SOME THEOREMS INVOLVING POWERS OF GENERALIZED FIBONACCI NUMBERS AT NON-EQUIDISTANT POINTS

Paul S. Bruckman
P.O. Box 150, Sointula, BC V0N 3E0 Canada

Ray S. Melham
Department of Mathematical Sciences, University of Technology,
Sydney P.O. Box 123, Broadway, NSW 2007 Australia

(Submitted September 2005)

ABSTRACT

The paper begins with a brief review of the generalized Fibonacci polynomials, satisfying the recurrence : $G_{n+2}(x) = xG_{n+1}(x) + G_n(x)$, with arbitrary initial values. After these introductory remarks, the paper points out that the plethora of identities that abound in the literature and that are associated with these sequences are based on evenly-spaced values of the subscripts. As implied in the title, the paper generalizes such identities by stating and proving various theorems and corollaries, giving identities satisfied by the m^{th} powers of the $G_n(x)$'s (or special values thereof). In these generalizations, m is fixed, but the different values of n are arbitrary and therefore not equidistant, in general. Essentially, two main theorems of this nature are presented in this paper, of which the others are either corollaries or auxiliary results leading thereto.

1. INTRODUCTION

Problem H-614 in this journal [5], proposed by one of the authors (Melham), stated the following:

$$\begin{aligned} &F_{a2-a3}F_{a2-a4}F_{a3-a4}F^4_{n+a1} + (-1)^{a1+a2+1}F_{a1-a3}F_{a1-a4}F_{a3-a4}F^4_{n+a2} \\ &\quad + (-1)^{a1+a2}F_{a1-a2}F_{a1-a4}F_{a2-a4}F^4_{n+a3} + (-1)^{a1+a2+a3+a4+1}F_{a1-a2}F_{a1-a3}F_{a2-a3}F^4_{n+a4} \\ &\quad = F_{a1-a2}F_{a1-a3}F_{a1-a4}F_{a2-a3}F_{a2-a4}F_{a3-a4}F_{4n+a1+a2+a3+a4}. \end{aligned}$$

Using the relation: $F_{-n} = (-1)^{n-1}F_n$, we may express this last identity in a nice symmetrical form:

$$\begin{aligned} &F^4_{n+a1}/\{F_{a1-a2}F_{a1-a3}F_{a1-a4}\} + F^4_{n+a2}/\{F_{a2-a1}F_{a2-a3}F_{a2-a4}\} \\ &\quad + F^4_{n+a3}/\{F_{a3-a1}F_{a3-a2}F_{a3-a4}\} \\ &\quad + F^4_{n+a4}/\{F_{a4-a1}F_{a4-a2}F_{a4-a3}\} = F_{4n+a1+a2+a3+a4}. \end{aligned} \tag{1.1}$$

In correspondence between the authors, Melham designated the statement of this problem as a "4-point" identity. The counterparts for 1, 2 and 3 points are the following (respectively):

$$F_{n+a1} \equiv F_{n+a1} \tag{1.2}$$

$$F^2_{n+a1}/F_{a1-a2} + F^2_{n+a2}/F_{a2-a1} = F_{2n+a1+a2} \tag{1.3}$$

$$F_{n+a1}^3/\{F_{a1-a2}F_{a1-a3}\} + F_{n+a2}^3/\{F_{a2-a1}F_{a2-a3}\} + F_{n+a3}^3/\{F_{a3-a1}F_{a3-a2}\} = F_{3n+a1+a2+a3}. \tag{1.4}$$

For background information on (1.3), and for a proof of (1.4), see [4]. In their correspondence, Melham informed Bruckman that he had been unable to prove (1.1).

Bruckman submitted an erroneous solution of H-614 to the Advanced Problems Editor. Indeed, the Advanced Problems Editor later informed Bruckman that (as of this writing) no one had been able to submit a complete solution to H-614. Looking at the form of (1.1), there appears to be nothing of theoretical difficulty in proving the identity, other than one of computational complication. Someone with an appropriate program (such as Mathematica), that is able to express products of Fibonacci numbers in terms of other Fibonacci or Lucas numbers should be able to insert the parameters into such a program and easily verify the result. Unfortunately, the computations become more laborious for higher powers of the Fibonacci numbers, and it therefore seemed appropriate to attempt a theoretical approach to the general case. The purpose of this paper, inter alia, (as inspired by the authors' abortive efforts), is to indicate a proper solution to H-614 and to obtain a generalization. Indeed, a more general result is suggested by this problem, namely the following:

$$\sum_{k=1}^{m}(F_{n+ak})^m/R(k,m) = F_{mn+a1+a2+\cdots+am}, \tag{1.5}$$

where

$$R(k,m) = \prod_{\substack{j=1 \\ j\neq k}}^{m} F_{ak-aj}. \tag{1.6}$$

As a matter of fact, a yet more general result in terms of "generalized Fibonacci Polynomials" applies, of which (1.5) is itself a special case. We will indicate this latter result in the form of an identity, comprising the third of the (seven) theorems alluded to in the title of this paper.

The generalized Fibonacci numbers have a vast literature attached to them, too extensive to indicate here. We merely indicate the appropriate definitions. These numbers, which are in fact, polynomials in the variable x, satisfy the following recurrence relation:

$$G_{n+2}(x) = xG_{n+1}(x) + G_n(x), n = 0, 1, 2, \cdots. \tag{1.7}$$

In this paper, x is taken to be a fixed quantity; for this reason, we find it convenient to omit the argument "(x)" in the notation, and will in fact take the same notation to denote what is usually indicated as the subscript.

Depending on the initial conditions, generally taken to be the given values of G_0 and G_1, different sequences are generated. If we take $G_0 = 0$, $G_1 = 1$, the resulting sequence is sometimes called the sequence of "generalized Fibonacci numbers of the first kind", and is usually denoted as $\{U_n\}$; if we take $G_0 = 2$, $G_1 = x$, the resulting sequence is sometimes called the sequence of "generalized Fibonacci numbers of the second kind", and is usually denoted as $\{V_n\}$. As we soon discover by repeated application of the recurrence in (1), the U_n's and V_n's are monic polynomials in x; the degree of U_n is $n-1$, while V_n has degree n (assuming $n \geq 1$). Note that by setting $x = 1$, U_n and V_n reduce to the "ordinary" Fibonacci numbers

F_n and Lucas numbers L_n, respectively. It is easily found that we may express G_n in terms of the U_n's, as follows:

$$G_n = G_1U_n + G_0U_{n-1}. \tag{1.8}$$

These sequences may also be extended "backwards", defining them for negative subscripts by means of the recurrence in (1). For example, we readily verify that $U_{-1} = U_1 = 1$. Indeed, we may show the following, for all integers $n \geq 0$:

$$U_{-n} = (-1)^{n-1}U_n, \quad V_{-n} = (-1)^nV_n. \tag{1.9}$$

In this paper, for the sake of typographical clarity, we will find it convenient to denote G_n, U_n, V_n, F_n, L_n as G(n), U(n), V(n), F(n) and L(n), respectively. Given these definitions, we may readily obtain the following explicit (Binèt) formulas, valid for all integers n:

$$U(n) = (\alpha^n - \beta^n)/(\alpha - \beta), \quad V(n) = \alpha^n + \beta^n, \tag{1.10}$$

where

$$\alpha = \alpha(x) = (x + D^{1/2})/2, \ \beta = \beta(x) = (x - D^{1/2})/2, \tag{1.11}$$

and

$$D = D(x) = (\alpha - \beta)^2 = x^2 + 4. \tag{1.12}$$

Some useful identities satisfied by the $U(n)$'s and $G(n)$'s are given next, without proof; they occur in the literature and may easily be verified by a variety of methods:

$$U(n+1)U(n-1) - U^2(n) = (-1)^n \tag{1.13}$$

$$G(m+n) = G(m+1)U(n) + G(m)U(n-1). \tag{1.14}$$

As previously asserted, the aim of this paper is to state and establish some theorems that are satisfied by the m^{th} powers of the generalized Fibonacci numbers. Theorems 2, 5 and 7 apply to the most general numbers $G(n)$; the remaining theorems apply only to the $U(n)$. In the sequel, we are given a set of *distinct* "points" $a_1, \ a_2, \ a_3, \ \ldots, \ a_m$ (i.e. given integers); also, m and n are integers, with $m \geq 0$ (except as indicated).

Unlike the plethora of identities in the literature for the $U(n)$'s and $V(n)$'s that deal with equidistant, or equally spaced points, the theorems presented in this paper are more general, in that they apply to non-equidistant points. In this respect, they bear some resemblance to comparable known theorems from the finite calculus, expressed as so-called "divided differences" of polynomials.

2. STATEMENT OF THE FIRST THREE THEOREMS

Theorem 1: Given an integer $m \geq 2$, and distinct integers $a_1, \ a_2, \ a_3, \ \cdots, \ a_m$, then, for all integers n:

$$\sum_{k=1}^{m} (-1)^{a_k} U^{m-2}(n + a_k)/P(k, m) = 0 \tag{2.1}$$

$$\text{where } P(k,m) = \prod_{\substack{j=i \\ j \neq k}}^{m} U(a_k - a_j). \tag{2.2}$$

Theorem 2: Given the hypothesis of Theorem 1:

$$\sum_{k=1}^{m} (-1)^{a_k} G^{m-2}(n + a_k)/P(k,m) = 0. \tag{2.3}$$

Theorem 3: Given the hypothesis of Theorem 1 (but with $m \geq 1$):

$$\sum_{k=1}^{m} U^m(n + a_k)/P(k,m) = U(mn + a_1 + a_2 + a_3 + \cdots + a_m). \tag{2.4}$$

Note that by setting $x = 1$ in (2.4), we obtain (1.5) as a special case; furthermore, for specific values of m, we obtain (1.1) (the solution of H-614), (1.2), (1.3) and (1.4).

These theorems are generalizations of the special case obtained where, e.g. we take $a_k = k - 1$, which results in recurrences involving "generalized Fibonomial coefficients". Some additional introductory remarks and definitions are needed at this point, to make the exposition clearer.

Given $n \geq 1$, the "generalized Fibonacci factorials" are defined as follows: $[n!]_U = U(1)\,U(2) \cdots U(n)$. Also, the "generalized Fibonomial coefficient" $\binom{n}{k}_U$ is defined as follows: $\binom{n}{k}_U = [n!]_U/\{[k!]_U[(n-k)!]_U\}$, provided $0 < k < n$. We may also define $\binom{n}{0}_U = \binom{n}{n}_U = 1$, and note that $\binom{n}{k}_U = \binom{n}{n-k}_U$, properties that are shared with the "ordinary" binomial coefficients. It is not difficult to show that these coefficients are, in fact, polynomials in x, and that their degree is $k(n-k)$. The ordinary binomial coefficients satisfy the recurrence: $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$; the "generalized Fibonomial" counterpart of this is the following recurrence:

$$U(n+1-k)\binom{n}{k}_U + U(k)\binom{n}{k+1}_U = \binom{n+1}{k+1}_U. \tag{2.5}$$

For an excellent discussion on generalized binomial coefficients see [1].

If we now set $a_k = k - 1$ in (2.3), we obtain the following recurrence, after a change of parameters and some manipulations:

$$\sum_{k=0}^{m+1} \binom{m+1}{k}_U (-1)^{k(k+1)/2} G^m(n + m + 1 - k) = 0. \tag{2.6}$$

This is a better-known relation, involving consecutive values of the G-function ($m + 2$ such values, for m^{th} powers of the function). For a generalization of (2.6), first given by Dov Jarden, but not required for our purposes, see (3) in [6].

We may arrive at (2.6) independently, by a different method. The first step in this method consists in proving (2.6) for $G(n) \equiv U(n)$. If we expand $U^m(n)$ in terms of powers of α and β, we obtain an expression whose typical terms are of the type $\alpha^{(m-j)n}\beta^{nj}$, for $j = 0, 1, \cdots, m$. By the theory of equations, the characteristic polynomial $P_m(E)$ for *consecutive-indexed* values of $U^m(n)$ is given by the following product:

$$P_m(E) = \prod_{j=0}^{m}(E - \alpha^{m-j}\beta^j). \tag{2.7}$$

Here, E is the unit right-shift operator of finite calculus, operating on the arguments of the function $U^m(n)$; that is, $P_m(E)(U^m(n)) = 0$. For example, we find that $P_1(E) = E^2 - xE - 1$, which implies the recurrence (1.7) (for $G(n) = U(n)$); for $m = 2$, we obtain: $P_2(E) = E^3 - (x^2+1)E^2 - (x^2+1)E + 1$, which implies the recurrence:

$$U^2(n+3) - (x^2+1)U^2(n+2) - (x^2+1)U^2(n+1) + U^2(n) = 0, \quad \text{etc.}$$

Note that $P_m(E)$ is a polynomial of degree $m+1$, which implies that there exist $m+2$ appropriate rational functions $C_1, C_2, \ldots, C_{m+2}$, such that for all integers n:

$$\sum_{k=1}^{m+2} C_k U^m(n+k) = 0. \tag{2.8}$$

This, in turn, implies that for a given sequence of $m+2$ distinct integers $a_1, a_2, \cdots, a_{m+2}$, there exist appropriate rational functions $D_1, D_2, \cdots, D_{m+2}$, such that for all integers n:

$$\sum_{k=1}^{m+2} D_k U^m(n+a_k) = 0. \tag{2.9}$$

This will therefore be assumed as our starting point in the proof of Theorem 1, in the sequel. In fact, as we may show, the "rational functions" indicated in (2.8) and (2.9) are really polynomials, and the more general recurrence indicated in (2.6) holds for consecutive values of the arguments.

The following recurrence follows readily from (2.7):

$$P_{m+4}(E) = (E - \alpha^{m+4})(E - \beta^{m+4})(E + \alpha^{m+2})(E + \beta^{m+2})P_m(E), \quad \text{or equivalently :}$$

$$P_{m+4}(E) = \{E^2 - V(m+4)E + (-1)^m\}\{E^2 + V(m+2)E + (-1)^m\}P_m(E).$$

Using this, we could independently derive (2.6) as a general result in its own right (with $G(n) = U(n)$), noting that the operand of $P_{m+4}(E)$ is $U^{m+4}(n)$ rather than $U^m(n)$. However, we proceed directly to prove the more general Theorem 1; we will show that this is, in turn, equivalent to Theorem 2, from which (2.6) will follow as a Corollary.

3. PROOF OF THEOREM 1

We will express the theorem in terms of $m-2$, rather than m, in order to display it in a slightly more elegant form. Given $m \geq 2$, we begin by postulating the existence of m appropriate quantities $A_1, A_2, \cdots, A_m$, such that for all integers n:

$$\sum_{k=1}^{m} A_k U^{m-2}(n+a_k) = 0. \tag{3.1}$$

Our task is to show that $A_k = (-1)^{a_k}/P(k,m)$, which will establish Theorem 1 in its symmetric form. Our comments in the last part of the previous section indicate that our postulation is appropriate. Note that the A_k's are functions of $a_1, a_2, \cdots, a_m$, as well as of x. Then (3.1) implies the following relations:

$$\sum_{j=0}^{m-2} \binom{m-2}{j} (-1)^j \sum_{k=1}^{m} A_k \alpha^{(m-2-j)a_k} \beta^{ja_k} = 0.$$

This will be satisfied if we set each of the inner sums equal to zero. This gives us a set of $m-1$ equations in m unknowns. The values $A_1, A_2, \cdots, A_{m-1}$, say, may be expressed as functions of A_m. Thus, we may fix A_m, trusting that we may obtain the other quantities by multiplying by an appropriate homogeneity constant. In other words, we may solve for the ratios $A_1/A_m, A_2/A_m, \cdots, A_{m-1}/A_m$. The system may be put into the following matrix form:

$$\begin{pmatrix} \alpha^{(m-2)a_1} & \alpha^{(m-2)a_2} & \alpha^{(m-2)a_3} & \cdots & \alpha^{(m-2)a_m} \\ \alpha^{(m-3)a_1}\beta^{a_1} & \alpha^{(m-3)a_2}\beta^{a_2} & \alpha^{(m-3)a_3}\beta^{a_3} & \cdots & \alpha^{(m-3)a_m}\beta^{a_m} \\ \alpha^{(m-4)a_1}\beta^{2a_1} & \alpha^{(m-4)a_2}\beta^{2a_2} & \alpha^{(m-4)a_3}\beta^{2a_3} & \cdots & \alpha^{(m-4)a_m}\beta^{2a_m} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha^{a_1}\beta^{(m-3)a_1} & \alpha^{a_2}\beta^{(m-3)a_2} & \alpha^{a_3}\beta^{(m-3)a_3} & \cdots & \alpha^{a_m}\beta^{(m-3)a_m} \\ \beta^{(m-2)a_1} & \beta^{(m-2)a_2} & \beta^{(m-2)a_3} & \cdots & \beta^{(m-2)a_m} \\ 0 & 0 & 0 & \cdots & 1/A_m \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{m-2} \\ A_{m-1} \\ A_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The m^*m matrix above is invertible, as we now show. Its determinant is equal to $(1/A_m)$ times the determinant of the $(m-1)^*(m-1)$ matrix that is the minor of the final term. Such minor is of "modified" Vandermonde type, hence (for distinct values $a_1, a_2, \cdots, a_m$), is invertible. Therefore, there is a unique solution for the ratios A_k/A_m. The inverse of the m^*m matrix above need not be determined in its entirety; we only need to determine its final column.

The following result is known about Vandermonde matrices. Suppose

$$W = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \cdot & \cdot & \cdot & \cdot \\ (z_1)^{n-1} & (z_2)^{n-1} & \cdots & (z_n)^{n-1} \end{pmatrix}, \text{ and let } Q(k, n+1) = \prod_{\substack{j=1 \\ j \neq k}}^{n} (z_k - z_j).$$

Then

$$W^{-1} = \begin{pmatrix} & 1/Q(1, n+1) \\ * & \\ & 1/Q(2, n+1) \\ & \vdots \\ & 1/Q(n+1, n+1) \end{pmatrix},$$

where only the final column is shown. In the case of our problem, we obtain terms such as $\alpha^{a_i}\beta^{a_j} - \alpha^{a_j}\beta^{a_i}$, which simplify to: $(-1)^{a_j}D^{1/2}U(a_i - a_j)$. After some additional manipulation (left for the reader to verify) , we determine the following result:

$$\begin{aligned} A_k/A_m &= -U(a_k - a_m)P(m,m)/\{U(a_m - a_k)P(k,m)\} \text{ (for } 1 \le k \le m-1) \\ &= (-1)^{a_k + a_m} P(m,m)/P(k,m). \end{aligned}$$

Fixing A_m as $(-1)^{a_m}/P(m,m)$, we thus conclude that $A_k = (-1)^{a_k}/P(k,m)$ $(1 \le k \le m)$. □

4. PROOF OF THEOREM 2

Clearly, Theorem 2 would imply Theorem 1, since the $U(n)$'s are a special case of the more general $G(n)$'s. What is more interesting is that Theorem 1 implies Theorem 2, i.e. the two theorems are equivalent, as we will now show.

We note that the $P(k,m)$'s in the statement of Theorem 1 are independent of n. Hence, setting $n = 1$: $\sum_{k=1}^{m}(-1)^{a_k}U^{m-2}(1 + a_k)/P(k,m) = 0$. From (1.7), $U(1 + a_k) = xU(a_k) + U(-1 + a_k)$. Substituting this into the last expression, we obtain:

$$0 = \sum_{j=0}^{m-2} \binom{m-2}{j} x^{m-2-j} \sum_{k=1}^{m} (-1)^{a_k} U^{m-2-j}(a_k) U^j(-1 + a_k)/P(k,m).$$

This must be true for all values of x; we therefore argue that the inner sums in the last expression must vanish; that is:

$$\sum_{k=1}^{m} (-1)^{a_k} U^{m-2-j}(a_k) U^j(-1 + a_k)/P(k,m) = 0, \; j = 0, \, 1, \, \cdots, \, m-2. \tag{4.1}$$

Now from (1.14), we have: $G(n + a_k) = G(n+1)U(a_k) + G(n)U(-1 + a_k)$; hence, after simplification,

$$\sum_{k=1}^{m} (-1)^{a_k} G^{m-2}(n + a_k)/P(k,m) = \sum_{j=0}^{m-2} \binom{m-2}{j} G^{m-2-j}(n+1)$$

$$G^j(n) \sum_{k=1}^{m} (-1)^{a_k} U^{m-2-j}(a_k) U^j(-1 + a_k)/P(k,m) = 0,$$

by (4.1). Thus, Theorem 1 implies Theorem 2 and vice-versa. Since Theorem 1 is true, so is Theorem 2. □

5. PROOF OF THEOREM 3

We begin by assuming the generic putative identity:

$$U(x_1 + x_2 + \cdots + x_m) = \sum_{k=1}^{m} B_k U^m(x_k), \tag{5.1}$$

to be solved for the unknown expressions B_k.

Why such an identity should hold in the first place is a question whose answer is not immediately clear; yet, as we will show, we may determine appropriate B_k's that satisfy (5.1), which will automatically validate our assumption on an ad hoc basis.

Making the substitution $x_k = a_k + n$, where n is fixed, will generate a form that resembles the statement of Theorem 3; accordingly, it is sufficient to establish the theorem for $n = 0$. Note that the substitutions of $a_k + n$ for a_k (for all k) leave $P(k, m)$ unchanged.

Note that (5.1) involves only $U(n)$'s, rather than the more general $G(n)$'s. Because the putative sum is not identically zero, as was the case with (3.1), we cannot expect that the proper cancellations occur to make (5.1) work for $G(n)$'s. In any event, it is easy to come up with counterexamples to Theorem 3 (once proven), if we attempt to change the $U(n)$'s in Theorem 3 to $G(n)$'s .

Expressing the U's in their Binèt form, we obtain the following system of equations from the assumed form of (5.1):

$$\sum_{j=0}^{m}\binom{m}{j}(-1)^j\sum_{k=1}^{m}B_k\alpha^{(m-j)a_k}\beta^{ja_k} = D^{(m-1)/2}\{\alpha^{s_m}-\beta^{s_m}\},$$

where $s_m = a_1 + a_2 + \cdots + a_m$. We further assume that the B_k's are such that the following m+1 identities hold:

$$\sum_{k=1}^{m}B_k\alpha^{ma_k} = D^{(m-1)/2}\alpha^{s_m};\quad \sum_{k=1}^{m}B_k\alpha^{(m-j)a_k}\beta^{ja_k} = 0,\ 0<j<m;$$

$$\sum_{k=1}^{m}B_k\beta^{ma_k} = (-1)^{m-1}D^{(m-1)/2}\beta^{s_m}.$$

There are $m+1$ equations in m unknowns (the B_k's); to overcome this difficulty, we introduce a "dummy" variable, say B_{m+1}, equal to zero. We may then express the foregoing system in matrix form:

$$\begin{pmatrix} \alpha^{ma_1} & \alpha^{ma_2} & \alpha^{ma_3} & \cdots & \alpha^{ma_m} & 1 \\ \alpha^{(m-1)a_1}\beta^{a_1} & \alpha^{(m-1)a_2}\beta^{a_2} & \alpha^{(m-1)a_3}\beta^{a_3} & \cdots & \alpha^{(m-1)a_m}\beta^{a_m} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta^{ma_1} & \beta^{ma_2} & \beta^{ma_3} & \cdots & \beta^{ma_m} & 1 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \\ 0 \end{pmatrix}$$

$$= D^{(m-1)/2}\begin{pmatrix} \alpha^{s_m} \\ 0 \\ \vdots \\ 0 \\ (-1)^{m-1}\beta^{s_m} \end{pmatrix}.$$

The square matrix in the foregoing is seen to be of "modified" Vandermonde type, say $Z = (p_{k:j})$, where $p_{k:j} = \alpha^{ma_j}(z_j)^{k-1}$, and $z_j = (\beta/\alpha)^{a_j}$; here, we set $a_{m+1} = 0$, hence $z_{m+1} = 1$. Thus Z is invertible. In order to solve for the B_k's, we only need to know the first and last columns of Z^{-1}. Suppose $Z^{-1} = (q_{k:j})$; using known results about Vandermonde matrices, we find the following:

$$q_{k:1} = (-1)^m \alpha^{-ma_k} \pi_{m+1}/\{z_k Q(k,m+1)\},\ q_{k:m+1} = \alpha^{-ma_k}/Q(k,m+1),\ k = 1,2,\cdots,m+1,$$

where

$$\pi_{m+1} = z_1 z_2 \cdots z_m z_{m+1} = z_1 z_2 \cdots z_m = \pi_m, \quad \text{and } Q(k,m+1) = \prod_{\substack{k=1 \\ k\neq j}}^{m+1} (z_k - z_j).$$

We may now solve for the B_k's:

$$\begin{aligned} B_k = {} & \alpha^{s_m} D^{(m-1)/2}(-1)^m (\alpha/\beta)^{a_k} \alpha^{-ma_k} \pi_{m+1}/Q(k,m+1) \\ & - \beta^{s_m} D^{(m-1)/2}(-1)^m \alpha^{-ma_k}/Q(k,m+1), \quad k = 1,2,\cdots,m. \end{aligned} \tag{5.2}$$

We also require that $(-1)^m D^{(m-1)/2}\{\alpha^{s_m}\pi_{m+1} - \beta^{s_m}\}/Q(m+1,m+1) = 0$; since, as we see, $\pi_{m+1} = (\beta/\alpha)^{s_m}$, this last condition follows at once. Our next task is to simplify the expressions in (5.2). Now $z_k - z_j = z_j\{(\beta/\alpha)^{a_k - a_j} - 1\} = -\beta^{a_j}\alpha^{-a_k} D^{1/2} U(a_k - a_j)$; therefore, $Q(k,m+1) = (-1)^m D^{m/2} \beta^{s_m - a_k} \alpha^{-ma_k} P(k,m+1)$, where $P(k,m+1)$ is defined by (2.2). Then from (5.2) and our last results, after simplification, $B_k = U(a_k)/P(k,m+1) = 1/P(k,m)$ (since $U(a_k - a_{m+1}) = U(a_k)$). It only remains to show that these solutions are consistent and satisfy our assumptions; this task is left for the reader to verify. This completes the proof of Theorem 3. □

6. COROLLARY OF THEOREM 3

In (2.6), we gave a corollary to Theorem 2, obtained by setting $a_k = k - 1$ (also making some other changes of parameter, and simplifying). Making the same substitutions in Theorem 3, and simplifying the result, we obtain the following corollary of Theorem 3:

$$[m!]_U * U((m+1)n + m(m+1)/2) = \sum_{k=0}^{m} \binom{m}{k}_U (-1)^{k(k-1)/2} U^{m+1}(n+m-k). \tag{6.1}$$

In fact (6.1) is an instance of (5) in [6]. We indicate some special cases of (6.1), obtained by setting $m = 1,2,3,4$, along with a shift of indices:

$$U(2n+1) = U^2(n+1) + U^2(n) \tag{6.2}$$

$$xU(3n) = U^3(n+1) + xU^3(n) - U^3(n-1) \tag{6.3}$$

$$x(x^2+1)U(4n+2) = U^4(n+2) + (x^2+1)U^4(n+1) - (x^2+1)U^4(n) - U^4(n-1) \tag{6.4}$$

$$x(x^2+1)(x^2+2)U(5n) = U^5(n+2) + x(x^2+2)U^5(n+1) - (x^2+1)(x^2+2)U^5(n) - x(x^2+2)U^5(n-1) + U^5(n-2). \tag{6.5}$$

In turn, even more special cases are obtained by setting $x = 1$ in (6.2)-(6.5):

$$F(2n+1) = F^2(n+1) + F^2(n) \tag{6.6}$$

$$F(3n) = F^3(n+1) + F^3(n) - F^3(n-1) \tag{6.7}$$

$$2F(4n+2) = F^4(n+2) + 2F^4(n+1) - 2F^4(n) - F^4(n-1) \tag{6.8}$$

$$6F(5n) = F^5(n+2) + 3F^5(n+1) - 6F^5(n) - 3F^5(n-1) + F^5(n-2). \tag{6.9}$$

The known identities (6.6)-(6.9) have been the impetus for other investigations in this journal. See, for example, [2] and [3].

7. FOUR ADDITIONAL THEOREMS

As we will now show, Theorems 1 and 2 may be further generalized, and are corollaries of the following theorems:

Theorem 4: Given the hypotheses of Theorem 1,

$$\sum_{k=1}^{m} (-1)^{ja_k} U^{m-2j}(n+a_k)/P(k,m) = 0, \tag{7.1}$$
$$j = 1, 2, \cdots, [m/2].$$

Theorem 5: Given the hypotheses of Theorem 1,

$$\sum_{k=1}^{m} (-1)^{ja_k} G^{m-2j}(n+a_k)/P(k,m) = 0, \tag{7.2}$$
$$j = 1, 2, \cdots, [m/2].$$

As we see, Theorem 1(2) would follow from Theorem 4(5) by setting $j = 1$. Theorem 5 follows from Theorem 4 in the same way that Theorem 2 followed from Theorem 1; consequently, we will omit its proof and only indicate the proof of Theorem 4.

Proof of Theorem 4: We begin with the statement of Theorem 3 (see (2.4)). Now the following recurrence relation holds for all n and m, as easily shown:

$$U(n+m) - V(m)U(n) + (-1)^m U(n-m) = 0. \tag{7.3}$$

Therefore, we see that $\sum_{k=1}^{m}\{U^m(1+a_k) - V(m)U^m(a_k) + (-1)^m U^m(1+a_k)\}/P(k,m) = U(mn+m+s_m) - V(m)U(mn+s_m) + (-1)^m U(mn-m+s_m) = 0$, by (7.3). Define

$$H_m(a) = U^m(1+a) + (-1)^m U^m(-1+a). \tag{7.4}$$

It is easily shown that the H_m's satisfy the following recurrence:

$$H_{m+1}(a) = xU(a)H_m(a) + U(a+1)U(a-1)H_{m-1}(a). \tag{7.5}$$

We determine the following initial values: $H_0(a) = 2$, $H_1(a) = xU(a)$, $H_2(a) = (x^2+2)U^2(a) + 2(-1)^a$, etc. By induction or otherwise: $H_m(a) = \sum_{j=0}^{[m/2]} \varphi_{j,m}(-1)^{ja}U^{m-2j}(a)$, where $\varphi_{j,m}$ is a function of j, m (and x), but is independent of a. It is not important at this stage to determine the $\varphi_{j,m}$, except for $j = 0$. Using the recurrence in (7.5), we find that $\varphi_{0,m+1} = x\varphi_{0,m} + \varphi_{0,m-1}$, with the initial values $\varphi_{0,0} = 2$ and $\varphi_{0,1} = x$.

Hence, $\varphi_{0,m} = V(m)$. It follows that $H_m(a) = V(m)U^m(a) + \sum_{j=1}^{[m/2]} \varphi_{j,m}(-1)^{ja}U^{m-2j}(a)$. This implies the following: $\sum_{k=1}^{m}\sum_{j=1}^{[m/2]} \varphi_{j,m}(-1)^{ja_k}U^{m-2j}(a_k)/P(k,m) = 0$. Therefore, $\sum_{j=1}^{[m/2]} \varphi_{j,m}\sum_{k=1}^{m}(-1)^{ja_k}U^{m-2j}(a_k)/P(k,m) = 0$. This last relation must be an identity true for all x; therefore, the inner sums must vanish identically, for $j = 1, 2, \cdots, [m/2]$. Now replacing a_k by $n + a_k$ for $k = 1, 2, \cdots, m$, leaves the $P(k,m)$ invariant, and this is the statement of Theorem 4. □

As we show next, there are two more theorems of an even more general nature than those thus far indicated. Both of these theorems generalize Theorem 3.

Theorem 6: Given the hypotheses of Theorem 1:

$$\sum_{k=1}^{m} U^j(1+a_k)U^{m-j}(a_k)/P(k,m) = U(s_m + j), \tag{7.6}$$
$$j = 0, 1, 2, \cdots, m.$$

Proof of Theorem 6: Let θ_j denote the expression in the left member of (7.6) (treating m as a constant). We note, using Theorem 3, that

$$\theta_0 = U(s_m);\ \theta_m = U(s_m + m). \tag{7.7}$$

We also note the following, from (1.13):

$$U(2+a_k)U(a_k) - U^2(1+a_k) = (-1)^{1+a_k}. \tag{7.8}$$

Now consider the expression $x\theta_j + \theta_{j-1}$, assuming $1 \le j < m$. Using (1.7), this becomes:

$$\sum_{k=1}^{m} U^{j-1}(1+a_k)U^{m-j}(a_k)\{xU(1+a_k) + U(a_k)\}/P(k,m)$$
$$= \sum_{k=1}^{m} U(2+a_k)U^{j-1}(1+a_k)U^{m-j}(a_k)/P(k,m).$$

Using (7.8), we see that $x\theta_j + \theta_{j-1} = \sum_{k=1}^{m} U^{j-1}(1+a_k)U^{m-j-1}(a_k)\{U^2(1+a_k) - (-1)^{a_k}\}/P(k,m) = \sum_{k=1}^{m} U^{j+1}(1+a_k)U^{m-j-1}(a_k)/P(k,m) - \sum_{k=1}^{m}(-1)^{a_k}U^{j-1}(1+a_k)U^{m-j-1}$

$(a_k)/P(k,m) = \theta_{j+1} - 0$, the latter result obtained from (4.1) (with j replaced by $m-1-j$). Therefore, we see that the θ_j's satisfy the following recurrence relation:

$$\theta_{j+1} = x\theta_j + \theta_{j-1}, \quad j = 1, 2, \cdots, m-1. \tag{7.9}$$

Essentially, this recurrence is the same as (1.7). Now using the boundary conditions of (7.7), we conclude that $\theta_j = U(s_m + j)$; note that this is also true for $j = 0$ and $j = m$, which is the statement of Theorem 6. □

Theorem 7: Given the hypotheses of Theorem 1,

$$\sum_{k=1}^{m} G^m(n+a_k)/P(k,m)$$
$$= D^{-1/2}\{\alpha^{mn+s_m}(G(1) - \beta G(0))^m - \beta^{mn+s_m}(G(1) - \alpha G(0))^m\}. \tag{7.10}$$

Proof of Theorem 7: Note that if $G(n) \equiv U(n)$, we obtain: $G(1) - \beta G(0) = G(1) - \alpha G(0) = 1$, and the expression in the right member of (7.10) simplifies to $U(mn + s_m)$, which is Theorem 3. Thus, Theorem 7 is a generalization of Theorem 3.

Let H denote the left member of (7.10). From (1.14), we see that $H = \sum_{k=1}^{m}\{G(n+1)U(a_k) + G(n)U(a_k - 1)\}^m/P(k,m) = \sum_{j=0}^{m}\binom{m}{j}G^{m-j}(n+1)G^j(n)\sum_{k=1}^{m}U^{m-j}(a_k)U^j(a_k - 1)/P(k,m)$. Using Theorem 6 (with j replaced by $m-j$ and the a_k's reduced by 1), we obtain: $H = \sum_{j=0}^{m}\binom{m}{j}G^{m-j}(n+1)G^j(n)U(s_m - j)$. Then by (1.10) and (1.12), $H = D^{-1/2}\sum_{j=0}^{m}\binom{m}{j}G^{m-j}(n+1)G^j(n)\{\alpha^{s_m-j} - \beta^{s_m-j}\} = D^{-1/2}\{\alpha^{s_m}(G(n+1) - \beta G(n))^m - \beta^{s_m}(G(n+1) - \alpha G(n))^m\}$. Now using (1.8), we may easily show that $G(n+1) - \beta G(n) = \alpha^n\{G(1) - \beta G(0)\}$ and $G(n+1) - \alpha G(n) = \beta^n\{G(1) - \alpha G(0)\}$. We then see that H simplifies to the expression indicated in (7.10). □

Corollary of Theorem 7:

$$\sum_{k=1}^{m} V^m(n+a_k)/P(k,m) = D^{(m-1)/2}\{\alpha^{mn+s_m} - (-1)^m\beta^{mn+s_m}\} \tag{7.11}$$
$$= D^{m/2}U(mn+s_m) \text{ if } m \text{ is even, or } D^{(m-1)/2}V(mn+s_m) \text{ if } m \text{ is odd.}$$

Proof: We set $G(n) \equiv V(n)$ in Theorem 7, noting that $V(1) - \beta V(0) = x - 2\beta = D^{1/2}$ and $V(1) - \alpha V(0) = x - 2\alpha = -D^{1/2}$. □

8. CONCLUSION

The two main theorems of this paper are Theorems 5 and 7. Clearly, these theorems are far-reaching and of greater generality than the usual identities involving generalized Fibonacci and Lucas numbers that abound in the literature (involving equidistant points). These two theorems lead to a variety of identities, some of which may not be familiar, even to long-time readers of this journal. For example, if we set $j = [m/2]$ in Theorem 5, we obtain the following special case:

$$\sum_{k=1}^{m}(-1)^{[m/2]a_k}G^{m-2[m/2]}(n+a_k)/P(k,m) = 0. \tag{8.1}$$

If m is even , say $m = 2r$, this becomes: $\sum_{k=1}^{2r}(-1)^{ra_k}/P(k,2r) = 0$; if $a_k = k-1$, this may be further specialized, after some manipulation, to the following:

$$\sum_{k=0}^{2r-1}\binom{2r-1}{k}_U(-1)^{rk+k(k-1)/2} = 0. \tag{8.2}$$

If, in the other hand, we set $m = 2r+1$ in (8.1), we obtain: $\sum_{k=1}^{2r+1}(-1)^{ra_k}G(n+a_k)/P(k,2r+1) = 0$; if $a_k = k-1$, this may be simplified to the following:

$$\sum_{k=0}^{2r}\binom{2r}{k}_U(-1)^{rk+k(k-1)/2}G(n+k) = 0. \tag{8.3}$$

Further research along these lines seems to be indicated.

REFERENCES

[1] H. W. Gould. "The Bracket Function and Fontene-Ward Generalized Binomial Coefficients with Application to Fibonomial Coefficients." *The Fibonacci Quarterly* **7.1** (1969): 23-40, 55.

[2] R. S. Melham. "Some Analogs of the Identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$." *The Fibonacci Quarterly* **37.4** (1999): 305-311.

[3] R. S. Melham. "Families of Identities Involving Sums of Powers of the Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **37.4** (1999): 315-319.

[4] R. S. Melham. "A Three-Variable Identity involving Cubes of Fibonacci Numbers." *The Fibonacci Quarterly* **41.3** (2003): 220-223.

[5] R. S. Melham, proposer of Advanced Problem H-614. *The Fibonacci Quarterly* **42.3** (2004): 283.

[6] R. F. Torretto and J. A. Fuchs. "Generalized Binomial Coefficients." *The Fibonacci Quarterly* **2.4** (1964): 296-302.

AMS Classification Numbers: 11B39, 11B37, 11C20

✠ ✠ ✠