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ON DISTRIBUTIONS OF FIRST PASSAGE TIMES AND OPTIMAL STOPPING OF AR(1) SEQUENCES*

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Abstract. Sufficient conditions for the exponential boundedness of first passage times of autoregressive (AR(1)) sequences are derived in this paper. An identity involving the mean of the first passage time is obtained. Further, this identity is used for finding a logarithmic asymptotic of the mean of the first passage time of Gaussian AR(1)-sequences from a strip. Accuracy of the asymptotic approximation is illustrated by Monte Carlo simulations. A corrected approximation is suggested to improve accuracy of the approximation. An explicit formula is derived for the generating function of the first passage time for the case of AR(1)-sequences generated by an innovation with the exponential distribution. The latter formula is used to study an optimal stopping problem.

Key words. first passage time, autoregressive sequences, martingales, exponential boundedness, optimal stopping

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1. Introduction. The AR(1)-sequence is defined as a solution of the equation

$$X_n = \lambda X_{n-1} + \eta_n, \qquad n = 1, 2, \dots, \quad X_0 = x, \tag{1}$$

where $\{\eta_n\}$ is a sequence of independent identically distributed random variables (*innovation*), and x and λ are nonrandom constants.

Set

$$\tau_a = \inf\{n \geqq 0\colon\ X_n > a\} \quad \text{and} \quad \tau_{a,b} = \inf\{n \geqq 0\colon\ X_n > a \text{ or } X_n < b\},$$

where we always assume $\inf\{\varnothing\} = \infty$.

For the case $\lambda = 1$, that is, when X_n is a random walk, there are many results about properties of the distributions of τ_a and the overshoot $X_{\tau_a} - a$. One may find expositions of the corresponding results (obtained mainly with the help of the Wiener–Hopf factorization technique) in [4], [11], and many other monographs.

The following result about exponential boundedness is known for the stopping time $\tau_{a,b}$ (see [24], [26]): if $\lambda = 1$, then

$$\mathbf{P}\{\eta_1 = 0\} < 1 \implies \tau_{a,b} \in \mathrm{Cr}, \tag{2}$$

where Cr denotes the class of nonnegative random variables with a finite exponential moment:

$$\xi \in \mathrm{Cr} \iff \mathbf{E}e^{\alpha\xi} < \infty \qquad \text{for some } \alpha > 0$$

(the Cramér condition).

Note that if $\lambda = 0$, then, obviously,

$$\mathbf{P}\{b \leqq \eta_1 \leqq a\} < 1 \implies \tau_{a,b} \in \mathrm{Cr}. \tag{3}$$

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Some results about the distribution of τ_a for the case $\lambda \in (0,1)$ were obtained in [14], [15], and [19]. In particular, it was shown in [19] that if

$$\mathbf{E}(\eta_1^-)^\delta < \infty \qquad \text{for some} \quad \delta > 0$$

(where $x^- = \max(-x, 0)$), then for $a > 0$

$$\mathbf{P}\{\eta_1 \leqq a(1-\lambda)\} < 1 \implies \tau_a \in \mathrm{Cr}; \tag{4}$$

furthermore, under the condition

$$\psi(u) := \log \mathbf{E} e^{u\eta_1} < \infty \qquad \text{for all} \quad u \in [0, \infty), \tag{5}$$

the following identity holds:

$$\mathbf{E}_x \tau_a = \frac{1}{|\log \lambda|} \int_0^\infty (\mathbf{E}_x e^{uX_{\tau_a}} - e^{ux})\, e^{-\phi(u)} \frac{du}{u}, \tag{6}$$

where $\phi(u) = \sum_{k=0}^\infty \psi(\lambda^k u)$ and the symbol of expectation $\mathbf{E}_x(\,)$ corresponds to the initial condition $X_0 = x$.

In this paper we present a series of generalizations of the results of [19]; in particular, for the case $|\lambda| < 1$ we present an analogue of the implication (4) for $\tau_{a,b}$ (see Theorem 1 in section 2) and also obtain a formula for the expectation $\tau_{a,b}$ (see Theorem 2 in section 2). As a simple consequence of the latter result we obtain a lower bound and a logarithmic asymptotic for $\mathbf{E}_x(\tau_{a,-a})$ as $a \to \infty$ for the case of Gaussian innovation $\{\eta_n\}$ with zero mean (an upper bound for $\mathbf{E}_x(\tau_{a,-a})$ in the case under consideration was obtained in [10]).

In section 3 we consider the case when the innovation $\{\eta_n\}$ has the exponential distribution $\mathbf{P}\{\eta_n > y\} = e^{-y}$, $y > 0$, and so assumption (5) does not hold. Here, solving a corresponding integral equation, we derive explicit formulas for $\mathbf{E}_x(\theta^{\tau_a})$, $\theta \in (0,1)$, and $\mathbf{E}_x(\tau_a)$ (see Theorem 3; this result was presented without a proof in the lecture notes [18]). Note that in [12], motivated by some applications in physics, an explicit formula for $\mathbf{E}_0(\tau_0)$ was obtained for the case when the innovation η_n has the two-sided exponential distribution $\mathbf{P}\{|\eta_n| > y\} = \frac{1}{2} e^{-y}$, $y > 0$.

Theorem 3 can be used for analysis of some statistical charts used in quality control when observed random variables have the exponential distribution (see some details in [2]). Another application of Theorem 3 is presented in section 4, where an optimal stopping problem is discussed for the case of AR(1)-sequences generated by the exponentially distributed innovation.

In this paper we mainly use the martingale technique to study the distribution of first passage times like τ_a. The same technique was used for Ornstein–Uhlenbeck (O-U) processes (i.e., autoregressive processes with continuous time) in [14], [15], [17], where some analytical approximations for the distribution of first passage time τ_a were derived. A survey of some results for the continuous time case was presented in [18]. Also, we would like to mention the recent papers [7] and [8] which contain more general results for AR(1)-sequences and O-U processes for the case when the innovation has a mixture-exponential distribution.

2. The case $|\lambda| < 1$.

The following result contains analogues of (2), (3), and (4).

THEOREM 1.

(a) *If* $|\lambda| \leqq 1$, $b < 0 < a$, *then*

$$\mathbf{P}\{b(1-\lambda) \leqq \eta_1 \leqq a(1-\lambda)\} < 1 \implies \tau_{a,b} \in \mathrm{Cr}.$$

(b) *If* $\lambda \in (-1,0)$, $a > 0$, *and for some* $\delta > 0$

$$\mathbf{E}|\eta_1|^\delta < \infty, \tag{7}$$

then

$$\mathbf{P}\{\eta_1 + \lambda\eta_2 \leqq a(1-\lambda^2)\} < 1 \implies \tau_a \in \mathrm{Cr}. \tag{8}$$

(c) *If* $\lambda \in (-1,0)$, $a > 0$, *and*

$$\mathbf{E}\log\big(1+|\eta_1|\big) < \infty, \tag{9}$$

then

$$\mathbf{P}\{\eta_1 + \lambda\eta_2 \leqq a(1-\lambda^2)\} < 1 \implies \mathbf{E}_x\tau_a < \infty. \tag{10}$$

Proof. (a) For the case $\lambda \in [-1,0]$ we have for any $n > 1$

$$\begin{aligned}\mathbf{P}\{\tau_{a,b} > n\} &= \mathbf{P}\{\tau_{a,b} > n-1,\, X_{n-1} \in [b,a],\, X_n \in [b,a],\, \eta_n = X_n - \lambda X_{n-1}\}\\ &\leqq \mathbf{P}\Big\{\tau_{a,b} > n-1,\ \eta_n \in \big[b(1-\lambda),\ a(1-\lambda)\big]\Big\}\\ &= \mathbf{P}\{\tau_{a,b} > n-1\}\,\mathbf{P}\{b(1-\lambda) \leqq \eta_n \leqq a(1-\lambda)\},\end{aligned}$$

where the last inequality holds due to independency X_k and η_n for $k < n$. By induction,

$$\mathbf{P}\{\tau_{a,b} > n\} \leqq \Big[\mathbf{P}\{b(1-\lambda) \leqq \eta_n \leqq a(1-\lambda)\}\Big]^n.$$

This implies $\tau_{a,b} \in \mathrm{Cr}$ in view of the assumption $\mathbf{P}\{b(1-\lambda) \leqq \eta_n \leqq a(1-\lambda)\} < 1$.

In the case $\lambda \in (0,1)$ we can use (4). Since $\tau_{a,b} \leqq \tau_a$, by (4),

$$\mathbf{P}\{\eta_1 \leqq a(1-\lambda)\} < 1 \implies \tau_{a,b} \in \mathrm{Cr}.$$

By the same consideration (and due to symmetry in notation) we obtain

$$\mathbf{P}\{\eta_1 \geqq b(1-\lambda)\} < 1 \implies \tau_{a,b} \in \mathrm{Cr}.$$

It remains to note that

$$\mathbf{P}\{b(1-\lambda) \leqq \eta_1 \leqq a(1-\lambda)\} < 1 \iff \mathbf{P}\{b(1-\lambda) \leqq \eta_1\}\,\mathbf{P}\{\eta_1 \leqq a(1-\lambda)\} < 1.$$

The case $\lambda = 1$ is described above in (2).

To prove (b) and (c) we note that (1) implies that

$$X_{2k} = \lambda^2 X_{2(k-1)} + \eta_{2k} + \lambda\eta_{2k-1}, \qquad k = 1,2,\dots. \tag{11}$$

The sequence $\{\eta_{2k} + \lambda\eta_{2k-1}\}$, $k = 1,2,\dots$, is the innovation for X_{2k}, $k = 1,2,\dots$. Since the first passage time

$$\widetilde{\tau}_a = \inf\{k \geqq 0\colon X_{2k} > a\}$$

is not less than $\tau_a/2$, applying (4) for $\tilde{\tau}_a$ we obtain (8). By the same consideration, relation (10) holds in view of Theorem 3 of [19]. The proof is complete.

Further, in this section we assume that $|\lambda| < 1, \lambda \neq 0$, and the following condition holds:

$$\psi(u) := \log \mathbf{E}e^{u\eta_1} < \infty \qquad \text{for all} \quad u \in (-\infty,\infty). \tag{12}$$

This condition implies, of course, the validity of condition (9).

Set

$$(13)\qquad Q(y)=\frac{1}{|\log\lambda^2|}\int_0^\infty\left[(e^{uy}-e^{ux})\,e^{-\phi(u)}+(e^{-uy}-e^{-ux})\,e^{-\phi(-u)}\right]\frac{du}{u},$$

where

$$(14)\qquad \phi(u)=\sum_{k=0}^{\infty}\psi(\lambda^k u)=\psi(u)+\phi(\lambda u),\qquad u\in(-\infty,\infty).$$

THEOREM 2. *Let* $|\lambda|<1$, $\lambda\neq 0$, $b<0<a$; *let condition* (12) *hold; and let*

$$(15)\qquad \mathbf{P}\{b(1-\lambda)\leqq\eta_1\}<1\quad\textit{and}\quad\mathbf{P}\{\eta_1\leqq a(1-\lambda)\}<1.$$

Then

$$\mathbf{E}_x\tau_{a,b}=\mathbf{E}_xQ(X_{\tau_{a,b}})<\infty.$$

Proof. Here we present just a sketch of the proof because it is close to the proof of Theorem 3 in [19].

At first we show that the process $Q(X_n)-n$ is a martingale with respect to the natural filtration $\mathcal{F}_n=\sigma\{X_0,X_1,\dots,X_n\}$. For the case $\lambda\in(0,1)$ this fact is implied by the relation

$$\begin{aligned}Q(X_n)-n&=\frac{1}{2}\left[\frac{1}{|\log\lambda|}\int_0^\infty(e^{uX_n}-e^{ux})\,e^{-\phi(u)}\,\frac{du}{u}-n\right]\\&\quad+\frac{1}{2}\left[\frac{1}{|\log\lambda|}\int_0^\infty(e^{-uX_n}-e^{-ux})\,e^{-\phi(-u)}\,\frac{du}{u}-n\right],\end{aligned}$$

where the processes in the square brackets are martingales (see Proposition 2 and Theorem 2 in [19]).

For the case $\lambda\in(-1,0)$ we note that by using Lemma 1 of [19] one can easily check that the integral in the definition of $Q(y)$ above is finite under conditions (12) and (15).

In view of (1), (12), (14), and by the Fubini theorem

$$\begin{aligned}&\mathbf{E}(Q(X_n)-n\mid\mathcal{F}_{n-1})\\&=\frac{1}{|\log\lambda^2|}\int_0^\infty\Big[(e^{u\lambda X_{n-1}+\psi(u)}-e^{ux})\,e^{-\phi(u)}\\&\qquad\qquad+(e^{-u\lambda X_{n-1}+\psi(-u)}-e^{-ux})\,e^{-\phi(-u)}\Big]\frac{du}{u}-n\\&=\frac{1}{|\log\lambda^2|}\int_0^\infty\left[(e^{u\lambda X_{n-1}}-e^{\lambda ux})\,e^{-\phi(\lambda u)}+(e^{-u\lambda X_{n-1}}-e^{-\lambda ux})\,e^{-\phi(-\lambda u)}\right]\frac{du}{u}-n\\&\quad+\int_0^\infty\frac{1}{|\log\lambda^2|}\left[e^{u\lambda x-\phi(\lambda u)}+e^{-u\lambda x-\phi(-\lambda u)}-e^{ux-\phi(u)}-e^{-ux-\phi(-u)}\right]\frac{du}{u}\\&=Q(X_{n-1})-n\\&\quad+\int_0^\infty\frac{1}{|\log\lambda^2|}\left[e^{u\lambda x-\phi(\lambda u)}+e^{-u\lambda x-\phi(-\lambda u)}-e^{ux-\phi(u)}-e^{-ux-\phi(-u)}\right]\frac{du}{u}.\end{aligned}$$

The last term is equal to 1 since it can be written in the form of the well-known Frullani integral (see [9, section 12.16]). This leads to the fact that the sequence $Q(X_n) - n$ is a martingale and, hence, by the martingale stopping theorem

$$\mathbf{E}_x\tau_{a,b} = \lim_{n\to\infty} \mathbf{E}_x Q(X_{\min(\tau_{a,b},n)}).$$

The final step of the proof consists of a verification of the equality

$$\lim_{n\to\infty} \mathbf{E}_x Q(X_{\min(\tau_{a,b},n)}) = \mathbf{E}_x Q(X_{\tau_{a,b}})$$

and it can be done analogously to a similar step in [19].

COROLLARY 1. *Let* $|\lambda| < 1$, $\lambda \neq 0$, $a > 0$, $\eta_1 \overset{d}{=} -\eta_1$ (*i.e., by distribution*), *and*

$$\mathbf{P}\{\eta_1 \leqq a(1-\lambda)\} < 1.$$

Then

$$\mathbf{E}_x\tau_{a,-a} = \frac{1}{|\log|\lambda||}\int_0^\infty \left(\mathbf{E}_x \cosh(uX_{\tau_{a,-a}}) - \cosh(ux)\right) e^{-\phi(u)}\,\frac{du}{u} \tag{16}$$

$$\geqq \frac{1}{|\log|\lambda||}\int_0^\infty \left(\cosh(ua) - \cosh(ux)\right) e^{-\phi(u)}\,\frac{du}{u}. \tag{17}$$

Proof. To verify (16) one needs only to note that $\phi(u) = \phi(-u)$ and then make simplifications in (13). Inequality (17) is due to the lower bound $\cosh(uX_{\tau_{a,-a}}) \geqq \cosh(ua)$.

Consider now the case of the Gaussian innovation $\eta_n \sim N(0,\varepsilon^2)$. For this case by direct calculations one can easily obtain that

$$\phi(u) = \frac{u^2\varepsilon^2}{2(1-\lambda^2)}.$$

The problem of finding an asymptotic expansion for $\mathbf{E}_x\tau_{a,-a}$ as $\varepsilon \to 0$ for the case of the Gaussian innovation was studied in [10], where, in particular, it was shown that for $|x| < 1$

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{E}_x\tau_{1,-1} \leqq \frac{1-\lambda^2}{2} \tag{18}$$

(see Theorem 2 in the corrected version of [10] and the abstract of the talk [21]). With the help of Corollary 1 we now obtain the following result.

COROLLARY 2. *Let* $|\lambda| < 1$, $\lambda \neq 0$, $|x| < 1$, $\eta_1 \sim N(0,\varepsilon^2)$. *Then*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{E}_x\tau_{1,-1} = \frac{1-\lambda^2}{2}. \tag{19}$$

Proof. By Corollary 1

$$\mathbf{E}_x\tau_{1,-1} \geqq \frac{1}{|\log|\lambda||}\int_0^\infty \left(\cosh u - \cosh(ux)\right) \exp\left\{-\frac{u^2\varepsilon^2}{2(1-\lambda^2)}\right\}\frac{du}{u} \tag{20}$$

$$= \frac{1}{|\log|\lambda||}\left(F\left(\frac{\sqrt{1-\lambda^2}}{\varepsilon}\right) - F\left(\frac{\sqrt{1-\lambda^2}}{\varepsilon}\,x\right)\right), \tag{21}$$

where (e.g., with help of the package Mathematica)

$$F(x) = \int_0^\infty \left(\cosh(ux) - 1\right) e^{-u^2/2} \frac{du}{u} = \int_0^x \int_0^\infty \sinh(uz)\, e^{-u^2/2}\, du\, dz$$
$$= \sqrt{\frac{\pi}{2}} \int_0^x e^{z^2/2} \operatorname{Erf}\left[\frac{z}{\sqrt{2}}\right] dz.$$

Since $\operatorname{Erf}[z/\sqrt{2}] \to 1$ as $z \to \infty$, (e.g., with help of the L'Hospital rule)

$$F(x) \sim \sqrt{\frac{\pi}{2}} \int_0^x e^{z^2/2}\, dz \sim \sqrt{\frac{\pi}{2x}}\, e^{x^2/2} \qquad \text{as} \quad x \to \infty.$$

The latter relation, together with (20) and (18), implies (19). Corollary 2 is proved.

Using the self-similarity property of the Gaussian distribution one can easily show that $\mathbf{E}_x(\tau_{1,-1})$ for the case $\varepsilon = 1/a$ is equal to $\mathbf{E}_{xa}(\tau_{a,-a})$ for the case $\varepsilon = 1$.

Tables 1 and 2 contain the results of Monte Carlo simulations for $\log(\mathbf{E}_0(\tau_{a,-a}))$ and the corresponding logarithms of the lower bound (20) for $\lambda = 0.75$ and 0.9 for the case of the Gaussian innovation $\eta_i \sim N(0,1)$.

For comparison we also included logarithms $F((a+0.5826)\sqrt{1-\lambda^2})/|\log|\lambda||$ (we call this the "corrected approximation" for $\mathbf{E}_0\tau_{a,-a}$). This approximation is based on the assumption that when a is large and λ is close to 1, the expectation of the overshoot of AR(1)-sequence X_n is close to the well-known constant $C = 0.5826\ldots$ (this is the limit of the expectation of the overshoot $(X_{\tau_a} - a)$ as $a \to \infty$ for the Gaussian random walk $\sum_{k=1}^n \eta_k$; see, e.g., [23]).

TABLE 1. $\lambda = 0.75$, $(1-\lambda^2)/2 = 0.219$.

a	4	5	6	7	8
$\log(\text{lowbound})/a^2$	0.257	0.234	0.224	0.219	0.217
$\log(\text{corrected appr.})/a^2$	0.317	0.283	0.265	0.255	0.248
$\log(\text{simulation})/a^2$	0.325	0.288	0.268	0.258	0.249

TABLE 2. $\lambda = 0.9$, $(1-\lambda^2)/2 = 0.095$.

a	4	6	8	10
$\log(\text{lowbound})/a^2$	0.203	0.140	0.116	0.106
$\log(\text{corrected appr.})/a^2$	0.234	0.157	0.129	0.116
$\log(\text{simulation})/a^2$	0.236	0.158	0.130	0.117

Tables 1 and 2 demonstrate that the "corrected approximation" is a better approximation for the results of simulation compared with the lower bound and, of course, much better compared with the limiting value $(1-\lambda^2)/2$.

3. The case of exponentially distributed innovation. We use here standard notation of the theory of q-series (see [1]) and, in particular, the q-Pochhammer symbol

$$\prod_{j=1}^m (1 - \rho\lambda^{j-1}) =: (\rho;\lambda)_m, \qquad (\rho;\lambda)_0 = 1.$$

Using this notation we have, in particular,

$$(\lambda;\lambda)_m = \prod_{j=1}^m (1 - \lambda^j).$$

Set

$$N_\rho(x) = \sum_{k=0}^{\infty} \frac{(\lambda x)^k (\rho;\lambda)_k}{k!}, \qquad 0<\rho<1, \tag{22}$$

$$H(x) = \sum_{k=1}^{\infty} \frac{x^k(\lambda;\lambda)_{k-1}}{k!}.$$

THEOREM 3. *Let* $0<\lambda<1$, $x\in[0,a]$, $a>0$, $\mathbf{P}\{\eta_1>y\}=e^{-y}$, $y>0$. *Then*
(a) *for any* $\rho\in(0,1)$

$$\mathbf{E}_x\rho^{\tau_a} = \frac{\rho N_\rho(x)}{N_\rho(a/\lambda)}; \tag{23}$$

(b) $\mathbf{E}_x\tau_a = H(a) - H(\lambda x) + 1$.
Proof. Set $\phi_\rho(x) = \mathbf{E}_x(\rho^{\tau_a})$ and note that

$$\phi_\rho(x) = \mathbf{P}_x\{X_1>a\}\,\rho + \mathbf{E}_x\big(I\{X_1\leqq a\}\,\rho^{\tau_a}\big),$$

where by a Markov property of X_n

$$\mathbf{E}_x\big(I\{X_1\leqq a\}\rho^{\tau_a}\big) = \mathbf{E}_x\big[I\{X_1\leqq a\}(E\rho^{\tau_a}\,|\,X_1)\big] = \rho\mathbf{E}_x\big(I\{X_1\leqq a\}\phi(X_1)\big).$$

Hence, $\phi_\rho(x)$ is a solution of the following integral equation: for $x\leqq a$

$$\phi_\rho(x) = \rho\mathbf{P}_x\{X_1>a\} + \rho\mathbf{E}_x\big(I\{X_1\leqq a\}\,\phi_\rho(X_1)\big).$$

Since $\mathbf{P}_x\{X_1>y\} = e^{\lambda x-y}$ for $y>\lambda x$, the latter equation can be written in the following form:

$$e^{-\lambda x}\phi_\rho(x) = \rho e^{-a} + \rho\int_{\lambda x}^{a}\phi_\rho(y)\,e^{-y}\,dy. \tag{24}$$

Differentiating both parts with respect to x for $x\in(0,a)$ we obtain

$$\frac{d}{dx}\big(e^{-\lambda x}\phi_\rho(x)\big) = -\lambda\rho\phi(\lambda x)\,e^{-\lambda x} \tag{25}$$

or, equivalently,

$$\phi_\rho'(x) = \lambda\phi_\rho(x) - \lambda\rho\phi_\rho(\lambda x). \tag{26}$$

We solve this equation by the power series method setting $\phi_\rho(x) = C\sum_{k=0}^{\infty} c_k x^k$, $c_0=1$, where C is a positive constant. After substituting this series into (26), we obtain

$$\sum_{k=0}^{\infty} c_{k+1}(k+1)\,x^k = \sum_{k=0}^{\infty} c_k\lambda(1-\rho\lambda^k)\,x^k.$$

Equating the coefficients in the left and right sides, we obtain $c_{k+1}(k+1) = c_k\lambda(1-\rho\lambda^k)$. This implies $c_k = \lambda^k(\rho;\lambda)_k/k!$ and, hence, according to our notation,

$$\phi_\rho(x) = CN_\rho(x).$$

To find the constant C, at first we note that in view of (24) with $x=0$ we have

$$C = \rho e^{-a} + \rho C \int_0^a N_\rho(y)\, e^{-y} dy. \tag{27}$$

To find $\int_0^a N_\rho(y)\, e^{-y} dy$ in an explicit form, note that the function $N_\rho(x)$ is an analytical function for all $|x| < \infty$ and satisfies (25), that is,

$$\frac{d}{dx}\left(e^{-\lambda x} N_\rho(x)\right) = -\lambda\rho\, N_\rho(\lambda x)\, e^{-\lambda x}.$$

Integrating both parts of this equation on the interval $x \in (x_1, x_2)$ we obtain

$$\rho \int_{\lambda x_1}^{\lambda x_2} N_\rho(y)\, e^{-y} dy = e^{-\lambda x_1} N_\rho(x_1) - e^{-\lambda x_2} N_\rho(x_2). \tag{28}$$

In particular, for $x_1 = 0$ and $x_2 = a/\lambda$ we get $\rho \int_0^a N_\rho(y)\, e^{-y} dy = 1 - e^{-a} N_\rho(a/\lambda)$. This and (27) imply $C = \rho / N_\rho(a/\lambda)$. Hence, (23) is proved.

In order to prove the second part of Theorem 3, consider the asymptotic expansion for $\mathbf{E}_x(\rho^{\tau_a})$ as $\rho \to 1$. Note that $(1 - \mathbf{E}_x(\rho^{\tau_a}))/(1-\rho) \to \mathbf{E}_x(\tau_a)$. On the other hand, note that $(\rho;\lambda)_k/(1-\rho) \to (\lambda;\lambda)_{k-1}$ and, hence,

$$N_\rho(x) = 1 + (1-\rho)\left(\lambda x + \sum_{k=2}^{\infty} \frac{(\lambda x)^k (\lambda;\lambda)_{k-1}}{k!}\right) + o(1-\rho).$$

This implies

$$\begin{aligned} \frac{1 - \mathbf{E}_x \rho^{\tau_a}}{1-\rho} &= \frac{\sum_{k=0}^{\infty} a^k (\rho;\lambda)_k / k! - \rho \sum_{k=0}^{\infty} (x\lambda)^k (\rho;\lambda)_k / k!}{(1-\rho) \sum_{k=0}^{\infty} a^k (\rho,\lambda)_k / k!} \\ &= \frac{1 + \sum_{k=1}^{\infty} (a^k (\lambda;\lambda)_{k-1}/k! - (x\lambda)^k (\lambda;\lambda)_{k-1}/k!) + o(1-\rho)}{1 + o(1)} \\ &\longrightarrow H(a) - H(\lambda x) + 1 = \mathbf{E}_x \tau_a. \end{aligned}$$

Theorem 3 is proved.

Remark 1. In [8] Jacobsen obtained closed-form representations for the joint Laplace transform of τ_a and the overshoot $X_{\tau_a} - a$ in the case when the innovation of AR(1)-sequences has a mixture-exponential distribution. For the particular case of innovation with exponential distribution, the representation for the joint Laplace transformation for τ_a and the overshoot $X_{\tau_a} - a$ obtained in [8] implies two important facts:

$$X_{\tau_a} - a \overset{d}{=} \eta_1; \qquad X_{\tau_a} \text{ and } \tau_a \text{ are independent.} \tag{29}$$

For the case of random walks (i.e., $\lambda = 1$) these results are known and can be found, e.g., in [4].

4. An optimal stopping problem for AR(1)-sequences. Here we consider the problem of finding the value function

$$V_{\mathcal{M}}(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x\big[\rho^\tau g(X_\tau)\, I\{\tau < \infty\}\big],$$

where $\rho \in (0,1)$, $g(x)$ is a nonnegative reward function, $\mathcal{M}$ is a class of stopping times with respect to the natural filtration $\mathcal{F}_n = \sigma\{X_0, X_1, \dots, X_n\}$, and X_n is an

AR(1)-sequence. Typically, $\mathcal{M}$ stands for $\overline{\mathcal{M}}$, that is, the class of all stopping times or for the class

$$\mathcal{M}_T = \{\tau \in \overline{\mathcal{M}} \colon \tau \leqq T\}.$$

The exposition of the general theory of optimal stopping problems with a number of examples is presented in [6]; see also [22], [5], [20]. In particular, it is known that $V_{\overline{\mathcal{M}}}(x)$ is a solution of the Wald–Bellman equation which for the case under consideration has the following form:

$$V_{\overline{\mathcal{M}}}(x) = \max\left\{g(x), \rho \mathbf{E}_x V_{\overline{\mathcal{M}}}(\lambda x + \eta_1)\right\}. \tag{30}$$

Further we consider the case of innovations $\{\eta_n\}$ with $\mathbf{P}\{\eta_1 > y\} = e^{-y}$, $y > 0$.

By Fatou's lemma

$$\begin{aligned} V_{\overline{\mathcal{M}}}(x) &\leqq \sup_{\tau\in\overline{\mathcal{M}}} \sup_{a>0} \mathbf{E}_x \left[\rho^{\min(\tau_a,\tau)} N_\rho(X_{\min(\tau_a,\tau)}) \frac{g(X_{\min(\tau_a,\tau)})}{N_\rho(X_{\min(\tau_a,\tau)})}\right] \\ &\leqq \sup_{\tau\in\overline{\mathcal{M}}} \sup_{a>0} \mathbf{E}_x \left[\rho^{\min(\tau_a,\tau)} N_\rho(X_{\min(\tau_a,\tau)})\right] A, \end{aligned}$$

where $A = \sup_{y>0}(g(y)/N_\rho(y))$. Note that setting $x_1 = x$, $x_2 = \infty$ in (28) and omitting the last term, we obtain the following inequality:

$$\rho \int_{\lambda x}^{\infty} N_\rho(y)\, e^{-y} dy \leqq e^{-\lambda x} N_\rho(x),$$

which is equivalent to the inequality $\rho \mathbf{E}_x N_\rho(X_1) \leqq N_\rho(x)$. The latter implies, obviously, the supermartingale property of the process $\rho^n N_\rho(X_n)$ and, thus, for any stopping time $\tau \in \overline{\mathcal{M}}$

$$\mathbf{E}_x\left[\rho^{\min(\tau_a,\tau)} N_\rho(X_{\min(\tau_a,\tau)})\right] \leqq N_\rho(x).$$

Using the inequalities $(\rho;\lambda)_\infty \exp\{\lambda y\} \leqq N_\rho(y) \leqq \exp\{\lambda y\}$ (these inequalities are easily derived from (22)), one can see now that the constant A is finite when the reward function $g(x)$ is continuous and satisfies the condition

$$\limsup_{x\to\infty} g(x) \exp\{-\lambda x\} < \infty. \tag{31}$$

To obtain a lower bound for $V_{\overline{\mathcal{M}}}(x)$, note

$$V_{\overline{\mathcal{M}}}(x) \geqq \sup_{a>0} \mathbf{E}_x\left[\rho^{\tau_a} g(X_{\tau_a})\right], \tag{32}$$

where, by using Theorem 3 and Remark 1 (see (29)), we have

$$\mathbf{E}_x\left[\rho^{\tau_a} g(X_{\tau_a})\right] = \mathbf{E}_x \rho^{\tau_a} \mathbf{E}_x g(X_{\tau_a}) = \rho N_\rho(x) \frac{\mathbf{E} g(a+\eta_1)}{N_\rho(a/\lambda)}. \tag{33}$$

Since $V_{\overline{\mathcal{M}}}(x) \geqq g(x)$, now (32) and (33) imply the following lower bound:

$$V_{\overline{\mathcal{M}}}(x) \geqq G(x) := \max\left\{g(x),\ \rho N_\rho(x) \sup_{a>0} \frac{\mathbf{E} g(a+\eta_1)}{N_\rho(a/\lambda)}\right\}. \tag{34}$$

The latter result motivates the conjecture that the optimal stopping time in the class $\overline{\mathcal{M}}$ (maybe under some additional limitations on $g(x)$) is the first passage time over the level a^* which is a maximum point of the function $B(a) = \mathbf{E}g(a+\eta_1)/N_\rho(a/\lambda)$. One of the possible ways to verify this conjecture consists of showing that the function $G(x)$ from the lower bound (34) is a unique solution (in a proper class of functions) of the Wald–Bellman equation (30). Another way consists of verifying that the lower bound (34) is also the upper bound for the value function (this approach is used, e.g., in [20] for random walks). Note that some results about the optimality of threshold-type stopping times in the class $\overline{\mathcal{M}}$ are known, for example, for generalized Ornstein–Uhlenbeck processes without positive jumps; see [3] (where the results of [13] and [16] were used).

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