q^{n}(\omega) \mathrm{e}^{n R_{n}}\right\}$.

Next, we employ Bahadur-Ranga Rao's concentration inequality, Theorem 9 in Appendix A, to further lower bound $\alpha\left(\mathcal{U} ; p^{n}\right)$ and $\beta\left(\mathcal{U} ; q^{n}\right)$. Before proceeding, we need to introduce some notation. We define the tilted distributions, for every $i \in[n], \omega \in \operatorname{supp}\left(p_{x_{i}}\right)$, and $t \in[0,1]$ by

$$
\begin{equation*}
\hat{q}_{x_{i}, t}(\omega):=\frac{p_{x_{i}}(\omega)^{1-t} q_{x_{i}}(\omega)^{t}}{\sum_{\omega \in \operatorname{supp}\left(p_{x_{i}}\right)} p_{x_{i}}(\omega)^{1-t} q_{x_{i}}(\omega)^{t}} \tag{18}
\end{equation*}
$$

Let

$$
\begin{align*}
& \Lambda_{0, x_{i}}(t):=\log \mathbb{E}_{p_{x_{i}}}\left[\mathrm{e}^{t \log \frac{q x_{i}}{p_{x_{i}}}}\right]  \tag{19}\\
& \Lambda_{1, x_{i}}(t):=\log \mathbb{E}_{q_{x_{i}}}\left[\mathrm{e}^{t \log \frac{p_{x_{i}}}{q x_{i}}}\right]
\end{align*}
$$

Since $p^{n}$ and $q^{n}$ are mutually absolutely continuous, the maps $t \mapsto \Lambda_{j, x_{i}}(t), j \in\{0,1\}$ are differentiable for all $t \in[0,1]$. One can immediately verify the following partial derivatives with respect to $t$ :

$$
\begin{align*}
& \Lambda_{0, x_{i}}^{\prime}(t)=\mathbb{E}_{\hat{q}_{x_{i}, t}}\left[\log \frac{q_{x_{i}}}{p_{x_{i}}}\right], \Lambda_{0, x_{i}}^{\prime \prime}(t)=\operatorname{Var}_{\hat{q}_{x_{i}, t}}\left[\log \frac{q_{x_{i}}}{p_{x_{i}}}\right] \\
& \Lambda_{0, x_{i}}^{\prime \prime}(t)=\operatorname{Var}_{\hat{q}_{x_{i}, t}}\left[\log \frac{q_{x_{i}}}{p_{x_{i}}}\right], \Lambda_{1, x_{i}}^{\prime}(t)=\mathbb{E}_{\hat{q}_{x_{i}, 1-t}}\left[\log \frac{p_{x_{i}}}{q_{x_{i}}}\right] \tag{20}
\end{align*}
$$

With $\Lambda_{j, x_{i}}(t)$ in Eq. (19), we can define

$$
\begin{array}{ll}
\Lambda_{j, P_{\mathbf{x}^{n}}}(t):=\sum_{x \in \mathcal{X}} P_{\mathbf{x}^{n}}(x) \Lambda_{j, x}(t), & j \in\{0,1\} \\
\Lambda_{j, P_{\mathbf{x}^{n}}}^{*}(z):=\sup _{t \in \mathbb{R}}\left\{t z-\Lambda_{j, P_{\mathbf{x}^{n}}}(t)\right\}, & j \in\{0,1\} \tag{22}
\end{array}
$$

where $\Lambda_{j, P_{\mathbf{x}^{n}}}^{*}(z)$ in Eq. (22) is the Fenchel-Legendre transform of $\Lambda_{j, P_{\mathbf{x}^{n}}}(t)$. The quantities $\Lambda_{j, P_{\mathbf{x}^{n}}}^{*}(z)$ would appear in the lower bounds of $\alpha\left(\mathcal{U} ; p^{n}\right)$ and $\beta\left(\mathcal{U} ; q^{n}\right)$ obtained by Bahadur-Randga Rao's inequality as shown later.

In the following, we relate the Fenchel-Legendre transform $\Lambda_{j, P_{\mathbf{x}^{n}}}^{*}(z)$ to the desired error-exponent function $\phi_{n}\left(R_{n}\right)$. Such a relationship is stated in Lemma 7; the proof is provided in Appendix B.3.

Lemma 7. Under the prevailing assumptions and for all $R_{n} \in\left(R_{\infty}, C\right)$, the following holds:
(i) $\Lambda_{0, P_{\times^{n}}}^{*}\left(\phi_{n}\left(R_{n}\right)-R_{n}\right)=\phi_{n}\left(R_{n}\right)$;
(ii) $\Lambda_{1, P_{\mathbf{x}^{n}}}^{*}\left(R_{n}-\phi_{n}\left(R_{n}\right)\right)=R_{n}$;
(iii) There exists a unique $t^{\star}=\frac{s^{\star}}{1+s^{\star}} \in(0,1)$, such that $\Lambda_{0, P_{\mathbf{x}^{n}}}^{\prime}\left(t^{\star}\right)=\phi_{n}\left(R_{n}\right)-R_{n}$, where $s^{\star}:=$ $\left.\frac{\partial \phi_{n}(r)}{\partial r}\right|_{r=R_{n}}$.
Item (iii) in Lemma 7 shows that the optimizer $t$ in Eq. (22) always lies in the compact set [0, 1]. Further, Eqs. (19) and (20) ensure that $\Lambda_{0, x_{i}}(t)=\Lambda_{1, x_{i}}(1-t), \Lambda_{0, x_{i}}^{\prime}(t)=-\Lambda_{1, x_{i}}^{\prime}(1-t), \Lambda_{0, x_{i}}^{\prime \prime}(t)=\Lambda_{1, x_{i}}^{\prime \prime}(1-t)$. We define the following quantities:

$$
\begin{align*}
V_{\max } & :=\max _{t \in[0,1], x \in \mathcal{X}} \Lambda_{0, x}^{\prime \prime}(t)  \tag{23}\\
V_{\min } & :=\min _{t \in[0,1], x \in \mathcal{X}} \Lambda_{0, x}^{\prime \prime}(t)  \tag{24}\\
T_{\max } & :=\max _{t \in[0,1], x \in \mathcal{X}} T_{0, x}(t)  \tag{25}\\
T_{0, x}(t) & :=\mathbb{E}_{\hat{q}_{x, t}}\left[\left|\log \frac{q_{x}}{p_{x}}-\Lambda_{0, x}^{\prime}(t)\right|^{3}\right] \tag{26}
\end{align*}
$$

$T_{1, x}(t):=T_{0, x}(1-t) ;$ and $K_{\max }:=15 \sqrt{2 \pi} T_{\max } / V_{\min }$. Note that for every $x \in \mathcal{X}, \Lambda_{0, x}^{\prime \prime}(\cdot)$ and $T_{0, x}(\cdot)$ are continuous functions on $[0,1]$ from the definitions in Eqs. (20), (26) (see also [10, Lemma 9]). The maximization and minimization in the above definitions are well-defined and finite. Moreover, Lemma 8 guarantees that $V_{\min }$ is bounded away from zero.

Lemma 8 (Positivity). For any $R_{n} \in\left(R_{\infty}, C\right)$ and $P_{\mathbf{x}^{n}} \in \mathcal{P}(\mathcal{X}), \Lambda_{0, P_{\mathbf{x}}^{n}}^{\prime \prime}(t)>0$, for all $t \in[0,1]$.
Proof. Assume $\Lambda_{0, P_{x}^{n}}^{\prime \prime}(t)$ is zero for some $t \in[0,1]$. This is equivalent to

$$
\begin{equation*}
p_{x_{i}}(\omega)=q_{x_{i}}(\omega) \cdot \mathrm{e}^{-\Lambda_{0, x_{i}}^{\prime}(t)}, \quad \forall \omega \in p_{x_{i}}, \quad \forall i \in[n] . \tag{27}
\end{equation*}
$$

Summing the right-hand side of Eq. (27) over $\omega \in p_{x_{i}}$ gives $1=\operatorname{Tr}\left[p_{x_{i}}^{0} q_{x_{i}}\right] \mathrm{e}^{-\Lambda_{0, x_{i}}^{\prime}(t)}, \quad \forall i \in[n]$. Then, Eqs. (27) and the above equation imply that

$$
\begin{aligned}
\phi_{n}\left(R_{n}\right) & =\sup _{0<\alpha \leq 1} \frac{\alpha-1}{\alpha}\left(R_{n}+\sum_{x \in \mathcal{X}} P_{\mathbf{x}^{n}}(x) \log \operatorname{Tr}\left[p_{x}^{0} q_{x}\right]\right) \\
& =0,
\end{aligned}
$$

where we use the fact that $R_{n}>R_{\infty}=-\sum_{x \in \mathcal{X}} P_{\mathbf{x}^{n}}(x) \log \operatorname{Tr}\left[p_{x}^{0} q_{x}\right]$; see Eq. (3)). However, Lemma 5 implies that $\phi_{n}\left(R_{n}\right)=E_{\mathrm{sp}}\left(R_{n}\right)>0$, which leads to a contradiction.

Now, we are ready to derive the lower bounds to $\alpha\left(\mathcal{U} ; p^{n}\right)$ and $\beta\left(\mathcal{U} ; q^{n}\right)$. Let $N_{2} \in \mathbb{N}$ be sufficiently large such that for all $n \geq N_{2}$,

$$
\begin{equation*}
\sqrt{n} \geq \frac{1+\left(1+K_{\max }\right)^{2}}{\sqrt{V_{\min }}} \tag{28}
\end{equation*}
$$

Applying Bahadur-Randga Rao's inequality (Theorem 9) to $Z_{i}=\log q_{i}-\log p_{i}$ with the probability measure $\lambda_{i}=p_{i}$, and $z=R_{n}-\phi_{n}\left(R_{n}\right)$ gives

$$
\begin{align*}
\alpha\left(\mathcal{U} ; p^{n}\right) & =\operatorname{Pr}\left\{\frac{1}{n} \sum_{i=1}^{n} Z_{i} \geq R_{n}-\phi_{n}\left(R_{n}\right)\right\}  \tag{29}\\
& \geq \frac{2 A}{\sqrt{n}} \exp \left\{-n \Lambda_{0, P_{\mathbf{x}^{n}}}^{*}\left(\phi_{n}\left(R_{n}\right)-R_{n}\right)\right\} \tag{30}
\end{align*}
$$

where $A:=\frac{\mathrm{e}^{-K_{\max }}}{\sqrt{4 \pi V_{\max }}}$. Similarly, applying Theorem 9 to $Z_{i}=\log p_{i}-\log q_{i}$ with the probability measure $\lambda_{i}=q_{i}$, and $z=\phi_{n}\left(R_{n}\right)-R_{n}$ yields

$$
\begin{align*}
\beta\left(\mathcal{U} ; q^{n}\right) & =\operatorname{Pr}\left\{\frac{1}{n} \sum_{i=1}^{n} Z_{i} \geq \phi_{n}\left(R_{n}\right)-R_{n}\right\}  \tag{31}\\
& \geq \frac{2 A}{\sqrt{n}} \exp \left\{-n \Lambda_{1, P_{\mathbf{x}^{n}}}^{*}\left(R_{n}-\phi_{n}\left(R_{n}\right)\right)\right\} . \tag{32}
\end{align*}
$$

Continuing from Eq. (30) and item (i) in Lemma 7 gives

$$
\begin{equation*}
\alpha\left(\mathcal{U} ; p^{n}\right) \geq \frac{2 A}{\sqrt{n}} \exp \left\{-n \phi_{n}\left(R_{n}\right)\right\} . \tag{33}
\end{equation*}
$$

Eq. (32) together with item (iii) in Lemma 7 yields

$$
\begin{equation*}
\beta\left(\mathcal{U} ; q^{n}\right) \geq \frac{2 A}{\sqrt{n}} \exp \left\{-n R_{n}\right\}=2 A n^{\gamma} \exp \{-n R\} \tag{34}
\end{equation*}
$$

Let $N_{3} \in \mathbb{N}$ such that $A n^{\gamma}>1$, for all $n \geq N_{3}$. Then Eq. (34) implies that $\beta\left(\mathcal{U} ; q^{n}\right)>2 \exp \{-n R\}$. Thus, we can bound the left-hand side of Eq. (17) from below by $\frac{A}{\sqrt{n}} \mathrm{e}^{-n \phi_{n}\left(R_{n}\right)}$. For any test $0 \leq Q_{n} \leq \mathbb{1}$ such that $\beta\left(Q_{n} ; \sigma_{R}^{\star \otimes n}\right) \leq \exp \{-n R\}$, we have

$$
\begin{align*}
& \widehat{\alpha}_{\exp \left\{-n R_{n}\right\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma_{R}^{\star \otimes n}\right)=\alpha\left(Q_{n} ; \rho^{n}\right) \\
& \geq \frac{A}{\sqrt{n}} \exp \left\{-n \phi_{n}\left(R_{n}\right)\right\}=\frac{A}{\sqrt{n}} \exp \left\{-n E_{\mathrm{sp}}\left(R_{n}\right)\right\}, \tag{35}
\end{align*}
$$

where the last equality follows from Lemma 5 .

Finally, it remains to remove the back-off term $R_{n}=R-\gamma_{n}$ in Eq. (35). By Taylor's theorem, we have

$$
\begin{equation*}
E_{\mathrm{sp}}\left(R-\gamma_{n}\right)=E_{\mathrm{sp}}(R)-\gamma_{n} E_{\mathrm{sp}}^{\prime}(R)+\frac{\gamma_{n}^{2}}{2} E_{\mathrm{sp}}^{\prime \prime}(\bar{R}), \tag{36}
\end{equation*}
$$

for some $\bar{R} \in(R-\xi, R)$ and $E_{\mathrm{sp}}^{\prime \prime}(\bar{R}):=\left.\frac{\partial^{2} E_{\mathrm{sp}}^{(1)}\left(r, U_{\mathcal{X}}\right)}{\partial r^{2}}\right|_{r=\bar{R}}$. Further, one can calculate that

$$
\begin{align*}
E_{\mathrm{sp}}^{\prime \prime}(\bar{R}) & =-\left(\left.\frac{\partial^{2} E_{0}\left(s, U_{\mathcal{X}}\right)}{\partial s^{2}}\right|_{s=\bar{s}}\right)^{-1}  \tag{37}\\
& =\frac{(1+\bar{s})^{3}}{\Lambda_{0, U_{\mathcal{X}}}^{\prime \prime}\left(\frac{\bar{s}}{1+\bar{s}}\right)} \leq \frac{(1+\bar{s})^{3}}{V_{\min }}=: \Upsilon, \tag{38}
\end{align*}
$$

where $\bar{s}=\frac{1-\alpha_{\bar{R}}^{\star}}{\alpha_{\bar{R}}^{\star}}$. From item (iii) in Lemma 3, it follows that both $\bar{s}$ and $\left|E_{\mathrm{sp}}^{\prime}(R)\right|=s^{\star}$ are both positive and finite for $\bar{R} \in\left(R_{\infty}, C\right)$ and $R \in\left(R_{\infty}, C\right)$. Together with the fact that $V_{\min }>0$, we have $\Upsilon \in \mathbb{R}_{>0}$. We apply Taylor's expansion on the function $n^{-(\cdot)}$ again to yield

$$
\begin{align*}
n^{-\frac{1}{2}\left(1+\left|E_{\mathrm{sp}}^{\prime}(R)\right|\right)-\gamma_{n} \Upsilon} & =n^{-\frac{1}{2}\left(1+\left|E_{\mathrm{sp}}^{\prime}(R)\right|\right)} \cdot\left(1-\frac{\log n}{\left.n^{\bar{x} \Gamma} \gamma_{n} \Upsilon\right)}\right. \\
& =n^{-\frac{1}{2}\left(1+\left|E_{\mathrm{sp}}^{\prime}(R)\right|\right)} \cdot(1-o(1)), \tag{39}
\end{align*}
$$

where the first equality holds for some $\bar{x} \in\left(0, \gamma_{n}\right)$, and the last line follows from the definition $\gamma_{n}=$ $\left(\frac{1}{2}+\gamma\right) \frac{\log }{n}$. Finally, by combining Eqs. (16), (35), and (39), we obtain the desired Eq. (5) for sufficiently large $n \geq N_{0}:=\max \left\{N_{1}, N_{2}, N_{3}\right\}$.

## 5. Discussion

In this work, we establish a sphere-packing bound with a refined polynomial pre-factor that coincides with the best classical results [10, Theorem 1] to date. As discussed by Altuğ and Wagner [10, Sec. VII], the pre-factor is correct for binary symmetric channels but slightly worse for binary erasure channels (in the order of $1 / \sqrt{n}$ ). On the other hand, our pre-factor matches the recent result of the random coding upper bound [21, Theorem 2], where the pre-factor has been shown to be exact. Hence, we conjecture that the established result is optimal for general symmetric $\mathrm{c}-\mathrm{q}$ channels.

This work admits variety of potential extensions. First, the symmetric c-q channel studied in this paper is a covariant channel with a cyclic group:

$$
\begin{equation*}
W_{\mathcal{U}_{\text {in }}(g) x \mathcal{U}_{\text {in }}(g)^{\dagger}}=\mathcal{U}_{\text {out }}(g) W_{x} \mathcal{U}_{\mathrm{out}}(g)^{\dagger}, \quad \forall g, x \in \mathcal{X} \tag{40}
\end{equation*}
$$

where $\mathcal{U}_{\text {in }}$ and $\mathcal{U}_{\text {out }}$ are the unitary representations on $\mathcal{X}$ and $\mathcal{S}(\mathcal{H})$ such that $\mathcal{U}_{\text {in }}(g) x \mathcal{U}_{\text {in }}(g)^{\dagger}=(x+$ $g) \bmod |\mathcal{X}|$ and $\mathcal{U}_{\text {out }}(g)=V^{g}$. It would be interesting to investigate whether the refined sphere-packing bound can be extended to covariant quantum channels $\mathcal{N}: \mathcal{S}\left(\mathcal{H}_{\text {in }}\right) \rightarrow \mathcal{S}\left(\mathcal{H}_{\text {out }}\right)$ with arbitrary compact groups. Second, the random coding bound in the quantum case has been proved only for pure-state channels [16]. It is promising to prove the bound for this class of c-q channels by employing the symmetry property. Finally, the refinement provides a new possibility for moderate deviation analysis in c-q channels [13], which is left as future work.

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## Appendix A. A Tight Concentration Inequality

Let $\left(Z_{i}\right)_{i=1}^{n}$ be a sequence of independent, real-valued random variables whose probability measures are $\lambda_{i}$. Let $\Lambda_{i}(t):=\log \mathbb{E}\left[\mathrm{e}^{t Z_{i}}\right]$ and define the Fenchel-Legendre transform of $\frac{1}{n} \sum_{i=1}^{n} \Lambda_{i}(\cdot)$ to be: $\Lambda_{n}^{*}(z):=$ $\sup _{t \in \mathbb{R}}\left\{z t-\frac{1}{n} \sum_{i=1}^{n} \Lambda_{i}(t)\right\}, \quad \forall z \in \mathbb{R}$. Thenb there exists a real number $t^{\star} \in(0,1]$ for every $z \in \mathbb{R}$ such that $z=\frac{1}{n} \sum_{i=1}^{n} \Lambda_{i}^{\prime}\left(t^{\star}\right)$ and $\Lambda_{n}^{*}(z)=z t^{\star}-\frac{1}{n} \sum_{i=1}^{n} \Lambda_{i}\left(t^{\star}\right)$. Define the probability measure $\tilde{\lambda}_{i}$ via $\frac{\mathrm{d} \tilde{\lambda}_{i}}{\mathrm{~d} \lambda_{i}}\left(z_{i}\right):=\mathrm{e}^{t^{\star} z_{i}-\Lambda_{i}\left(t^{\star}\right)}$, and let $\bar{Z}_{i}:=Z_{i}-\mathbb{E}_{\tilde{\lambda}_{i}}\left[Z_{i}\right]$. Furthermore, define $m_{2, n}:=\sum_{i=1}^{n} \operatorname{Var}_{\tilde{\lambda}_{i}}\left[\bar{Z}_{i}\right], m_{3, n}:=$ $\sum_{i=1}^{n} \mathbb{E}_{\tilde{\lambda}_{i}}\left[\left|\bar{Z}_{i}\right|^{3}\right]$, and $K_{n}\left(t^{\star}\right):=\frac{15 \sqrt{2 \pi} m_{3, n}}{m_{2, n}}$. With these definitions, we can now state the following sharp concentration inequality for $\frac{1}{n} \sum_{i=1}^{n} Z_{i}$ :
Theorem 9 (Bahadur-Ranga Rao's Concentration Inequality [11, Proposition 5], [28]). Given

$$
\begin{equation*}
\sqrt{m_{2, n}} \geq 1+\left(1+K_{n}\left(t^{\star}\right)\right)^{2} \tag{41}
\end{equation*}
$$

it follows that

$$
\begin{equation*}
\operatorname{Pr}\left\{\frac{1}{n} \sum_{i=1}^{n} Z_{i} \geq z\right\} \geq \mathrm{e}^{-n \Lambda_{n}^{*}(z)} \frac{\mathrm{e}^{-K_{n}\left(t^{\star}\right)}}{2 \sqrt{2 \pi m_{2, n}}} \tag{42}
\end{equation*}
$$

## Appendix B. Proofs of Miscellaneous Lemmas

B.1. Proof of Lemma 3. Let $R>R_{\infty}$ and $P \in \mathcal{P}(\mathcal{X})$ be arbitrary. It is convenient to reparameterize the function $F_{R, P}$ by the substitution $\alpha=\frac{1}{1+s}$ :

$$
\begin{equation*}
\left.F_{R, P}(\alpha, \sigma)\right|_{\alpha=\frac{1}{1+s}}=-s R+s D_{\frac{1}{1+s}}(W \| \sigma \mid P)=: K_{R, P}(s, \sigma) . \tag{43}
\end{equation*}
$$

In the following, we prove the existence of a saddle-point of $K_{R, P}(\cdot, \cdot)$ on $\mathbb{R}_{\geq 0} \times \mathcal{S}_{P, W}(\mathcal{H})$, where $\mathbb{R}_{\geq 0}:=$ $[0, \infty)$. By Ref. [29, Lemma 36.2], $\left(s^{\star}, \sigma^{\star}\right)$ is a saddle point of $K_{R, P}(\cdot, \cdot)$ if and only if the supremum in

$$
\begin{equation*}
\sup _{s \in \mathbb{R}_{\geq 0}} \inf _{\sigma \in \mathcal{S}_{P, W}(\mathcal{H})} K_{R, P}(s, \sigma) \tag{44}
\end{equation*}
$$

is attained at $s^{\star}$, the infimum in

$$
\begin{equation*}
\inf _{\sigma \in \mathcal{S}_{P, W}(\mathcal{H})} \sup _{s \in \mathbb{R}_{\geq 0}} K_{R, P}(s, \sigma) \tag{45}
\end{equation*}
$$

is attained at $\sigma^{\star}$, and the two extrema in Eqs. (44), (45) are equal and finite. We first claim that

$$
\begin{equation*}
\forall s \in \mathbb{R}_{\geq 0}, \quad \inf _{\sigma \in \mathcal{S}_{P, W}(\mathcal{H})} K_{R, P}(s, \sigma)=\inf _{\sigma \in \mathcal{S}(\mathcal{H})} K_{R, P}(s, \sigma) \tag{46}
\end{equation*}
$$

To see that, observe that for any $s \in \mathbb{R}_{\geq 0}$, the definition of the $\alpha$-Rényi divergence yields

$$
\begin{equation*}
\forall \sigma \in \mathcal{S}(\mathcal{H}) \backslash \mathcal{S}_{P, W}(\mathcal{H}), \quad D_{\frac{1}{1+s}}(W \| \sigma \mid P)=+\infty, \tag{47}
\end{equation*}
$$

which, in turn, implies

$$
\begin{equation*}
\forall \sigma \in \mathcal{S}(\mathcal{H}) \backslash \mathcal{S}_{P, W}(\mathcal{H}), \quad K_{R, P}(s, \sigma)=+\infty \tag{48}
\end{equation*}
$$

Hence, Eq. (46) yields

$$
\begin{equation*}
\sup _{s \in \mathbb{R} \geq 0} \inf _{\sigma \in \mathcal{S}_{P, W}(\mathcal{H})} K_{R, P}(s, \sigma)=\sup _{s \in \mathbb{R} \geq 0} \inf _{\sigma \in \mathcal{S}(\mathcal{H})} K_{R, P}(s, \sigma)=\sup _{s \in \mathbb{R} \geq 0} \min _{\sigma \in \mathcal{S}(\mathcal{H})} K_{R, P}(s, \sigma), \tag{49}
\end{equation*}
$$

where the last equality in Eq. (49) follows from the lower semi-continuity of the map $\sigma \mapsto D_{1 /(1+s)}(W \| \sigma \mid P)$ [24, Corollary III.25] and the compactness of $\mathcal{S}(\mathcal{H})$. Further, by the fact $R>R_{\infty}$ and the definition of $E_{\mathrm{sp}}^{(2)}$, we have

$$
\begin{equation*}
E_{\mathrm{sp}}^{(2)}(R, P)=\sup _{s \in \mathbb{R} \geq 0} \min _{\sigma \in \mathcal{S}(\mathcal{H})} K_{R, P}(s, \sigma)<+\infty, \tag{50}
\end{equation*}
$$

which guarantees the supremum in the right-hand side of Eq. (49) is attained at some $s \in \mathbb{R}_{\geq 0}$, i.e.,

$$
\begin{equation*}
\sup _{s \in \mathbb{R} \geq 0} \inf _{\sigma \in \mathcal{S}_{P, W}(\mathcal{H})} K_{R, P}(s, \sigma)=\max _{s \in \mathbb{R} \geq 0} \min _{\sigma \in \mathcal{S}(\mathcal{H})} K_{R, P}(s, \sigma)<+\infty \tag{51}
\end{equation*}
$$

Thus, we complete our claim in Eq. (44). It remains to show that the infimum in Eq.(45) is attained at some $\sigma^{\star} \in \mathcal{S}_{P, W}(\mathcal{H})$ and the supremum and infimum are exchangeable. To achieve this, we will show that $\left(\mathbb{R}_{\geq 0}, \mathcal{S}_{P, W}(\mathcal{H}), K_{R, P}\right)$ is a closed saddle-element (see Definition 10 below) and apply Rockafellar's saddle-point result, Theorem 11, to conclude our claim.
Definition 10 (Closed Saddle-Element [29]). The triple $(\mathcal{A}, \mathcal{B}, F)$ is called a closed saddle-element if for any ${ }^{1} x \in \operatorname{ri}(\mathcal{A})($ resp. $y \in \operatorname{ri}(\mathcal{B})):$
(a) $\mathcal{B}$ (resp. $\mathcal{A}$ ) is convex;
(b) $F(x, \cdot)$ (resp. $F(\cdot, y)$ ) is convex (resp. concave) and lower (resp. upper) semi-continuous; and
(c) any accumulation point of $\mathcal{B}$ (resp. $\mathcal{A}$ ) that does not belong to $\mathcal{B}$ (resp. $\mathcal{A}$ ), say $y_{o}$ (resp. $x_{o}$ ) satisfies $\lim _{y \rightarrow y_{o}} F(x, y)=+\infty\left(\right.$ resp. $\left.\lim _{x \rightarrow x_{o}} F(x, y)=-\infty\right)$.
Theorem 11 (The Existence of Saddle-Points [29, Theorem 8], [30, Theorem 37.3]). Let $(\mathcal{A}, \mathcal{B}, F)$ be any closed saddle-element on $\mathbb{R}^{m} \times \mathbb{R}^{n}$.
(I) No non-zero $x_{0}$ has the property that, for all $x \in \operatorname{ri}(\mathcal{A})$ and $y \in \operatorname{ri}(\mathcal{B})$, the half-line $\left\{x+t x_{0}: t \geq 0\right\}$ is contained in $\mathcal{A}$ and $F\left(x+t x_{0}, y\right)$ is a non-zero and non-decreasing function for $t \geq 0$.
(II) No non-zero $y_{0}$ has the property that, for all $x \in \operatorname{ri}(\mathcal{A})$ and $y \in \operatorname{ri}(\mathcal{B})$, the half-line $\left\{y+t y_{0}: t \geq 0\right\}$ is contained in $\mathcal{B}$ and $F\left(x, y+t y_{0}\right)$ is a non-increasing function for $t \geq 0$.
If condition (I) is satisfied, then

$$
\begin{equation*}
\max _{x \in \mathcal{A}} \inf _{y \in \mathcal{B}} F(x, y)=\inf _{y \in \mathcal{B}} \sup _{x \in \mathcal{A}} F(x, y)<+\infty . \tag{52}
\end{equation*}
$$

If condition (II) is satisfied, then

$$
\begin{equation*}
-\infty<\sup _{x \in \mathcal{A}} \inf _{y \in \mathcal{B}} F(x, y)=\min _{y \in \mathcal{B}} \sup _{x \in \mathcal{A}} F(x, y) \tag{53}
\end{equation*}
$$

If (I) and (II) are both satisfied, then $F$ has a saddle-point on $\mathcal{A} \times \mathcal{B}$.
Fix an arbitrary $s \in \operatorname{ri}\left(\mathbb{R}_{\geq 0}\right)=\mathbb{R}_{>0}$. We check that $\left(\mathcal{S}_{P, W}(\mathcal{H}), K_{R, P}(s, \cdot)\right)$ fulfills the three items in Definition 10. (a) The set $\mathcal{S}_{P, W}(\mathcal{H})$ is clearly convex. (b) Since the map $\sigma \mapsto D_{1 /(1+s)}(W \| \sigma \mid P)$ is convex (owing to Lieb's concavity theorem [31]) and lower semi-continuous on $\mathcal{L}(\mathcal{H})_{+}$[24, Corollary III.25], by Eq. (43), $\sigma \mapsto K_{R, P}(\alpha, \sigma)$ is also convex and lower semi-continuous on $\mathcal{S}_{P, W}(\mathcal{H})$. (c) Due to the compactness of $\mathcal{S}(\mathcal{H})$, any accumulation point of $\mathcal{S}_{P, W}(\mathcal{H})$ that does not belong to $\mathcal{S}_{P, W}(\mathcal{H})$, say $\sigma_{o}$, satisfies $\sigma_{o} \in \mathcal{S}(\mathcal{H}) \backslash \mathcal{S}_{P, W}(\mathcal{H})$. By Eqs. (47), (48), one finds $K_{R, P}\left(\alpha, \sigma_{o}\right)=+\infty$.

Next, fix an arbitrary $\sigma \in \operatorname{ri}\left(\mathcal{S}_{P, W}(\mathcal{H})\right)$. Owing to the convexity of $\mathcal{S}_{P, W}(\mathcal{H})$, it follows that ri $\left(\mathcal{S}_{P, W}(\mathcal{H})\right)$ $=\operatorname{ri}\left(\operatorname{cl}\left(\mathcal{S}_{P, W}(\mathcal{H})\right)\right.$ ) (see e.g. [30, Theorem 6.3]). We first claim $\operatorname{cl}\left(\mathcal{S}_{P, W}(\mathcal{H})\right)=\mathcal{S}(\mathcal{H})$. To see this, observe that $\mathcal{S}(\mathcal{H})_{++} \subseteq \mathcal{S}_{P, W}(\mathcal{H})$ since a full-rank density operator is not orthogonal with every $W_{x}, x \in \mathcal{X}$. Hence,

$$
\begin{equation*}
\mathcal{S}(\mathcal{H})=\mathrm{cl}\left(\mathcal{S}(\mathcal{H})_{++}\right) \subseteq \mathrm{cl}\left(\mathcal{S}_{P, W}(\mathcal{H})\right) . \tag{54}
\end{equation*}
$$

On the other hand, the fact $\mathcal{S}_{P, W}(\mathcal{H}) \subseteq \mathcal{S}(\mathcal{H})$ leads to

$$
\begin{equation*}
\mathrm{cl}\left(\mathcal{S}_{P, W}(\mathcal{H})\right) \subseteq \mathrm{cl}(\mathcal{S}(\mathcal{H}))=\mathcal{S}(\mathcal{H}) . \tag{55}
\end{equation*}
$$

By Eqs. (54) and (55), we deduce that

$$
\begin{equation*}
\operatorname{ri}\left(\mathcal{S}_{P, W}(\mathcal{H})\right)=\operatorname{ri}\left(\operatorname{cl}\left(\mathcal{S}_{P, W}(\mathcal{H})\right)\right)=\operatorname{ri}(\mathcal{S}(\mathcal{H}))=\mathcal{S}(\mathcal{H})_{++}, \tag{56}
\end{equation*}
$$

where the last equality in Eq. (56) follows from [32, Proposition 2.9]. Hence, we obtain

$$
\begin{equation*}
\forall \sigma \in \operatorname{ri}\left(\mathcal{S}_{P, W}(\mathcal{H})\right) \quad \text { and } \quad \forall x \in \mathcal{X}, \quad \sigma \gg W_{x} . \tag{57}
\end{equation*}
$$

Now, we verify that $\left(\mathbb{R}_{\geq 0}, K_{R, P}(\cdot, \sigma)\right)$ satisfies the three items in Definition 10. (a) The set $\mathbb{R}_{\geq 0}$ is obviously convex. (b) From Eqs. (57) and the definition of the Rényi divergence, the map $s \mapsto D_{1 /(1+s)}(W \| \sigma \mid P)$ is continuous on $\mathbb{R}_{\geq 0}$. Further, $s \mapsto s D_{1 /(1+s)}(W \| \sigma \mid P)$ is concave on $\mathbb{R}_{\geq 0}$ [24, Appendix B]. By Eq. (43), the map $s \mapsto K_{R, P}(s, \sigma)$ is concave and continuous on $\mathbb{R}_{\geq 0}$. (c) Since $\mathbb{R}_{\geq 0}$ is closed, there is no accumulation point of $\mathbb{R}_{\geq 0}$ that does not belong to $\mathbb{R}_{\geq 0}$.

[^1]We are now in a position to prove item (i) of this Proposition. Since the set $\mathcal{S}_{P, W}(\mathcal{H})$ is bounded, condition (II) is satisfied. Equation (53) in Theorem 11 implies that

$$
\begin{equation*}
-\infty<\sup _{s \in \mathbb{R} \geq 0} \inf _{\sigma \in \mathcal{S}_{P, W}(\mathcal{H})} K_{R, P}(s, \sigma)=\min _{\sigma \in \mathcal{S}_{P, W}(\mathcal{H})} \sup _{s \in \mathbb{R} \geq 0} K_{R, P}(s, \sigma) . \tag{58}
\end{equation*}
$$

Then Eqs. (51) and (58) lead to the existence of a saddle-point of $K_{R, P}(\cdot, \cdot)$ on $\mathbb{R}_{\geq 0} \times \mathcal{S}_{P, W}(\mathcal{H})$. Note that $K_{R, P}(s, \sigma)=F_{R, P}(1 /(1+s), \sigma)$. We conclude the existence of a saddle-point of $F_{R, P}(\cdot, \cdot)$ on $(0,1] \times$ $\mathcal{S}_{P, W}(\mathcal{H})$. Hence, item (i) is proved.

We postpone the proof of the uniqueness of the optimizer to later and now show item (iii). Given any $R \in\left(R_{\infty}, C\right)$ and $P \in \mathcal{P}_{R}(\mathcal{X})$, one finds

$$
\begin{equation*}
\min _{\sigma \in \mathcal{S}(\mathcal{H})} \sup _{0<\alpha \leq 1} F_{R, P}(\alpha, \sigma) \in(0,+\infty) . \tag{59}
\end{equation*}
$$

If $\alpha^{\star}=1$ and $\sigma^{\star}$ is a saddle point of $F_{R, P}(\cdot, \cdot)$, by Eq. (10) we deduce that $F_{R, P}\left(1, \sigma^{\star}\right)=0$ for every possible $\sigma^{\star}$, which contradicts Eq. (59). Hence, $\alpha^{\star}=1$ is not a saddle point of $F_{R, P}\left(\cdot, \sigma^{\star}\right)$.

For any saddle-point ( $\alpha^{\star}, \sigma^{\star}$ ) of $F_{R, P}(\cdot, \cdot)$, it holds that

$$
\begin{equation*}
F_{R, P}\left(\alpha^{\star}, \sigma^{\star}\right)=\min _{\sigma \in \mathcal{S}(\mathcal{H})} F_{R, P}\left(\alpha^{\star}, \sigma\right)=\frac{\alpha^{\star}-1}{\alpha^{\star}} R+\frac{1-\alpha^{\star}}{\alpha^{\star}} \min _{\sigma \in \mathcal{S}(\mathcal{H})} D_{\alpha^{\star}}(W \| \sigma \mid P) . \tag{60}
\end{equation*}
$$

We claim the minimizer of Eq. (60) must satisfy

$$
\begin{equation*}
\sigma^{\star}=\frac{\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_{x}^{\alpha^{\star}}}{\operatorname{Tr}\left[W_{x}^{\alpha^{\star}}\left(\sigma^{\star}\right)^{1-\alpha^{\star}}\right]}\right)^{\frac{1}{\alpha^{\star}}}}{\operatorname{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_{x}^{\alpha^{\star}}}{\operatorname{Tr}\left[W_{x}^{\alpha^{\star}}\left(\sigma^{\star}\right)^{1-\alpha^{\star}}\right]}\right)^{\frac{1}{\alpha^{\star}}}\right]}=\frac{\left(\sum_{x \in \mathcal{X}} P(x) W_{x}^{\alpha^{\star}} \mathrm{e}^{\left(1-\alpha^{\star}\right) D_{\alpha^{\star}}\left(W_{x} \| \sigma\right)}\right)^{\frac{1}{\alpha^{\star}}}}{\operatorname{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) W_{x}^{\alpha^{\star}} \mathrm{e}^{\left(1-\alpha^{\star}\right) D_{\alpha^{\star}}\left(W_{x} \| \sigma\right)}\right)^{\frac{1}{\alpha^{\star}}}\right]} \tag{61}
\end{equation*}
$$

for every $\alpha^{\star} \in(0,1)$. Our approach closely follows from Hayashi and Tomamichel [33, Lemma 5]. For two density operators $\sigma, \omega \in \mathcal{S}(\mathcal{H})$ and a $\operatorname{map} G: \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})_{\text {sa }}$ (where $\mathcal{L}(\mathcal{H})_{\text {sa }}$ denotes the self-adjoint operators on $\mathcal{H}$ ), define the Fréchet derivative (see e.g. [33, Appendix C], [34] ${ }^{2}$ )

$$
\begin{equation*}
\partial_{\omega} G(\sigma):=\mathrm{D} G(\sigma)[\omega-\sigma] . \tag{62}
\end{equation*}
$$

By letting

$$
\begin{equation*}
g_{\alpha}(\sigma):=\sum_{x \in \mathcal{X}} P(x) \log \operatorname{Tr}\left[W_{x}^{\alpha} \sigma^{1-\alpha}\right], \tag{63}
\end{equation*}
$$

it follows that

$$
\begin{equation*}
\sigma^{\star}=\underset{\sigma \in \mathcal{S}(\mathcal{H})}{\arg \min } D_{\alpha}(W \| \sigma \mid P)=\underset{\sigma \in \mathcal{S}(\mathcal{H})}{\arg \max } g_{\alpha}(\sigma), \quad \forall \alpha \in(0,1) . \tag{64}
\end{equation*}
$$

Since the map $\sigma \mapsto g_{\alpha}(\sigma)$ is strictly concave for every $\alpha \in(0,1)$ [31], a sufficient and necessary condition for $\sigma$ to be an optimizer of Eq. (64) is $\partial_{\omega} g_{\alpha}(\sigma)=0$ for all $\omega \in \mathcal{S}(\mathcal{H})$. Direct calculation shows that

$$
\begin{equation*}
\partial_{\omega} g_{\alpha}(\sigma)=\operatorname{Tr}\left[\sum_{x \in \mathcal{X}} P(x) \frac{W_{x}^{\alpha}}{\operatorname{Tr}\left[W_{x}^{\alpha} \sigma^{1-\alpha}\right]} \partial_{\omega} \sigma^{1-\alpha} .\right] \tag{65}
\end{equation*}
$$

Next, we check that the fixed-points of the following map achieves the optimum:

$$
\begin{equation*}
\sigma \mapsto \frac{\left(\sum_{x \in \mathcal{X}} P(x) \frac{W^{\alpha}}{\operatorname{Tr}\left[W_{x}^{\alpha} \sigma^{1-\alpha}\right]}\right)^{\frac{1}{\alpha}}}{\operatorname{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_{x}^{\alpha}}{\operatorname{Tr}\left[W_{x}^{\alpha} \sigma^{1-\alpha}\right]}\right)^{\frac{1}{\alpha}}\right]} . \tag{66}
\end{equation*}
$$

[^2]Let

$$
\begin{equation*}
\chi_{\alpha}(\sigma):=\operatorname{Tr}\left[\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_{x}^{\alpha}}{\operatorname{Tr}\left[W_{x}^{\alpha} \sigma^{1-\alpha}\right]}\right)^{\frac{1}{\alpha}}\right]>0, \quad \forall \alpha \in(0,1), \tag{67}
\end{equation*}
$$

and let $\bar{\sigma}$ be a fix-point of the map in Eq. (66). Then, by Eqs. (66), (67), we have

$$
\begin{equation*}
\chi_{\alpha}(\bar{\sigma}) \cdot \bar{\sigma}=\left(\sum_{x \in \mathcal{X}} P(x) \frac{W_{x}^{\alpha}}{\operatorname{Tr}\left[W_{x}^{\alpha} \bar{\sigma}^{1-\alpha}\right]}\right)^{\frac{1}{\alpha}} \tag{68}
\end{equation*}
$$

Substituting Eq. (68) into Eq. (65) yields

$$
\begin{align*}
\partial_{\omega} g_{\alpha}(\bar{\sigma}) & =\operatorname{Tr}\left[\chi_{\alpha}(\bar{\sigma})^{\alpha} \bar{\sigma}^{\alpha} \partial_{\omega} \bar{\sigma}^{1-\alpha}\right]=\operatorname{Tr}\left[\chi_{\alpha}(\bar{\sigma})^{\alpha} \bar{\sigma}^{\alpha}(1-\alpha) \bar{\sigma}^{-\alpha}(\omega-\bar{\sigma})\right] \\
& =(1-\alpha) \chi_{\alpha}(\bar{\sigma})^{\alpha} \operatorname{Tr}[\omega-\bar{\sigma}]=0 . \tag{69}
\end{align*}
$$

By Brouwer's fixed-point theorem, the map in Eq. (66) is indeed the optimizer for Eq. (64). Further, from Eq. (61), it is clear that

$$
\begin{equation*}
\sigma^{\star} \gg W_{x}, \quad \forall x \in \operatorname{supp}(P), \tag{70}
\end{equation*}
$$

and thus item (iii) is proved.
Lastly, we show the uniqueness of the saddle-point. Since the map $\sigma \mapsto D_{\alpha}(W \| \sigma \mid P)$ is strictly concave [31], the minimizer of Eq. (59) is unique for any $\alpha \in(0,1)$. Then, it remains to prove the uniqueness of the maximizer. Let $\sigma^{\star}$ attain the minimum in Eq. (59). By using the reparameterization again, we have

$$
\begin{align*}
K_{R, P}\left(s, \sigma^{\star}\right) & =-s R+s D_{\frac{1}{1+s}}\left(W \| \sigma^{\star} \mid P\right)  \tag{71}\\
& =-s R+s \sum_{x \in \mathcal{X}} P(x) D_{\frac{1}{1+s}}\left(p_{x} \| q_{x}\right), \tag{72}
\end{align*}
$$

where $p_{x}, q_{x}$ are the Nussbaum-Szkoła distributions of $W_{x}$ and $\sigma^{\star}$. The second-order partial derivative can be calculated as

$$
\begin{equation*}
\frac{\partial^{2} K_{R, P}\left(s, \sigma^{\star}\right)}{\partial s^{2}}=-\frac{1}{(1+s)^{3}} \sum_{x \in \mathcal{X}} P(x) \operatorname{Var}_{\hat{q}_{1 \frac{1}{1+s}}, x}\left[\log \frac{q_{x}}{p_{x}}\right] \tag{73}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{q}_{t, x}(\omega):=\frac{p_{x}(\omega)^{1-t} q_{x}(\omega)^{t}}{\sum_{\omega \in \operatorname{supp}\left(p_{x}\right) \cap \operatorname{supp}\left(q_{x}\right)} p_{x}(\omega)^{1-t} q_{x}(\omega)^{t}}, \quad \forall \omega \in \operatorname{supp}\left(p_{x}\right) \cap \operatorname{supp}\left(q_{x}\right), t \in[0,1] . \tag{74}
\end{equation*}
$$

Now, we assume the right-hand side of Eq. (73) is zero, which is equivalent to

$$
\begin{equation*}
p_{x}(\omega)=c_{x} \cdot q_{x}(\omega), \quad \forall \omega \in \operatorname{supp}\left(p_{x}\right) \cap \operatorname{supp}\left(q_{x}\right) \tag{75}
\end{equation*}
$$

for some constant $c_{x}>0$ and $x \in \operatorname{supp}(P)$. From Eq. (70), one finds $p_{x} \ll q_{x}$. Summing the right-hand side of Eq. (75) over $\omega \in p_{x}^{0}$ yields

$$
\begin{equation*}
1=c_{x} \cdot \operatorname{Tr}\left[p_{x}^{0} q_{x}\right], \quad \forall x \in \operatorname{supp}(P) \tag{76}
\end{equation*}
$$

By combining Eqs. (75) and (76), one can verify

$$
\begin{equation*}
\sup _{s \in \mathbb{R}>0}\left\{-s R+s \sum_{x \in \mathcal{X}} P(x) D_{\frac{1}{1+s}}\left(p_{x} \| q_{x}\right)\right\}=\sup _{s \in \mathbb{R}>0}\left\{-s R-s \sum_{x \in \mathcal{X}} P(x) \log \operatorname{Tr}\left[p_{x}^{0} q_{x}\right]\right\}=0 \tag{77}
\end{equation*}
$$

where we rely the fact $R>R_{\infty}(W) \geq-\sum_{x \in \mathcal{X}} P(x) \log \operatorname{Tr}\left[p_{x}^{0} q_{x}\right]$ from Eq. (3). However, Eq. (77) contradicts the assumption $P \in \mathcal{P}_{R}(\mathcal{X})$, which in turn implies that the right-hand side of Eq. (73) is strictly negative. Therefore, the map $s \rightarrow K_{R, P}\left(s, \sigma^{\star}\right)$ is strictly concave for all $s \in \mathbb{R}_{>0}$ and thus the maximizer of Eq. (59) is unique.
B.2. Proof of Lemma 6. Let $\mathbf{x}^{n}(m)$ be the codeword encoding the message $m \in\{1, \ldots, \exp \{n r\}\}$. We define a binary hypothesis testing problem as:

$$
\begin{align*}
& \mathrm{H}_{0}: W_{\mathbf{x}^{n}(m)}^{\otimes n} ;  \tag{78}\\
& \mathrm{H}_{1}: \sigma^{n}:=\bigotimes_{i=1}^{n} \sigma_{i}, \tag{79}
\end{align*}
$$

where $\sigma^{n} \in \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)$ can be viewed as a dummy channel output. Since $\sum_{m=1}^{M} \beta\left(\Pi_{n, m} ; \sigma^{n}\right)=1$ for any POVM $\Pi_{n}=\left\{\Pi_{n, 1}, \ldots, \Pi_{n, \exp \{n r\}}\right\}$, and $\beta\left(\Pi_{n, m} ; \sigma^{\otimes n}\right) \geq 0$ for every $m \in \mathcal{M}$, there must exist a message $m \in \mathcal{M}$ for any code $\mathcal{C}_{n}$ such that $\beta\left(\Pi_{n, m} ; \sigma^{n}\right) \leq \exp \{-n r\}$. Let $\mathbf{x}^{n}:=\mathbf{x}^{n}(m)$ be the codeword for that message $m$. Then

$$
\begin{equation*}
\epsilon_{\max }\left(\mathcal{C}_{n}\right) \geq \epsilon_{m}\left(\mathcal{C}_{n}\right)=\alpha\left(\Pi_{n, m} ; W_{\mathbf{x}^{n}}^{\otimes n}\right) \geq \widehat{\alpha}_{\exp \{-n r\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma^{n}\right) \tag{80}
\end{equation*}
$$

Since the above inequality (80) holds for every $\sigma^{n} \in \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)$, it follows that

$$
\begin{equation*}
\epsilon_{\max }\left(\mathcal{C}_{n}\right) \geq \max _{\sigma \in \mathcal{S}(H)} \widehat{\alpha}_{\exp \{-n r\}}\left(W_{\mathbf{x}^{n}}^{\otimes n} \| \sigma^{\otimes n}\right) \tag{81}
\end{equation*}
$$

B.3. Proof of Lemma 7. This lemma closely follows from Altuğ and Wagner's [11, Lemma 9]. However, the major difference is that we prove the claim using the expression $\phi_{n}$ as the error-exponent instead of the discrimination function: $\min \left\{\mathbb{D}(\tau \| \rho): \mathbb{D}(\tau \| \sigma) \leq R_{n}\right\}$. This expression is crucial to obtaining the sphere-packing bound in Theorem 1 in the strong form of Gallager's expression

For convenience, we shorthand $r=R_{n}$. From Lemma 5 , it can be verified that

$$
\begin{align*}
E_{0}(s) & :=-\frac{1+s}{n} \log \operatorname{Tr}\left[\left(p^{n}\right)^{\frac{1}{1+s}}\left(q^{n}\right)^{\frac{s}{1+s}}\right]  \tag{82}\\
& =-(1+s) \Lambda_{0, P_{\mathbf{x}^{n}}}\left(\frac{s}{1+s}\right) \tag{83}
\end{align*}
$$

where Eq. (83) follows from the definition of $\Lambda_{0, P_{\mathbf{x}^{n}}}$ in Eq. (21). Then, we rewrite the error-exponent function $\phi_{n}(r)$ by the Legendre-Fenchel transform of $E_{0}(s)$, i.e.,

$$
\begin{align*}
\phi_{n}(r) & =\sup _{\alpha \in(0,1]}\left\{\frac{\alpha-1}{\alpha}\left(r-\sum_{x \in \mathcal{X}} P_{\mathbf{x}^{n}}(x) D_{\alpha}\left(p_{x} \| q_{x}\right)\right)\right\}  \tag{84}\\
& =\sup _{s \geq 0}\left\{-s r+E_{0}(s)\right\} \tag{85}
\end{align*}
$$

Direct calculation shows that

$$
\begin{align*}
\frac{\partial E_{0}(s)}{\partial s} & =-\Lambda_{0, P_{x^{n}}}\left(\frac{s}{1+s}\right)-\frac{1}{1+s} \Lambda_{0, P_{x^{n}}}^{\prime}\left(\frac{s}{1+s}\right)  \tag{86}\\
\frac{\partial^{2} E_{0}(s)}{\partial s^{2}} & =-\frac{1}{(1+s)^{3}} \Lambda_{0, P_{x^{n}}}^{\prime \prime}\left(\frac{s}{1+s}\right) \tag{87}
\end{align*}
$$

Now assume the second-order derivative $\Lambda_{0, P_{x^{n}}}^{\prime \prime}(t)$ in right-hand side of Eq. (87) is zero for some $t \in[0,1]$. This is equivalent to

$$
\begin{equation*}
p_{x}(\omega)=q_{x}(\omega) \cdot \mathrm{e}^{-\Lambda_{0, x}^{\prime}(t)}, \quad \forall \omega \in p_{x}, \quad \forall x \in \operatorname{supp}\left(P_{\mathbf{x}^{n}}\right) \tag{88}
\end{equation*}
$$

Summing the right-hand side of Eq. (88) over $\omega \in p_{x}$ gives

$$
\begin{equation*}
1=\operatorname{Tr}\left[p_{x}^{0} q_{x}\right] \mathrm{e}^{-\Lambda_{0, x}^{\prime}(t)} \tag{89}
\end{equation*}
$$

Then, Eqs. (88) and (89) imply that

$$
\begin{align*}
\phi_{n}(r) & =\sup _{0<\alpha \leq 1} \frac{\alpha-1}{\alpha}\left(r-\sum_{x \in \mathcal{X}} P_{\mathbf{x}^{n}}(x) D_{\alpha}\left(p_{x} \| q_{x}\right)\right)  \tag{90}\\
& =\sup _{0<\alpha \leq 1} \frac{\alpha-1}{\alpha}\left(r+\sum_{x \in \mathcal{X}} P_{\mathbf{x}^{n}}(x) \log \operatorname{Tr}\left[p_{x}^{0} q_{x}\right]\right)=0, \tag{91}
\end{align*}
$$

where in Eq. (91) we use the fact that $r>-\sum_{x \in \mathcal{X}} P_{\mathbf{x}^{n}}(x) \log \operatorname{Tr}\left[p_{x}^{0} q_{x}\right]$; see Eq. (3). However, from Lemma 5 we know that $\phi_{n}(r)=E_{\mathrm{sp}}(R)>0$, which leads to a contradiction. Hence, we obtain

$$
\begin{equation*}
\Lambda_{0, P_{x^{n}}}^{\prime \prime}(t)>0, \quad \forall t \in[0,1], \tag{92}
\end{equation*}
$$

and prove item (i).
From Eqs. (87) and (92), the objective function $-s r+E_{0}(s)$ in Eq. (85) is strictly concave in $s$ for $s \in \mathbb{R}_{+}$. Further, by recalling that $\phi_{n}(r)=E_{\mathrm{sp}}(R)>0, s=0$ will not be an optimum in Eq. (85). We deduce that there exists a unique maximizer $s^{\star} \in \mathbb{R}_{>0}$ such that

$$
\begin{align*}
r & =\left.\frac{\partial E_{0}(s)}{\partial s}\right|_{s=s^{\star}}  \tag{93}\\
\phi_{n}(r) & =E_{0}\left(s^{\star}\right)-\left.s^{\star} \frac{\partial E_{0}(s)}{\partial s}\right|_{s=s^{\star}}, \tag{94}
\end{align*}
$$

if $r$ lies in the range:

$$
\begin{equation*}
-\frac{1}{n} \log \operatorname{Tr}\left[\left(p^{n}\right)^{0} q^{n}\right]=\lim _{s \rightarrow+\infty} \frac{\partial E_{0}(s)}{\partial s} \leq r \leq\left.\frac{\partial E_{0}(s)}{\partial s}\right|_{s=0}=\frac{1}{n} \mathbb{D}\left(p^{n} \| q^{n}\right) \tag{95}
\end{equation*}
$$

where the boundary values $-\frac{1}{n} \log \operatorname{Tr}\left[\left(p^{n}\right)^{0} q^{n}\right]$ and $\frac{1}{n} \mathbb{D}\left(p^{n} \| q^{n}\right)$ can be obtained from Eqs. (86), (19) and (20). Substituting Eq. (93) into (86) gives

$$
\begin{equation*}
r=-\Lambda_{0, P_{\mathbf{x}^{n}}}\left(\frac{s^{\star}}{1+s^{\star}}\right)-\frac{1}{1+s^{\star}} \Lambda_{0, P_{\mathbf{x}^{n}}}^{\prime}\left(\frac{s^{\star}}{1+s^{\star}}\right) . \tag{96}
\end{equation*}
$$

Further, Eqs. (85), (83), (96) imply that

$$
\begin{align*}
\phi_{n}(r) & =-s^{\star} r+E_{0}\left(s^{\star}\right)  \tag{97}\\
& =\frac{s^{\star}}{1+s^{\star}} \Lambda_{0, P_{x^{n}}}^{\prime}\left(\frac{s^{\star}}{1+s^{\star}}\right)-\Lambda_{0, P_{x^{n}}}\left(\frac{s^{\star}}{1+s^{\star}}\right) . \tag{98}
\end{align*}
$$

By comparing Eqs. (96) and (98), we obtain

$$
\begin{equation*}
\Lambda_{0, P_{\mathbf{x}^{n}}}^{\prime}\left(\frac{s^{\star}}{1+s^{\star}}\right)=\phi_{n}(r)-r \tag{99}
\end{equation*}
$$

which is exactly the optimum solution to the Fenchel-Legendre transform $\Lambda_{0, P_{x^{n}}}^{*}(z)$ in Eq. (22) with

$$
\begin{align*}
t^{\star} & =\frac{s^{\star}}{1+s^{\star}} \in(0,1),  \tag{100}\\
z & =\phi_{n}(r)-r . \tag{101}
\end{align*}
$$

From Eqs. (22), (99) and (98), we conclude the item (i) of Lemma 7:

$$
\begin{align*}
\Lambda_{0, P_{\mathbf{x}^{n}}}^{*}\left(\phi_{n}(r)-r\right) & =t^{\star} z-\Lambda_{0, P_{\mathbf{x}^{n}}}\left(t^{\star}\right)  \tag{102}\\
& =\frac{s^{\star}}{1+s^{\star}}\left(\phi_{n}(r)-r\right)-\Lambda_{0, P_{\mathbf{x}^{n}}}\left(\frac{s^{\star}}{1+s^{\star}}\right)  \tag{103}\\
& =\frac{s^{\star}}{1+s^{\star}} \Lambda_{0, P_{\mathbf{x}^{n}}}^{\prime}\left(\frac{s^{\star}}{1+s^{\star}}\right)-\Lambda_{0, P_{\mathbf{x}^{n}}}\left(\frac{s^{\star}}{1+s^{\star}}\right)  \tag{104}\\
& =\phi_{n}(r) . \tag{105}
\end{align*}
$$

Item (ii) follows from item (i), the symmetry $\Lambda_{0, x_{i}}(t)=\Lambda_{1, x_{i}}(1-t)$ and $\Lambda_{0, x_{i}}^{\prime}(t)=-\Lambda_{1, x_{i}}^{\prime}(1-t)$, and Eq. (22). $\Lambda_{1, P_{x^{n}}}^{*}(r-\phi(r))=r$.

For the item (iii), the positivity of $\Lambda_{0, P_{x^{n}}}^{\prime \prime}(t)$, for $t \in[0,1]$, implies that the objective function $t z-$ $\Lambda_{0, P_{x^{n}}}(t)$ in Eq. (22) is strictly concave in $t$ for $t \in[0,1]$. Hence, by Eq. (100), the optimizer $t^{\star} \in(0,1)$ exists uniquely. By recalling Eq. (99), we complete the claim in item (iii).

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[^1]:    ${ }^{1}$ We denote by ri and cl the relative interior and the closure of a set, respectively.

[^2]:    ${ }^{2}$ We note that the Fréchet derivative of functions involving matrices has other applications in quantum information theory; see e.g. [35, 36, 37].

