Strong Predictor-Corrector Euler Methods for Stochastic Differential Equations

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Dedicated to the 70th Birthday of Ludwig Arnold.

Abstract. This paper introduces a new class of numerical schemes for the pathwise approximation of solutions of stochastic differential equations (SDEs). The proposed family of strong predictor-corrector Euler methods are designed to handle scenario simulation of solutions of SDEs. It has the potential to overcome some of the numerical instabilities that are often experienced when using the explicit Euler method. This is of importance, for instance, in finance where martingale dynamics arise for solutions of SDEs with multiplicative diffusion coefficients. Numerical experiments demonstrate the improved asymptotic stability properties of the proposed symmetric predictor-corrector Euler methods.

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1 Introduction

During recent years simulation methods for the approximate solution of stochastic differential equations (SDEs) have become indispensable tools in many areas of applications, in particular, in finance. Monographs describing such methods include, for instance, Kloeden & Platen (1999), Milstein (1995), Kloeden, Platen & Schurz (2003), Jäckel (2002) and Glasserman (2004). A major problem is the control of the propagation of errors during simulation. If the impact of naturally arising errors is not dampened over time, then the approximate path simulated may diverge substantially from the exact solution. An illustration of such a situation is provided by the Black-Scholes Itô SDE

$$dX_t = X_t \, \sigma \, dW_t \tag{1.1}$$

for $t \geq 0$ starting at $X_0 > 0$. Here $\sigma > 0$ denotes the volatility of the asset price process $X = \{X_t, t \geq 0\}$, which is in our case chosen to be a martingale, as is typical in finance when using an appropriate numeraire and pricing measure. The SDE (1.1) is driven by a standard Wiener process $W = \{W_t, t \geq 0\}$ and has a multiplicative diffusion coefficient.

For certain time step sizes one observes that strong explicit methods, in particular, the widely used Euler method, work unreliably and sometimes generate large errors. This phenomenon is visualized in Figure 1.1, where we generate the exact

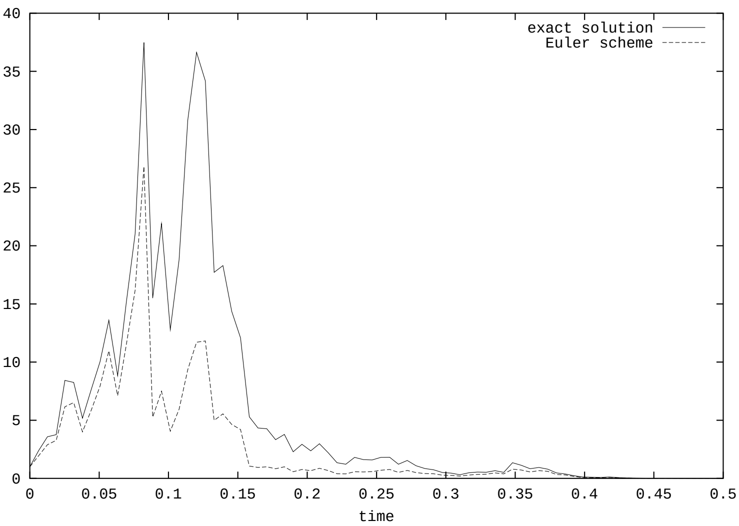

Figure 1.1: Exact solution and approximate solution by Euler scheme.

path of a solution of the SDE (1.1) for $\sigma = 5$ and $X_0 = 1$, and compare it with the one generated by the corresponding Euler scheme

$$Y_{n+1} = Y_n + Y_n \, \sigma \, \Delta W_n$$

for $n \in \{0, 1, \ldots\}$, with $Y_0 = X_0$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ and time step size $\Delta = 0.00625$.

As in the deterministic case, implicit methods can sometimes be used to control the propagation of errors. We may refer for this approach, for instance, to papers by Talay (1982), Klauder & Petersen (1985), Milstein (1988), Hernandez & Spigler (1992, 1993), Saito & Mitsui (1993a, 1993b), Kloeden & Platen (1992, 1999), Milstein, Platen & Schurz (1998), Higham (2000) and Alcock & Burrage (2006). For general SDEs it is not straightforward to introduce implicitness into the approximation of the diffusion terms since *ad hoc* attempts lead typically to terms involving the inverse of Gaussian random variables, which can lead to explosions and, thus, makes numerically no sense. In some applications, as given by the SDE (1.1), there is no point in making some drift terms in a scheme implicit since it would not change anything. For these reasons, balanced implicit methods have been developed in Milstein, Platen & Schurz (1998) and Alcock & Burrage (2006). These methods introduce some implicitness into the approximation of the diffusion term, which can significantly improve the numerical stability of the scheme. However, the disadvantage of implicit methods is that, in general, an algebraic equation has to be solved at each time step. This may cost significant computational time and is sometimes not reliably accomplished.

It has been known for the case of deterministic ordinary differential equations that predictor-corrector methods can achieve improved numerical stability when compared with standard explicit methods, see Hairer, Nørsett & Wanner (1987). One first calculates for a time step a predictor using a standard explicit method. This predicted value is then improved in a corrector step, which mimics an implicit method but uses the predicted value instead of the unknown new value. This method does not require to solve an algebraic equation in each time step, but may inherit some of the numerical stability properties of the corresponding implicit scheme. In Platen (1995) and Kloeden & Platen (1999), predictor-corrector methods have been proposed as weak discrete time approximations. These methods can be used in Monte Carlo simulation. Predictor-corrector methods for Stratonovich SDEs have been proposed in Burrage & Tian (2002) along the lines of Runge-Kutta methods for deterministic ordinary differential equations.

The aim of this paper is to introduce a family of strong predictor-corrector methods and to provide a picture of the numerical stability properties of these schemes. It will introduce and investigate systematically the proposed family of strong predictor-corrector Euler methods, which are shown to converge with strong order 0.5. These schemes are designed for scenario simulation and do not require to solve at each time step an algebraic equation numerically. Their numerical stability will be examined under the concept of asymptotic stability using a family of linear test equations with multiplicative noise.

2 Strong Predictor-Corrector Euler Schemes

Within this section we propose a family of strong predictor-corrector Euler schemes. First, let us consider the solution $X = \{X_t = (X_t^1, \ldots, X_t^d)^\top, t \geq 0\}$ of the d-dimensional SDE

$$X_t = X_0 + \int_0^t a(s, X_s)\, ds + \sum_{j=1}^m \int_0^t b^j(s, X_s)\, dW_s^j \tag{2.1}$$

for $t \geq 0$. Here $X_0 \in \Re^d$ denotes the deterministic initial value. The function $a : [0, \infty) \times \Re^d \to \Re^d$ is the d-dimensional drift function with $a^k(\cdot, \cdot)$ as its kth component. The function $b^j : [0, \infty) \times \Re^d \to \Re^d$ denotes the d-dimensional diffusion coefficient function with respect to the jth Wiener process $W^j = \{W_t^j, t \geq 0\}$, $j \in \{1, 2, \ldots, m\}$, where $b^{k,j}(\cdot, \cdot)$ is its kth component. Additionally, we will need the function $\bar{a}_\eta$ for $\eta \in [0, 1]$ with kth component

$$\bar{a}_\eta^k(t, x) = a^k(t, x) - \eta_k \sum_{j_1, j_2 = 1}^m \sum_{i=1}^d b^{i, j_1}(t, x) \frac{\partial b^{k, j_2}(t, x)}{\partial x^i}, \tag{2.2}$$

for $(t, x) \in [0, \infty) \times \Re^d$.

To ensure existence and uniqueness of the solution of the SDE (2.1) on a finite interval $[0, T]$, $T < \infty$, and convergence of the schemes, we propose the Lipschitz condition

$$|a(t, x) - a(t, y)| + |\bar{a}_\eta(t, x) - \bar{a}_\eta(t, y)| + \sum_{j=1}^m |b^j(t, x) - b^j(t, y)| \leq K\, |x - y|, \tag{2.3}$$

the linear growth condition

$$|a(t, x)| + |\bar{a}_\eta(t, x)| + \sum_{j=1}^m |b^j(t, x)| \leq K\, (1 + |x|) \tag{2.4}$$

and the continuity condition

$$|a(s, x) - a(t, x)| + |\bar{a}_\eta(s, x) - \bar{a}_\eta(t, x)| + \sum_{j=1}^m |b^j(s, x) - b^j(t, x)| \leq K\, (1 + |x|)\, |s - t|^{\frac{1}{2}} \tag{2.5}$$

for all $s, t \in [0, T]$, $\eta \in [0, 1]$ and $x, y \in \Re^d$ with some finite constant $K < \infty$.

The following strong predictor-corrector schemes can potentially provide some improved numerical stability, while avoiding to solve an algebraic equation in each time step, as is required by implicit methods. At the nth time step, first the so-called *predictor* is constructed by using an explicit Euler scheme which predicts a value $\bar{Y}_{n+1}$. Second, the so-called, *corrector* scheme is used, which is in its structure similar to an implicit Euler scheme and corrects the predicted value. Not only is the predictor step explicit, but the corrector step is also explicit since it uses the predicted value. With this two-step predictor-corrector procedure one can introduce some stabilizing effect not only in the approximation of the drift term but also in the approximation of the diffusion term. The gain in numerical stability that will be achievable via such predictor-corrector method will be modest, since the scheme will still remain an explicit scheme. Nevertheless, such predictor-corrector methods are of practical interest as alternatives to implicit methods. The strong predictor-corrector Euler methods that we are going to introduce appear to be more straightforward in their design than balanced implicit methods. The latter require skillful selection of some functions to construct the balancing terms.

Let us consider a time discretization $0 = t_0 < t_1 < \ldots$ with maximum time step size $\Delta \geq \Delta_n = t_{n+1} - t_n$, $n \in \{0, 1, \ldots\}$. The corresponding increments of the jth driving Wiener process are denoted by $\Delta W_n^j = W_{t_{n+1}}^j - W_{t_n}^j$, $n \in \{0, 1, \ldots\}$. We denote by $Y_n = Y_{t_n}$ the value of a discrete time approximation at time t_n, and by $n_t = \max\{n \in \{0, 1, \ldots\} : t_n \geq t\}$ the largest integer n for which t_n does not exceed $t \geq 0$.

In the one-dimensional case, $d = m = 1$, the *family of strong predictor-corrector Euler schemes*, we are proposing, is given by the *corrector*

$$\begin{aligned} Y_{n+1} &= Y_n + \left\{\theta\, \bar{a}_\eta(t_{n+1}, \bar{Y}_{n+1}) + (1-\theta)\bar{a}_\eta(t_n, Y_n)\right\} \Delta_n \\ &\quad + \left\{\eta\, b^1(t_{n+1}, \bar{Y}_{n+1}) + (1-\eta)\, b^1(t_n, Y_n)\right\} \Delta W_n^1, \end{aligned} \tag{2.6}$$

and by the *predictor*

$$\bar{Y}_{n+1} = Y_n + a(t_n, Y_n)\, \Delta_n + b^1(t_n, Y_n)\, \Delta W_n^1. \tag{2.7}$$

Here we need to use the corrected drift function $\bar{a}_\eta = a - \eta\, b\, b'$, where b' denotes the first spatial derivative of b. We call the parameters $\theta, \eta \in [0, 1]$ the degree of implicitness in the drift and the diffusion coefficients, respectively. For the case $\eta = \theta = 0$ we recover the well-known Euler scheme. We remark that with the choice of $\eta > 0$ one may obtain a scheme with some type of stabilizing effect in the approximation of the diffusion term. A major advantage of the above family of schemes is that they can be implemented with flexible parameters η and θ. This allows comparing simulated trajectories for different degrees of implicitness. If these trajectories diverge significantly from each other over time, then some numerical stability problem is likely to be present and one has to focus on those degrees of implicitness that provide numerical stability.

For the general multi-dimensional case, the kth component of the proposed family of strong predictor-corrector Euler schemes, is given by the corrector

$$\begin{aligned} Y_{n+1}^k &= Y_n^k + \left\{\theta_k\, \bar{a}_\eta^k(t_{n+1}, \bar{Y}_{n+1}) + (1-\theta_k)\, \bar{a}_\eta^k(t_n, Y_n)\right\} \Delta_n \\ &\quad + \sum_{j=1}^{m} \left\{\eta_k\, b^{k,j}(t_{n+1}, \bar{Y}_{n+1}) + (1-\eta_k)\, b^{k,j}(t_n, Y_n)\right\} \Delta W_n^j \end{aligned} \tag{2.8}$$

for $\theta_k, \eta_k \in [0, 1]$, $k \in \{1, 2, \ldots, d\}$, and the predictor

$$\bar{Y}^k_{n+1} = Y^k_n + a^k(t_n, Y_n)\,\Delta_n + \sum_{j=1}^{m} b^{k,j}(t_n, Y_n)\,\Delta W^j_n, \tag{2.9}$$

where $\bar{a}^k_\eta$ is given in (2.2). Note that one can, in practice, also experiment with degrees of implicitness outside the interval $[0, 1]$, which we do not consider for simplicity.

Now, let us identify the order of strong convergence, in the sense of Kloeden & Platen (1999), for the above described new family of schemes.

Theorem 2.1 *The above strong predictor-corrector Euler methods converge with strong order 0.5, that is,*

$$E\left(|X_T - Y_{n_T}|\right) \leq K\,\Delta^{\frac{1}{2}} \tag{2.10}$$

for a constant K that is not depending on the maximum time step size Δ.

The proof of this result is given in the Appendix.

3 Numerical Stability

Roundoff and truncation errors arise naturally during most simulations on a digital computer. It is essential to understand the propagation of such errors in a scenario simulation. In practice, the ability of a numerical method to control the propagation of errors plays an important role and decides about its practical applicability. It is crucial to recognize that the numerical stability of a scheme must have priority over a potentially higher order of convergence.

In some sense, the stability of a numerical scheme refers to the conditions under which the impact of an error vanishes asymptotically over time. More precisely, stability of a numerical scheme refers to the property that the numerical approximate solution tends to zero with the true solution. In general, it is difficult to quantify the notion of numerical stability. However, various concepts of numerical stability for schemes approximating the solution of an SDE have been developed in the literature. These often use specially designed test equations, see, for instance, Kloeden & Platen (1999). Under such concept one can systematically analyze the stability properties of a given numerical scheme for the given family of test equations.

Typically, regions of stability are identified, describing the time step sizes for the given scheme where the propagation of errors is under control in a well-defined sense. In some cases authors use solutions of complex valued linear SDEs with additive noise as test dynamics, see Milstein (1995), Kloeden & Platen (1999) and Hernandez & Spigler (1992, 1993). However, for applications in finance this is sometimes not sufficient, for instance, when studying the stability of numerical schemes for the dynamics of asset prices that have no drift and some level dependent diffusion coefficient. For such problems test SDEs with multiplicative noise have been suggested in real valued and complex valued form, see Saito & Mitsui (1993a, 1993b, 1996), Hofmann & Platen (1994, 1996) and Higham (2000).

The aim of this paper is to use a concept of numerical stability that provides stability regions that are rather wide and easy to interpret. In particular, this concept makes sense for the study of the numerical stability of strong schemes when simulating typical dynamics of asset prices in finance.

Similar to Hofmann & Platen (1994) we use in this paper a *linear test SDE* with *multiplicative noise*. Its explicit solution is of the form

$$X_t = X_0 \exp\left\{(1-\alpha)\,\lambda\, t + \sqrt{\alpha\,|\lambda|}\, W_t\right\} \tag{3.1}$$

for $t \geq 0$ and $\alpha, \lambda \in \Re$. It follows from (3.1) by the law of the iterated logarithm and the law of large numbers that

$$P\left(\lim_{t\to\infty} X_t = 0\right) = 1 \tag{3.2}$$

if and only if $(1-\alpha)\lambda < 0$, see Protter (2004). This means, that in the case $\lambda < 0$ and $\alpha \in [0,1]$ perturbations of the initial value X_0 of the solution X_t have negligible impact in the long term. There is not much point in trying to identify numerically stable schemes for unstable test dynamics. Therefore, we will consider the family of test dynamics given by (3.1) for negative parameter $\lambda < 0$ and $\alpha \in [0,1]$. The process $X = \{X_t, t \geq 0\}$ satisfies the linear Itô SDE with multiplicative noise

$$dX_t = \left(1 - \frac{3}{2}\,\alpha\right) \lambda\, X_t\, dt + \sqrt{\alpha\,|\lambda|}\, X_t\, dW_t, \tag{3.3}$$

where $X_0 > 0$, $\lambda < 0$, $\alpha \in [0,1]$, $t \geq 0$. The corresponding Stratonovich SDE has here the form

$$dX_t = (1-\alpha)\,\lambda\, X_t\, dt + \sqrt{\alpha\,|\lambda|}\, X_t \circ dW_t. \tag{3.4}$$

Here "$\circ$" denotes the Stratonovich stochastic integral, see Kloeden & Platen (1999).

For the two equivalent real valued SDEs (3.3) and (3.4) the parameter $\alpha \in [0,1]$ describes the degree of stochasticity of the test dynamics. Obviously, for $\alpha = 0$ there is no randomness in the solution of (3.3) and (3.4). We note for the case $\alpha = 1$ that the Stratonovich SDE (3.4) has no drift. In the case $\alpha = \frac{2}{3}$ it turns out that the Itô SDE (3.3) has no drift, and that X_t is a martingale. This case models a typical Black-Scholes asset price dynamics in finance when the price is expressed in units of an appropriate numeraire under the corresponding pricing measure. We remark, that when choosing the so-called numeraire portfolio as numeraire, the pricing measure is simply the real world probability measure, see Platen & Heath (2006). In the case when the numeraire is the savings account then the pricing measure is, under appropriate assumptions, the risk neutral probability measure.

Let us now introduce a well-known and rather weak concept of stability.

Definition 3.1 *A process* $Y = \{Y_t, t \geq 0\}$ *is called* asymptotically stable *if*

$$P\left(\lim_{t\to\infty} |Y_t| = 0\right) = 1. \tag{3.5}$$

Consequently, a process is asymptotically stable if in the long run its value vanishes. From (3.2) it follows that for all $\alpha \in [0, 1)$ and $\lambda < 0$ the solution process X, given in (3.1), is *asymptotically stable.*

From the numerical point of view it is desirable to have a discrete time approximation Y of X with comparable asymptotic stability properties as X itself. In the case of a strong scheme, which generates an asymptotically stable discrete time approximation, the impact of perturbations through roundoff and truncation errors declines asymptotically over time for the test SDE (3.3) when $\lambda < 0$ and $\alpha \in [0, 1)$.

As we will see, a numerical scheme Y that approximates X may behave rather differently for different choices of the parameters α and λ, and different time step sizes Δ. Therefore, it is appropriate to introduce for a given discrete time approximation the notion of a stability region which allows visualization of its numerical stability properties.

Definition 3.2 *The* stability region Γ *is determined by those pairs* $(\lambda\Delta, \alpha) \in (-\infty, 0) \times [0, 1)$ *for which the discrete time approximation* Y *with time step size* Δ*, when applied to the test equation* (3.4)*, is asymptotically stable.*

As already emphasized, there exists a wide range of numerical stability concepts. For instance, it has been common to use second moments for identifying some type of mean-square stability, see Saito & Mitsui (1996), Higham (2000), Higham & Kloeden (2005) and Alcock & Burrage (2006). This may lead for some schemes to mathematically convenient characterizations of the resulting regions of mean-square stability. However, in general, stability regions are calculated numerically. The stability regions under stronger stability concepts, such as mean square stability, are usually smaller than those that result when asking whether a given numerical scheme is asymptotically stable. For scenario simulation we typically need to control error propagation in a pathwise sense and not in a sense related to particular moments. Furthermore, when using some moment related stability concept, there are moment requirements to satisfy for the underlying processes. For numerical stability analysis in a scenario simulation, mean square stability, for instance, may eventually request too much when compared with what is actually needed to control the pathwise propagation of errors. What only matters is the impact of errors over long time periods on the simulated trajectory. If this impact is dampened, as is the case when asymptotic stability is guaranteed, then the method is likely to be useful in long term scenario simulation.

For the above reasons we will concentrate in this paper on the concept of asymptotic stability for strong discrete time approximations using the test equation (3.3) for $\lambda < 0$ and $\alpha \in [0, 1)$. This approach also provides in some sense the widest stability regions. Our aim is now to study for given strong discrete time methods the corresponding stability regions. This analysis provides guidance for the choice of particular schemes.

We will see that by application of most discrete time approximations Y with time step size $\Delta > 0$ to the test equation (3.3) for $\lambda < 0$ and a given degree of stochasticity $\alpha \in [0, 1)$, the ratio

$$\left| \frac{Y_{n+1}}{Y_n} \right| = G_{n+1}(\lambda\,\Delta, \alpha) \tag{3.6}$$

is of major interest, where $n \in \{0, 1, \ldots\}$ and $Y_n > 0$. We call the random variable $G_{n+1}(\lambda\,\Delta, \alpha)$ the *transfer function* of the method Y at time t_n. It transfers the previous approximate value Y_n into the next approximate value Y_{n+1}.

Furthermore, let us assume from now on that for a given scheme and $\lambda < 0$, $\Delta \in (0, 1)$ and $\alpha \in [0, 1)$ the random variables $G_{n+1}(\lambda\,\Delta, \alpha)$ are for $n \in \{0, 1, \ldots\}$ nonnegative, independent and identically distributed with $E((\ln(G_{n+1}(\lambda\,\Delta, \alpha)))^2) < \infty$. This assumption is satisfied for a wide range of schemes, and will allow us to visualize corresponding stability regions. For this purpose we will employ the following result, which has been derived in Higham (2000), as a consequence of the law of iterated logarithm and the law of large numbers.

Lemma 3.3 *A discrete time approximation is for given $\lambda\,\Delta < 0$ and $\alpha \in [0, 1)$ asymptotically stable if and only if*

$$E(\ln(G_{n+1}(\lambda\,\Delta, \alpha))) < 0 \tag{3.7}$$

for all $n \in \{0, 1, \ldots\}$.

It is obvious from (3.6) that if $G_{n+1}(\lambda\,\Delta, \alpha) < 1$, almost surely, for all $n \in \{0, 1, \ldots\}$, then because $\ln(G_{n+1}(\lambda\,\Delta, \alpha)) < 0$ there is no propagation of errors. By Lemma 3.3 it turns out that there is still asymptotic stability if $\ln(G_{n+1}(\lambda\,\Delta, \alpha))$ is just on average negative. We now define the A-stability region for given schemes by identifying the set of those pairs $(\lambda\,\Delta, \alpha)$ for $\lambda < 0$, $\Delta > 0$ and $\alpha \in [0, 1)$ for which the inequality (3.7) is satisfied.

4 A-Stability Regions

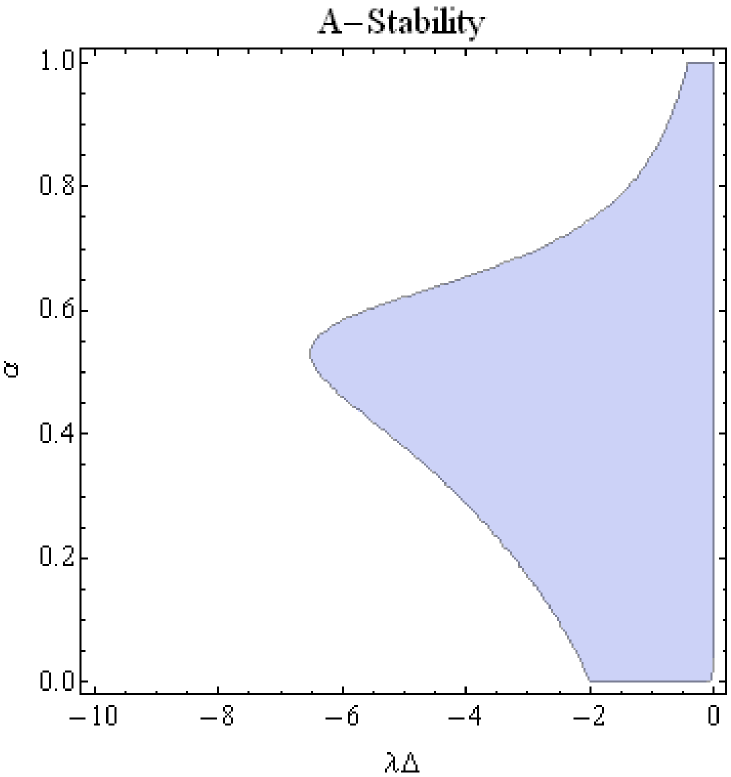

Figure 4.1: A-stability region for the Euler scheme.

For the Euler scheme, which results for $\theta = \eta = 0$ in (2.6), it follows by (3.6) that

$$G_{n+1}(\lambda\,\Delta, \alpha) = \left|1 + \left(1 - \frac{3}{2}\,\alpha\right)\lambda\,\Delta + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right|, \tag{4.1}$$

where $\Delta W_n \sim \mathcal{N}(0, \Delta)$ is a Gaussian distributed random variable with mean zero and variance Δ. The transfer function (4.1) yields the A-stability region that is shown as shaded area in Figure 4.1. It is the region where $E(\ln(G_{n+1}(\lambda\,\Delta, \alpha))) < 0$. This A-stability region has been obtained numerically by identifying for each $\alpha \in [0, 1)$ those values $\lambda\Delta$ for which the inequality (3.7) holds. One notes that for the purely deterministic method, that is $\alpha = 0$, the A-stability region covers the interval $(-2, 0)$. For an increasing stochasticity parameter of up to about $\alpha \approx 0.55$ the A-stability region covers an interval of increasing length of up to about 6.5. For further increased stochasticity parameter α the A-stability region declines in Figure 4.1.

Let us now consider the semi-drift-implicit predictor-corrector Euler method with

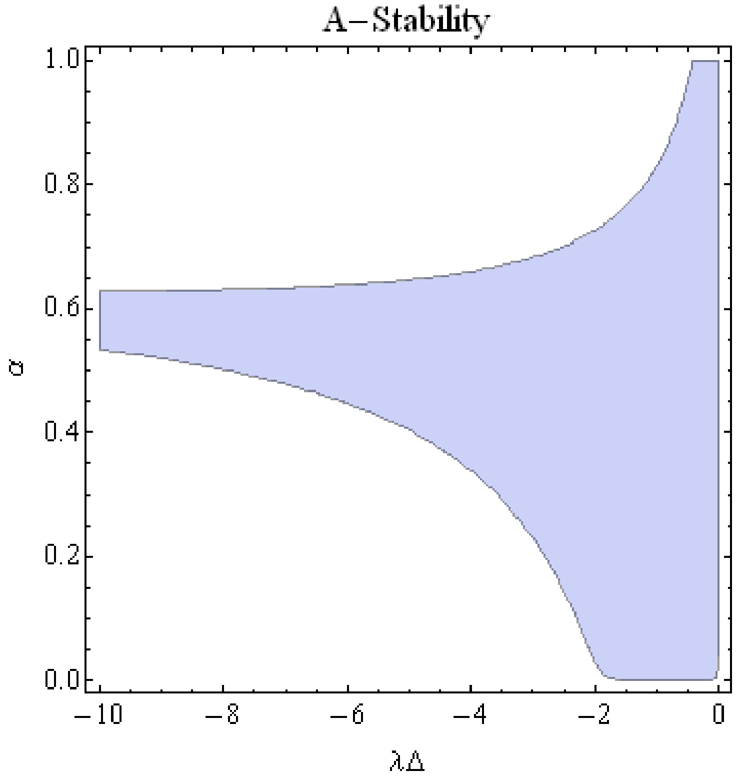

Figure 4.2: A-stability region for semi-drift-implicit predictor-corrector Euler method.

$\theta = \frac{1}{2}$ and $\eta = 0$ in (2.6). The transfer function for this method equals

$$G_{n+1}(\lambda\,\Delta, \alpha) = \left|1 + \lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right)\left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\} + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right|.$$

Its stability region is shown in Figure 4.2. We note that it extends for most values of α near 0.6 the A-stability region considerably further than was the case for the Euler scheme. This is welcome since it provides for this type of stochasticity still an asymptotically numerically stable method when the Euler scheme fails. Unfortunately, for the stochasticity parameter value $\alpha = \frac{2}{3}$, that is when X forms a martingale, the A-stability region is not as wide as for $\alpha = 0.6$.

Similarly, we obtain for the drift-implicit predictor-corrector Euler method with $\theta = 1$ and $\eta = 0$ in (2.6) the transfer function

$$G_{n+1}(\lambda\,\Delta, \alpha) = \left|1 + \lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right)\left\{1 + \lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right\} + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right|.$$

The corresponding A-stability region is plotted in Figure 4.3. It does not seem that the stability region has significantly increased above the one for the previous choice $\theta = \frac{1}{2}$. Semi-drift implicitness seems to provide a reasonable balance.

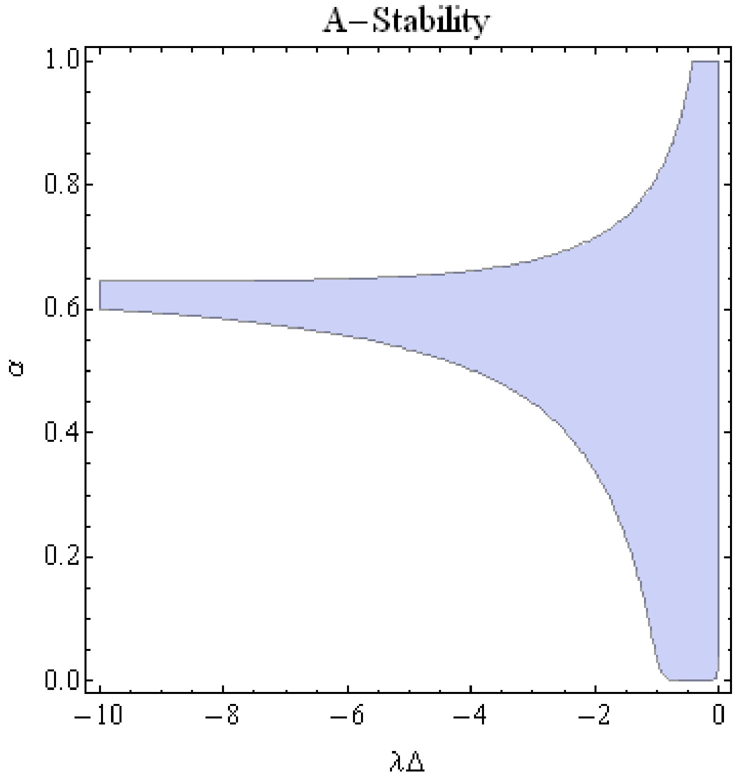

Figure 4.3: A-stability region for drift-implicit predictor-corrector Euler method.

Now, let us now study the impact of making the diffusion term implicit in a predictor-corrector Euler method. First we consider a predictor-corrector Euler method with semi-implicit diffusion term where $\theta = 0$ and $\eta = \frac{1}{2}$. Its transfer function has the form

$$G_{n+1}(\lambda\,\Delta, \alpha) = \Bigg| 1 + \lambda\,\Delta\,(1-\alpha) + \sqrt{-\alpha\,\lambda}\,\Delta W_n \times \left\{ 1 + \frac{1}{2}\left(\lambda\,\Delta \left(1 - \frac{3}{2}\,\alpha \right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n \right) \right\} \Bigg| .$$

The corresponding A-stability region is shown in Figure 4.4. It is rather restricted when compared with the one for the semi-drift implicit predictor-corrector method. However, it is important to note that for the martingale case, that is $\alpha = \frac{2}{3}$, the A-stability region is here wider than for the Euler scheme, which is relevant for applications in finance. Furthermore, we observe for, say $\alpha = 0.7$ that for about $\lambda\Delta \approx -3$ the method is asymptotically stable, whereas for smaller step sizes for instance $\lambda\Delta \approx -1$, the method is no longer asymptotically stable for $\alpha = 0.7$.

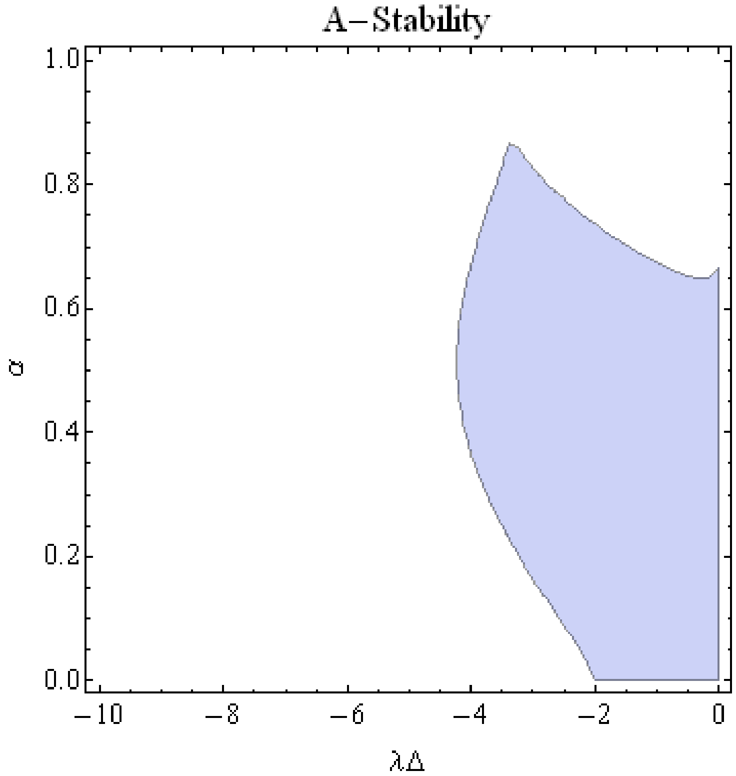

Figure 4.4: A-stability region for the predictor-corrector Euler method with $\theta = 0$ and $\eta = \frac{1}{2}$.

This means, by reducing the time step size the method could enter an area of numerical instability. This seems to be counter intuitive because reduced time step size is typically connected with improved numerical stability in deterministic numerical analysis. Here we meet an important phenomenon that can arise in a stochastic setting.

An interesting scheme is the symmetric predictor-corrector Euler method with $\theta = \eta = \frac{1}{2}$ in (2.6), which has transfer function

$$G_{n+1}(\lambda\,\Delta, \alpha) \;=\; \Bigg| 1 + \lambda\,\Delta\,(1-\alpha)\left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\} + \sqrt{-\alpha\,\lambda}\,\Delta W_n\left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\}\Bigg|.$$

Its A-stability region is shown in Figure 4.5. In particular, for the martingale case $\alpha = \frac{2}{3}$ it has a rather large interval of asymptotic stability when compared with the Euler scheme. It is remarkable that for α close to one and small step sizes there is a very small area where this scheme is not asymptotically stable but still stable for larger step sizes. As mentioned earlier, this can happen in a stochastic setting.

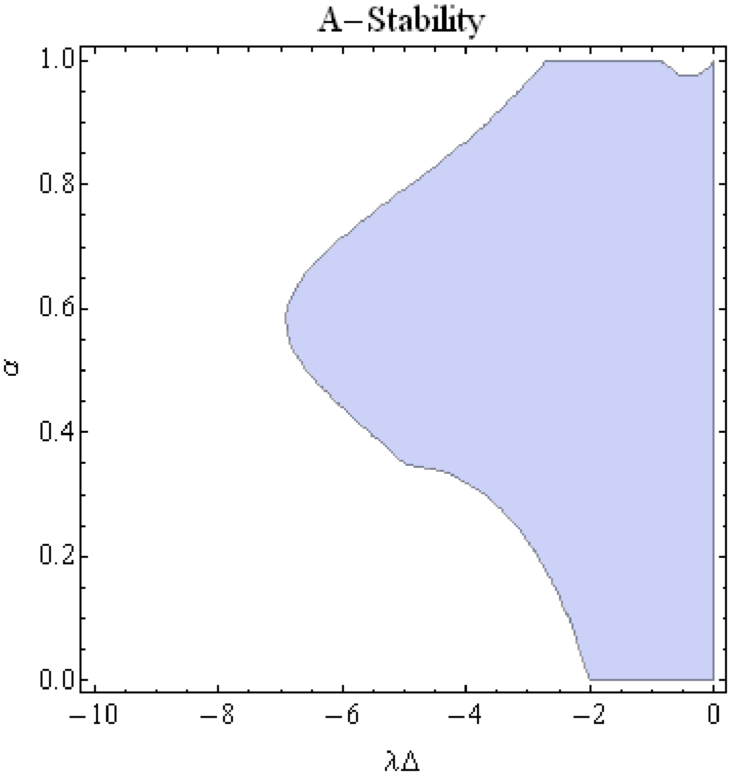

Figure 4.5: A-stability region for the symmetric predictor-corrector Euler method.

Let us now check what A-stability region we obtain for a fully implicit predictor-corrector Euler method when setting both degrees of implicitness to $\theta = \eta = 1$ in (2.6). The corresponding transfer function is then of the form

$$\begin{aligned} G_{n+1}(\lambda\,\Delta, \alpha) \;&=\; \Bigg| 1 + \lambda\,\Delta \left(1 - \frac{1}{2}\,\alpha\right) \left\{ 1 + \lambda\,\Delta \left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n \right\} \\ &\quad + \sqrt{-\alpha\,\lambda}\,\Delta W_n \left\{ 1 + \lambda\,\Delta \left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n \right\} \Bigg| \\ &= \Bigg| 1 + \left\{ 1 + \lambda\,\Delta \left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n \right\} \\ &\quad \times \left\{ \lambda\,\Delta \left(1 - \frac{1}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n \right\} \Bigg| . \end{aligned}$$

The resulting A-stability region is shown in Figure 4.6. We note that this scheme has a rather small stability region. It seems that a moderate degree of implicitness increases the stability region, however, too large degrees of implicitness may reduce the A-stability region. This is confirmed by further systematic study of other predictor-corrector Euler schemes.

The most extensive A-stability region which we considered in the previous figures

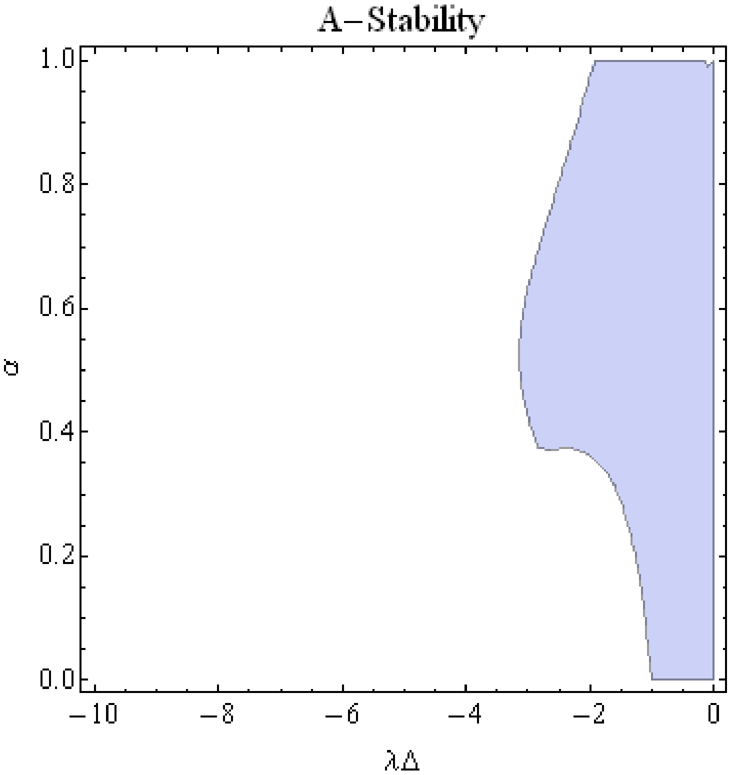

Figure 4.6: A-stability region for fully implicit predictor-corrector Euler method.

appears to be approximately that of the symmetric predictor-corrector Euler method shown in Figure 4.5. The symmetry in the terms of the symmetric predictor-corrector Euler methods balances well its numerical stability properties and makes it the appropriate choice for a range of strong simulation tasks. In practice, it can be expected to have better numerical stability properties than the Euler scheme in most situations.

5 Martingale Dynamics

Dynamics that form martingales are of paramount importance in asset price models in finance. As mentioned previously, for the case $\alpha = \frac{2}{3}$ the solution X_t of the test SDE (3.3) is a martingale. Since in this case we have no drift term in the Itô SDE, at a first glance it may not seem to matter how one chooses in a predictor-corrector Euler scheme the degree of drift implicitness parameter θ. However, as we will see, it turns out that it matters when giving the approximation of the diffusion coefficient some degree of implicitness, that is, choosing $\eta \in (0, 1]$. For any choice $\eta \in (0, 1]$ we have to adjust the respective drift in the scheme according to the adjusted drift formula (2.2).

The observed A-stability regions for various predictor-corrector Euler methods are rather illuminating. They can guide the selection of appropriate strong schemes when simulating approximate solutions of SDEs. To illustrate this further we select the martingale case for the test equation (3.3) by setting $\alpha = \frac{2}{3}$. We showed in Figure 1.1 for $\sigma = 5$ and $X_0 = 1$ the exact path together with a diverging one generated by the Euler scheme. Now we compare in Figure 5.1 for the same trajectory of the driving Wiener process as in Figure 1.1 the exact

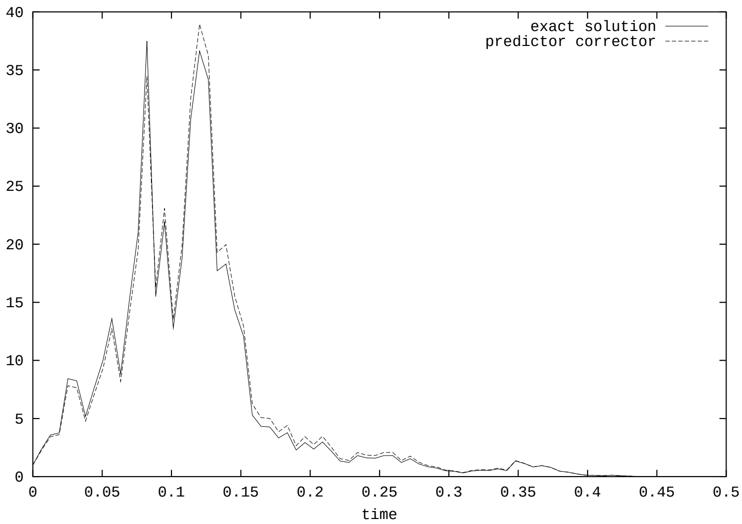

Figure 5.1: Exact solution and approximate solution generated by the symmetric predictor-corrector Euler scheme.

solution (3.1) with the one generated by the symmetric predictor-corrector Euler method. When we use the symmetric predictor-corrector Euler method instead of the Euler scheme, the previously observed discrepancy vanishes. This indicates some superior numerical stability of the new symmetric predictor-corrector Euler scheme.

The above findings demonstrate that by the use of appropriate strong predictor-corrector Euler methods, one can improve the asymptotic stability of a scenario simulation. The propagation of errors over long periods of time can be better controlled.

There is a convenient way of potentially checking whether some predictor-corrector Euler schemes may be not numerically stable. One can easily implement the entire family of these methods by keeping $\theta \in [0,1]$ and $\eta \in [0,1]$ flexible. In the simulation experiment one generates the trajectories for various choices of θ and η in the predictor-corrector Euler schemes with the same driving Wiener process path. If these trajectories differ much for some schemes from those of other schemes, then the numerical stability of some of the employed schemes is questionable. This approach guides the selection of numerically stable schemes within the given class. Finally, it should be mentioned that the degrees of implicitness θ

and η need not to be strictly confined to the interval $[0,1]$. For instance, negative or rather large degrees of implicitness in the drift or diffusion terms could have a stabilizing effect, and the following proof of Theorem 2.1 can be extended to cover such cases.

Appendix

A. Proof of Theorem 2.1

The proof exploits results presented in Kloeden & Platen (1999). We aim to use here the same notation as in this book and refer to it for unexplained notions or results. Furthermore, our assumptions in Section 2 are assumed to be valid.

Let us first consider the one-dimensional case, $d = m = 1$, where the strong predictor-corrector Euler scheme is given by the corrector

$$\begin{aligned} Y_{n+1} &= Y_n + \left\{\theta\, \bar{a}_\eta(t_{n+1}, \bar{Y}_{n+1}) + (1-\theta)\, \bar{a}_\eta(t_n, Y_n)\right\} \Delta_n \\ &\quad + \left\{\eta\, b^1(t_{n+1}, \bar{Y}_{n+1}) + (1-\eta)\, b^1(t_n, Y_n)\right\} \Delta W_n^1 \end{aligned} \tag{6.1}$$

and the predictor

$$\bar{Y}_{n+1} = Y_n + a(t_n, Y_n)\, \Delta_n + b(t_n, Y_n)\, \Delta W_n^1 \tag{6.2}$$

with $\bar{a}_\eta = a - \eta\, b\, b'$ for $\theta, \eta \in [0,1]$, see (2.8) – (2.9).

This scheme can be written in the form

$$Y_{n+1} = Y_n + a(t_n, Y_n)\, \Delta_n + b^1(t_n, Y_n)\, \Delta W_n^1 + \sum_{\ell=1}^{4} R_{\ell,n}. \tag{6.3}$$

Here we have

$$R_{1,n} = \theta \left\{ a(t_{n+1}, \bar{Y}_{n+1}) - a(t_n, Y_n) \right\} \Delta_n, \tag{6.4}$$

$$R_{2,n} = -\theta\, \eta \left\{ b^1(t_{n+1}, \bar{Y}_{n+1})\, b^{1\prime}(t_{n+1}, \bar{Y}_{n+1}) - b^1(t_n, Y_n)\, b^{1\prime}(t_n, Y_n) \right\} \Delta_n, \tag{6.5}$$

$$R_{3,n} = -\eta\, b^1(t_n, Y_n)\, b^{1\prime}(t_n, Y_n) \Delta_n, \tag{6.6}$$

and

$$R_{4,n} = -\eta \left\{ b^1(t_{n+1}, \bar{Y}_{n+1}) - b^1(t_n, Y_n) \right\} \Delta W_n^1. \tag{6.7}$$

The following second moment estimate on the numerical approximation Y_n can be derived by using similar steps as those described in the proof of Lemma 10.8.1 and Theorem 10.2.2 in Kloeden & Platen (1999), where

$$E\left(\max_{0 \le n \le n_T - 1} |Y_n|^2 \right) \le K\Big(1 + E(|Y_0|^2)\Big). \tag{6.8}$$

By the Cauchy-Schwarz inequality, the Lipschitz condition on the drift coefficient and equation (6.2), we then obtain

$$
\begin{aligned}
&E\left(\max_{1\le n\le n_T}\left|\sum_{0\le k\le n-1} R_{1,k}\right|^2\right)\\
&= E\left(\max_{1\le n\le n_T}\left|\sum_{0\le k\le n-1}\theta\left\{a(t_{k+1},\bar{Y}_{k+1})-a(t_k,Y_k)\right\}\Delta_k\right|^2\right)\\
&\le E\left(\max_{1\le n\le n_T}\left(\sum_{0\le k\le n-1}|\theta\Delta_k|^2\right)\left(\sum_{0\le k\le n-1}\left|a(t_{k+1},\bar{Y}_{k+1})-a(t_k,Y_k)\right|^2\right)\right)\\
&\le K\Delta E\left(\left(\sum_{0\le k\le n_T-1}\Delta_k\right)\left(\sum_{0\le k\le n_T-1}\left|a(t_{k+1},\bar{Y}_{k+1})-a(t_k,Y_k)\right|^2\right)\right)\\
&\le K\,\Delta\,E\left(\sum_{0\le k\le n_T-1}\left|a(t_{k+1},\bar{Y}_{k+1})-a(t_k,Y_k)\right|^2\right)\\
&\le K\,\Delta\,E\left(\sum_{0\le k\le n_T-1}\left|\bar{Y}_{k+1}-Y_k\right|^2\right) \qquad (6.9)\\
&\le K\,\Delta\,E\left(\sum_{0\le k\le n_T-1}\left(E\left(|a(t_k,Y_k)\,\Delta|^2\;\middle|\;\mathcal{A}_{t_k}\right)+E\left(\left|b^1(t_k,Y_k)\,\Delta W_k^1\right|^2\;\middle|\;\mathcal{A}_{t_k}\right)\right)\right).
\end{aligned}
$$

By using the Cauchy-Schwarz inequality, the Itô isometry and the linear growth conditions (2.4), we obtain

$$
\begin{aligned}
E\left(\max_{1\le n\le n_T}\left|\sum_{0\le k\le n-1} R_{1,k}\right|^2\right) &\le K\Delta\,E\left(\int_0^{t_{n_T}} E\left((1+|Y_{n_z}|^2)\middle|\mathcal{A}_{t_{n_z}}\right)dz\right)\\
&\le K\Delta\int_0^{t_{n_T}} E\left(1+\max_{0\le n\le n_T}|Y_n|^2\right)dz\\
&\le K\Delta\left(1+E\left(|Y_0|^2\right)\right)T\\
&\le K\Delta. \qquad (6.10)
\end{aligned}
$$

With similar steps as in (6.9) and (6.10) we obtain

$$
E\left(\max_{1\le n\le n_T}\left|\sum_{0\le k\le n-1} R_{2,k}\right|^2\right)\le K\,\Delta. \qquad (6.11)
$$

By the linear growth condition on the coefficient $\bar{a}_\eta$ and the second moment estimate (6.8), one can show that

$$E\left(\max_{1\le n\le n_T}\left|\sum_{0\le k\le n-1} R_{3,k}\right|^2\right) \le K\,\Delta. \tag{6.12}$$

By Doob's inequality, Itô's isometry, the Lipschitz condition on the diffusion coefficient and (6.2), we obtain

$$\begin{aligned}
&E\left(\max_{1\le n\le n_T}\left|\sum_{0\le k\le n-1} R_{4,k}\right|^2\right)\\
&\quad = E\left(\max_{1\le n\le n_T}\left|\sum_{0\le k\le n-1}\eta\int_{t_k}^{t_{k+1}}\big(b^1(t_{k+1},\bar{Y}_{k+1})-b^1(t_k,Y_k)\big)dW_z\right|^2\right)\\
&\quad \le 4E\left(\left|\int_0^T\big(b^1(t_{n_z+1},\bar{Y}_{n_z+1})-b^1(t_{n_z},Y_{n_z})\big)dW_z\right|^2\right)\\
&\quad = 4\int_0^T E\left(\left|b^1(t_{n_z+1},\bar{Y}_{n_z+1})-b^1(t_{n_z},Y_{n_z})\right|^2\right)dz\\
&\quad \le K\int_0^T E\left(\left|\bar{Y}_{n_z+1}-Y_{n_z}\right|^2\right)dz.
\end{aligned}\tag{6.13}$$

By the Lipschitz condition and the second moment estimate (6.8), we have

$$\begin{aligned}
&E\left(\max_{1\le n\le n_T}\left|\sum_{0\le k\le n-1} R_{4,k}\right|^2\right)\\
&\qquad \le K\int_0^T E\left(\int_{t_{n_z}}^{t_{n_z+1}} E\left(1+|Y_{t_{n_z}}|^2\big|\mathcal{A}_{t_{n_z}}\right)ds\right)dz\\
&\qquad \le K\int_0^T E\left(\int_{t_{n_z}}^{t_{n_z+1}} E\left(1+\max_{0\le n\le n_T}|Y_{t_n}|^2\big|\mathcal{A}_{t_{n_z}}\right)ds\right)dz\\
&\qquad \le K\,\Delta\,\big(1+E(|Y_0|^2)\big)\,T\\
&\qquad \le K\Delta.
\end{aligned}\tag{6.14}$$

In view of the Lipschitz condition (2.2) and the linear growth condition (2.3) it is straightforward to show that

$$E\left(\sup_{0\le s\le T}|X_s|^2\,\big|\,\mathcal{A}_0\right) \le K\left(1+E\left(|X_0|^2\right)\right). \tag{6.15}$$

Furthermore, by (6.3) we can write

$$Y(t) = Y_n + \int_{t_n}^{t} a(t_n, Y_n)\, ds + \int_{t_n}^{t} b^1(t_n, Y_n)\, dW_s^1 + \sum_{\ell=1}^{4} R_{\ell,n}, \tag{6.16}$$

where $Y(t_n) = Y_n$ for $t \in [t_n, t_{n+1}]$, $n \in \{0, 1, \ldots, n_T - 1\}$. Then we obtain

$$\begin{aligned} Z(t) &= E\left(\sup_{0\le s\le t} |X_s - Y(s)|^2 \,\middle|\, \mathcal{A}_0 \right) \\ &\le K \left\{ E\left(|X_0 - Y_0|^2 + S_0(t) + S_1(t) \right) + \sum_{\ell=1}^{4} \tilde{R}_\ell(t) \right\}, \end{aligned} \tag{6.17}$$

where the terms being summed will be defined below and upper bounds determined for them.

From (2.1) and (6.16) we have

$$\begin{aligned} S_0(t) &= E\left(\sup_{0\le s\le t} \left| \sum_{n=0}^{n_s-1} \int_{t_n}^{t_{n+1}} \left(a(t_n, X_{t_n}) - a(t_n, Y_n)\right) ds \right.\right. \\ &\qquad \left.\left. + \int_{t_{n_s}}^{s} \left(a(t_{n_s}, X_{t_{n_s}}) - a(t_{n_s}, Y_{t_{n_s}})\right) ds \right|^2 \,\middle|\, \mathcal{A}_0 \right) \\ &\le K \int_0^t E\left(\sup_{0\le s\le u} \left| a(t_{n_s}, X_{t_{n_s}}) - a(t_{n_s}, Y_{t_{n_s}}) \right|^2 \,\middle|\, \mathcal{A}_0 \right) du \\ &\le K \int_0^t Z(u)\, du \end{aligned} \tag{6.18}$$

for $t \in [0, T]$. Similarly, we obtain

$$\begin{aligned} S_1(t) &= E\left(\sup_{0\le s\le t} \left| \sum_{n=0}^{n_s-1} \int_{t_n}^{t_{n+1}} \left(b^1(t_n, X_{t_n}) - b^1(t_n, Y_n)\right) dW_s^1 \right.\right. \\ &\qquad \left.\left. + \int_{t_{n_s}}^{s} \left(b^1(t_{n_s}, X_{t_{n_s}}) - b^1(t_{n_s}, Y_{t_{n_s}})\right) dW_s^1 \right|^2 \,\middle|\, \mathcal{A}_0 \right) \\ &\le K \int_0^t E\left(\sup_{0\le s\le u} \left| b^1(t_{n_s}, X_{t_{n_s}}) - b^1(t_{n_s}, Y_{t_{n_s}}) \right|^2 \,\middle|\, \mathcal{A}_0 \right) du \\ &\le K \int_0^t Z(u)\, du \end{aligned} \tag{6.19}$$

for $t \in [0, T]$. Furthermore, we have due to (6.10), (6.11), (6.12) and (6.14)

$$\tilde{R}_\ell(t) = E\left(\max_{1\le n\le n_t} \left| \sum_{k=0}^{n-1} R_{\ell,k} \right|^2 \right) \le K\,\Delta \tag{6.20}$$

for $\ell \in \{1,2,3,4\}$ and $t \in [0,T]$.

Consequently, by combining (6.17) – (6.20) and applying the Gronwall inequality, see Lemma 4.5.1 in Kloeden & Platen (1999), we obtain

$$Z(t) \leq K\,\Delta,$$

for $t \in [0,T]$, which is the claim of the theorem. The proof for the general multi-dimensional case driven by several Wiener processes is analogous. $\square$

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