International Journal of Differential Equations and Applications
Volume 8 No. 3 2003, 257-270

ASYMPTOTICS OF THE GLOBAL SOLUTION FOR A NONLINEAR TELEGRAPH EQUATION

Yong Hong Wu[1 §], Shaoyong Lai[2], Guangquan Zhang[3]
[1,2,3]Department of Mathematics and Statistics
Curtin University of Technology
GPO Box U 1987, Perth 6845, WA, AUSTRALIA
[1]e-mail: yhwu@maths.curtin.edu.au

Abstract: In this paper, we consider an initial-boundary value problem for the following nonlinear telegraph equation

$$u_{tt} - u_{xx} + 2au_t + bu = \beta(u^2)_{xx},$$

where $t > 0$, a, b and β are constants. For the case $b > a^2$, we establish a global solution of the equation in the form of a Fourier series. The coefficients of the series are related to a small parameter present in the initial conditions and are expressed as uniformly convergent series of the parameter. The long time asymptotics of the global solution is found to decay exponentially in time.

AMS Subject Classification: 35Q20, 76B15
Key Words: global solution, initial-boundary value problem, nonlinear telegraph equations

1. Introduction

In recent years, various telegraph equations have attracted the attention of many mathematicians, engineers and physists [1], [2], [4]-[8]. One of the classical linear telegraph equations is, as follows:

Received: March 10, 2003

§Correspondence author

$$u_{tt} - u_{xx} + 2au_t + bu = 0, \tag{1}$$

where a and b are constants. The equation arises in the study of many real world problems such as the propagation of electrical signals in transmission lines [2-3], the propagation of pressure waves in pulsatile blood flow through arteries [1] and the one dimensional random motion of bugs along hedge [2].

For nonlinear telegraph equations, the common version can be expressed as

$$u_{tt} - u_{xx} + 2au_t + bu = \beta f(u, \ u_t, u_x), \tag{2}$$

where a, b and β are constants, and $f(u, u_t, u_x)$ denotes a nonlinear function of u and its first order derivatives. An initial boundary value problem for the above equation has been studied in [2] under certain assumptions on the nonlinear function f. The asymptotic theory of the solution to an initial value problem associated with a perturbed telegraph equation has been established in a Sobolev space by Lai [3]. The application of the asymptotic theory is also given in details in reference [3]. For more information of research on telegraph equations, the reader is referred to references [4]-[10].

In this paper, an initial boundary value problem for the following telegraph equation shall be discussed

$$u_{tt} - u_{xx} + 2au_t + bu = \beta(u^2)_{xx}. \tag{3}$$

We will mainly consider the asymptotics of the global solution to equation (3) in classical sense. For the case $b > a^2$, the classical solution of the initial-boundary value problem for equation (3) shall be constructed in the form of a Fourie series with coefficients in their own turn represented as series in terms of a small parameter present in the initial conditions. It will be shown that the new solution of the initial-boundary value problem for equation (3) is well-posed. Its long time asymptotics is obtained in explicit form. As far as the authors know, it is the first attempt to obtain the long time asymptotics for the telegraph equation (3). It will also be demonstrated that the asymptotics shows the presence of both time and space oscillations and the exponential decay of the solution in time as $t \to \infty$ due to dissipation. This is in contrast with the linear growth in time of the major term of the long time asymptotic representation of the solution to the problem presented in reference [3].

2. Theorem of Existence, Uniqueness and Asymptotics of Solutions

The aim of this section is to establish the well-posedness of the following initial-boundary value problem :

$$\begin{cases} u_{tt} - u_{xx} + 2au_t + bu = \beta(u^2)_{xx}, & x \in (0,\pi), t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u_{xx}(0,t) = u_{xx}(\pi,t) = 0, & t > 0, \\ u(x,0) = \varepsilon^2\varphi(x), u_t(x,0) = \varepsilon^2\psi(x), & x \in (0,\pi), \end{cases} \tag{4}$$

where a, b and ε are positive, $\beta \in R^1$.

Definition 1. The function $u(x)$ is said to be in $C^n(0,\pi), n \geq 1$, if $u(0) = u(\pi) = u'(0) = u'(\pi) = \cdots = u^{(n-1)}(0) = u^{(n-1)}(\pi) = 0$ and $u^{(n)}(x) \in L^2(0,\pi)$.

Definition 2. The function $u(x,t)$ defined on $[0,\pi] \times [0,+\infty)$ is said to be the classical solution of problem (4), if it has two continuous derivatives on $[0,\pi] \times [0,+\infty)$ and satisfies the differential equation and all the boundary and initial conditions of (4).

Theorem 1. *If $b > a^2, \varphi(x) \in C^4(0,\pi), \psi(x) \in C^3(0,\pi)$, then there exists a $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, there exists a unique classical solution to the problem defined by system (4) and the solution can be expressed in the form of*

$$u(x,t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(x,t), \tag{5}$$

where the function $u^{(N)}(x,t)$ is as that defined by (19) in the proof below. This series and the series for the derivatives of $u(x,t)$ involved *in (4) converge absolutely and uniformly with respect to* $x \in [0, \pi], t \geq 0, \varepsilon \in [0, \varepsilon_0]$. *In addition, the solution of system (4) has the following long time asymptotics as* $t \to +\infty$

$$u(x,t) = e^{-at}[(A\cos\sigma t + B\sin\sigma t)\sin x] + O(e^{-(1+\eta)at}), \qquad (6)$$

where $\sigma = \sqrt{1 + b - a^2}$, $b > a^2$, $0 < \eta < \frac{1}{2}$, *and the coefficients* A *and* B *are defined by (23). The estimate of the residual term is uniform in* $x \in [0, \pi]$, $\varepsilon \in [0, \varepsilon_0]$.

3. The Proof of the Theorem

The proof include three parts, namely the existence of a solution, the uniqueness of the solution and the asymptotics of the solution.

3.1. Existence of Solution

Firstly, we make an odd extension of $u(x,t)$ in x to the interval $[-\pi, 0]$ and represent u in the form of a complex Fourie series, namely

$$u(x,t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \widehat{u}_n(t) e^{inx}, \quad \widehat{u}_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x,t)e^{-inx}dx, \qquad (7)$$

where $\widehat{u}_{-n}(t) = -\widehat{u}_n(t)$ for $n \geq 1$. In the following, we shall use the fact that $u(x,t)$ belongs to the space $L^2(-\pi, \pi)$ for every fixed $t > 0$ and we shall denote the corresponding norm by

$$\|u(t)\| = \|u(t)\|_{L^2(-\pi,\pi)} = \left(\int_{-\pi}^{\pi} |u(x,t)|^2 dx\right)^{\frac{1}{2}}.$$

From (7), we obtain, through a simple calculation, that

$$u(x,t) = 2i\sum_{n=1}^{\infty} \widehat{u}_n(t)\sin nx, \qquad \widehat{u}_n(t) = \frac{1}{i\pi}\int_0^{\pi} u(x,t)\sin nx dx. \qquad (8)$$

Similarly the initial functions can be represented in the form of a complex series in $[-\pi, \pi]$

$$\varphi(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \widehat{\varphi}_n e^{inx}, \quad \psi(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \widehat{\psi}_n e^{inx}, \tag{9}$$

where $\widehat{\varphi}_{-n} = \widehat{\varphi}_n, \widehat{\psi}_{-n} = \widehat{\psi}_n$ for $n \geq 1$. Again, through a simple calculation, we have for $x \in [0, \pi]$,

$$\varphi(x) = 2i \sum_{n=1}^{\infty} \widehat{\varphi}_n \sin nx, \qquad \psi(x) = 2i \sum_{n=1}^{\infty} \widehat{\psi}_n \sin nx, \tag{10}$$

$$\widehat{\varphi}_n = \tfrac{1}{i\pi} \int_0^{\pi} \varphi(x) \sin nx dx, \quad \widehat{\psi}_n = \tfrac{1}{i\pi} \int_0^{\pi} \psi(x) \sin nx dx.$$

Using integrating by parts for the integrals in (10) and noting the smoothness assumption of the initial data, we obtain the following inequalities

$$|\widehat{\varphi}_n| \leq C_1 n^{-4}, \quad |\widehat{\psi}_n| \leq C_1 n^{-3}, \quad n \geq 1, \tag{11}$$

where C_1 is a positive constant. Substituting (7) and (9) into (4), we deduce the following Cauchy problem for the function $\widehat{u}_n(t), n \in Z$,

$$\widehat{u}_n''(t) + 2a\widehat{u}_n'(t) + (n^2 + b)\widehat{u}_n(t) = -\beta n^2 P\left(\widehat{u}_n(t)\right), \tag{12}$$

$$\widehat{u}_n(0) = \varepsilon^2 \widehat{\varphi}_n, \widehat{u}_n'(0) = \varepsilon^2 \widehat{\psi}_n, \tag{13}$$

where

$$P(\widehat{u}_n(t)) = \sum_{\substack{g=-\infty \\ g\neq 0,n}}^{\infty} \widehat{u}_{n-g}(t)\widehat{u}_g(t) \text{ and } \widehat{u}_{-n}(t) = -\widehat{u}_n(t) \quad \text{for } n \geq 1.$$

For $n = 1$, we have from the above formula that

$$P\left(\widehat{u}_1(t)\right) = 2\sum_{g=1}^{\infty} \widehat{u}_{-g}(t)\widehat{u}_{1+g}(t) = -2\sum_{g=1}^{\infty} \widehat{u}_g(t)\widehat{u}_{1+g}(t).$$

Similarly, for $n \geq 2$,

$$P\left(\widehat{u}_n(t)\right) = \sum_{g=1}^{n-1} \widehat{u}_{n-g}(t)\widehat{u}_g(t) - 2\sum_{g=1}^{\infty} \widehat{u}_g(t)\widehat{u}_{n+g}(t).$$

Letting $\Phi_n = \varepsilon\widehat{\varphi}_n, \Psi_n = \varepsilon\widehat{\psi}_n$, and solving the ordinary differential equation (12) subject to initial conditions (13), we get

$$\begin{aligned}\widehat{u}_n(t) &= \varepsilon e^{-at}\left\{\left[\cos(\sigma_n t) + a\cdot\frac{\sin(\sigma_n t)}{\sigma_n}\right]\Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n}\Psi_n\right\} \\ &\quad - \frac{\beta n^2}{\sigma_n}\int_0^t \exp\left[-a(t-\tau)\right]\sin\left[\sigma_n(t-\tau)\right]P(\widehat{u}_n(\tau))d\tau,\end{aligned} \tag{14}$$

where

$$\sigma_n = \sqrt{n^2+b-a^2}, \qquad n \geq 1, \quad b > a^2.$$

Now, in order to solve the above equation, we use the perturbation theory. Firstly, we express $\widehat{u}_n(t), n \geq 1$, as a formal series in ε

$$\widehat{u}_n(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1}\widehat{\xi}_n^{(N)}(t). \tag{15}$$

Then, by substituting (15) into (14) and equating the coefficients of the corresponding powers of ε, we get the following formulas for $N = 0$ and $N \geq 1$ respectively,

$$\widehat{\xi}_n^{(0)}(t) = \varepsilon e^{-at}\left\{\left[\cos(\sigma_n t) + a\cdot\frac{\sin(\sigma_n t)}{\sigma_n}\right]\Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n}\Psi_n\right\}, \tag{16}$$

$$\begin{aligned}\widehat{\xi}_n^{(N)}(t) &= -\frac{\beta n^2}{\sigma_n}\int_0^t \exp\left[-a(t-\tau)\right] \\ &\quad \times \sin\left[\sigma_n(t-\tau)\right]Q_N\left(\widehat{\xi}_n^{(j)}(\tau)\right)d\tau,\end{aligned} \tag{17}$$

where

$$Q_N(\widehat{\xi}_n^{(j)}(\tau)) = \varepsilon_n \sum_{g=1}^{n-1} \sum_{j=1}^{N} \widehat{\xi}_{n-g}^{(j-1)}(\tau)\widehat{\xi}_g^{(N-j)}(\tau) - 2\sum_{g=1}^{\infty} \sum_{j=1}^{N} \widehat{\xi}_{n+g}^{(j-1)}(\tau)\widehat{\xi}_g^{(N-j)}(\tau),$$

in which $\varepsilon_1 = 0$ and $\varepsilon_n = 1$ for $n \geq 2$.

Now we must prove that the formally constructed function (8), together with (14)-(15), does represent a solution of system (4). To do this, we shall show that the series

$$\begin{aligned} u(x,t) &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{inx} \sum_{N=0}^{\infty} \varepsilon^{N+1}\widehat{\xi}_n^{(N)}(t) \\ &= 2i\sum_{n=1}^{\infty} \sin nx \sum_{N=0}^{\infty} \varepsilon^{N+1}\widehat{\xi}_n^{(N)}(\tau) \end{aligned}$$

converges absolutely and uniformly. For this purpose, we firstly establish the following time estimates for $n \geq 1, t > 0, N \geq 0$, and $0 < \eta < \frac{1}{2}$

$$\left|\widehat{\xi}_n^{(N)}(t)\right| \leq C^N(N+1)^{-2}n^{-4}e^{-\eta a t}. \tag{18}$$

Here and in the sequel we denote by C any positive constant independent of N, n, ε, and t, but possibly depending on the coefficients of the equation and the initial functions.

We shall use the induction technique for the proof. For $N = 0$ and $0 < \eta < \frac{1}{2}$, we have from (11) and (16) that

$$\begin{aligned} \left|\widehat{\xi}_n^{(0)}(t)\right| &\leq \varepsilon e^{-at}\left[\left(1+\frac{a}{\sigma_n}\right)\left|\widehat{\Phi}_n\right| + \frac{1}{\sigma_n}\left|\Psi_n\right|\right] \\ &\leq \varepsilon n^{-4}e^{-a\eta t}. \end{aligned}$$

Assuming that (17) is valid for all $\widehat{\xi}_n^{(s)}(t)$ with $0 \leq s \leq N-1$, we shall then prove that (18) also holds for $s = N$. According to [1], for any integer $n \geq 1, g \geq 1$ and $g \neq n$, we have

$$\begin{aligned} |n-g|^{-4}g^{-4} &\leq 2^6 n^{-4}[g^{-4}+|n-g|^{-4}], \\ j^{-2}(N+1-j)^{-2} &\leq 2^2(N+1)^{-2}\left[j^{-2}+(N+1-j)^{-2}\right]. \end{aligned}$$

From (17) , we have

$$\begin{aligned} \left|\widehat{\xi}_n^{(N)}(t)\right| &\leq C|\beta|(N+1)^{-2}n^{-4}\sum_{g=1}^{\infty}(g^{-4}+|n-g|^{-4}) \\ &\quad \times\sum_{j=1}^{\infty}C^{j-1}C^{N-j}\left[(N+1-j)^{-2}+j^{-2}\right]|S_N(n,t)|, \end{aligned}$$

$$\begin{aligned} |S_N(n,t)| &= e^{-at}\int_0^t e^{(1-2\eta)a\tau}d\tau \\ &\leq \frac{e^{-at}(e^{(1-2\eta)at}-1)}{a(1-2\eta)} \\ &\leq \frac{(e^{-2\eta at}-e^{-at})}{a(1-2\eta)}, \\ &\leq \frac{e^{-2\eta at}}{a(1-2\eta)}. \end{aligned}$$

Therefore (18) holds for $0 < \eta < \frac{1}{2}$.

To derive (5), we recall (8) and (15) with $\widehat{\xi}_n^{(N)}$ defined by (16) and (17) and interchange the order of summations in the series, namely

$$\begin{aligned} u(x,t) &= 2i\sum_{n=1}^{\infty}\widehat{\rho}_n(t)\sin nx = 2i\sum_{n=1}^{\infty}\sin nx\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{\xi}_n^{(N)}(t) \\ &= \sum_{N=0}^{\infty}\varepsilon^{N+1}u^{(N)}(x,t), \end{aligned} \tag{19}$$

where

$$u^{(N)}(x,t) = 2i\sum_{n=1}^{\infty} \widehat{\xi}_n^{(N)} \sin nx.$$

The interchange of the order of summations is permissible as the series are absolutely and uniformly convergent for $x \in [0,\pi], t \geq 0$ and $\varepsilon \in [0,\varepsilon_0]$. The last statement in its own turn follows from (15) with sufficiently small ε. Differentiating (16)-(17), we get, for $k = 1, 2$, that

$$\begin{aligned} \partial_k^t \widehat{\xi}_n^{(0)}(t) &= \sum_{l=0}^{k} c_k^l (-1)^l a^l e^{-at} \partial_k^{k-l} \left\{ \left[\cos(\sigma_n t) + a\frac{\sin(\sigma_n t)}{\sigma_n} \right] \Phi_n \right. \\ &\quad \left. + \frac{\sin(\sigma_n t)}{\sigma_n} \Psi_n \right\}, \\ \partial_t^k \widehat{\xi}_n^{(N)}(t) &= -\frac{\beta n^2}{\sigma_n} \int_0^t g_k(n, t-\tau) Q_N \left(\widehat{\xi}_n^{(j)}(\tau) \right) d\tau + R_k(n,t), \end{aligned}$$

where

$$g_k(n,t) = \sum_{l=0}^{k} c_k^l (-1)^l \left(a\right)^l e^{-at} \sigma_n^{k-l} \sin\left[\sigma_n t + \frac{(k-l)\pi}{2} \right],$$

$Q_N\left(\widehat{\xi}_n^{(j)}(\tau)\right)$ is defined by (17), c_k^l are binomial coefficients, and $R_k(n,t)$ are obtained by differentiating the integral in (17) and are as follows,

$$R_1(n,t) = 0, R_2(n,t) = -\beta n^2 Q_N\left(\widehat{\xi}_n^{(j)}(t)\right).$$

Hence, using (18), we deduce that for $n \geq 1, N \geq 0, t > 0$ and $k = 0, 1, 2$,

$$\left| \partial_t^k \widehat{\xi}_n^{(N)}(t) \right| \leq C^N (N+1)^{-2} n^{k-4} e^{-\eta a t},$$

$$\left| \partial_t^k \widehat{\rho}_n(t) \right| \leq C n^{k-4} e^{-\eta a t}. \tag{20}$$

Using these estimates and calculating the necessary derivatives of (19), we can prove straightforwardly that (19) represents the classical solution of the initial-boundary value problem (4)

3.2. Uniqueness of the Solution

For proving the uniqueness of the constructed solution, we shall assume that there exist two classical solutions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ of system (4) and then deduce that $u^{(1)}(x,t)$ must equal to $u^{(2)}(x,t)$.

Making an odd extension of the two solutions to the segment $(-\pi, 0]$, we notice that both of them belong to the $L^2(-\pi,\pi)$ space and according to Definition 2, we have for any fixed time $t > 0$ that

$$\max_{x\in[-\pi,\pi]} \left|u^{(1)}(x,t)\right| < C_t, \quad \max_{x\in[-\pi,\pi]} \left|u^{(2)}(x,t)\right| < C_t,$$

where C_t is a constant depending on t. Let $w(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t)$ and make an even extension of $w(x,t)$ to ... $(-3\pi,-2\pi)$, $(-2\pi,-\pi)$, $(\pi, 2\pi)$, $(2\pi, 3\pi)$, ... Then, we have from (4) that

$$\begin{aligned} w_{tt} - w_{xx} + 2aw_t + bw &= \beta[w(x,t)(u^{(1)}(x,t) + u^{(2)}(x,t))]_{xx}, \\ w(x,0) = w_t(x,t) &= 0. \end{aligned}$$

Taking the Fourie transform in the interval $(-\infty, +\infty)$, namely

$$\widehat{w}(\xi,t) = \int_{-\infty}^{+\infty} e^{-ix\xi} w(x,t)dx,$$

we have

$$\widehat{w}''(\xi,t) + 2a\widehat{w}(\xi,t) + \left(\xi^2+1\right)\widehat{w}(\xi,t) = -\beta\xi^2\widehat{f}(\xi,t), \tag{21}$$

where

$$f(x,t) = w \times (u^{(1)} + u^{(2)})(x,t).$$

From (21), we have

$$\widehat{w}(\xi,t) = -\frac{\beta\xi^2}{\sigma_\xi}\int_0^t \exp\left[-a(t-\tau)\right]\sin\left(\sigma_\xi(t-\tau)\right)\widehat{f}(\xi,\tau)d\tau,$$

where

$$\sigma_\xi = \sqrt{\xi^2 + b - a^2}.$$

Thus,

$$
\begin{aligned}
|\widehat{w}(\xi,t)| &\le C\int_0^t \left|\exp\left[-a(t-\tau)\right]\widehat{f}(\xi,\tau)\right| d\tau \\
&\le C\left[\int_0^t \exp\left[-a(t-\tau)\right] d\tau\right]^{\frac{1}{2}} \left[\int_0^t \left|\widehat{f}(\xi,\tau)\right|^2 d\tau\right]^{\frac{1}{2}} \\
&\le C\left[\int_0^t \left|\widehat{f}(\xi,\tau)\right|^2 d\tau\right]^{\frac{1}{2}}. \qquad (22)
\end{aligned}
$$

It follows from (22) and the Parseval inequality that

$$
\begin{aligned}
\int_{-\infty}^{+\infty} |\widehat{w}(\xi,t)|^2 d\xi &\le C\int_{-\infty}^{+\infty}\int_0^t \left|\widehat{f}(\xi,\tau)\right|^2 d\tau d\xi \\
&\le C\int_0^t \left\|\widehat{f}(\xi,t)\right\|_{L^2}^2 d\tau \\
&\le C\int_0^t \left\|w(x,\tau)\left(u^{(1)}(x,\tau)+u^{(2)}(x,\tau)\right)\right\|_{L^2}^2 d\tau \\
&\le C\int_0^t C_\tau \left\|w(x,\tau)\right\|_{L^2}^2 d\tau.
\end{aligned}
$$

Using the Growall inequality, we obtain

$$w(x,t) = 0 \quad (\text{in } L^2),$$

which implies that $u^{(1)}(x,t) \equiv u^{(2)}(x,t)$

3.3. Long Time Asymptotics

To investigate the long-time asymptotic behavior of the constructed solution, we firstly determine a subtle asymptotic estimate of $\widehat{u_1}(t)$ which will contribute to the major term and then estimate the remaining terms $\sum\limits_{n=2}^{\infty} \widehat{u_n}(t) \sin nx$.

Substituting

$$\widehat{u_1}(t)=\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{\xi_1}^{(N)}(t),$$

into (16) and (17), we obtain

$$\widehat{\xi_1}^{(0)}(t)=e^{-at}\left[A^{(0)}\cos(\sigma t)+B^{(0)}\sin(\sigma t)\right],$$

$$\widehat{\xi_1}^{(N)}(t)=e^{-at}\{\left[A^{(N)}+R_A^{(N)}(t)\right]\cos(\sigma t)+\left[B^{(N)}+R_B^{(N)}(t)\right]\sin(\sigma t)\},\quad N\geq 1,\qquad (23)$$

where

$$\begin{aligned}
A^{(0)} &= \varepsilon\widehat{\varphi_1},\\
B^{(0)} &= \frac{\varepsilon}{\sigma}\left(a\widehat{\varphi_1}+\widehat{\psi_1}\right),\\
\sigma &= \sqrt{1+b-a^2}\\
A^{(N)} &= \frac{\beta}{\sigma}\int_0^t e^{a\tau}\sin(\sigma\tau)S_{u_n}(\tau)d\tau,\\
B^{(N)} &= -\frac{\beta}{\sigma}\int_0^t e^{a\tau}\cos(\sigma\tau)S_{u_n}(\tau)d\tau,\\
R_A^{(N)} &= \frac{\beta}{\sigma}\int_0^{+\infty} e^{a\tau}\sin(\sigma\tau)S_{u_n}(\tau)d\tau,\\
R_B^{(N)} &= -\frac{\beta}{\sigma}\int_0^{+\infty} e^{a\tau}\cos(\sigma\tau)S_{u_n}(\tau)d\tau,\\
S_{u_n}(t) &= \sum_{g=1}^{\infty}\sum_{j=1}^{N}\widehat{\xi}_{1+g}^{(j-1)}(t)\widehat{\xi}_g^{(N-j)}(t),\quad N\geq 1,
\end{aligned}$$

and the functions $\widehat{\xi}^{(j)}(t), j = 0, 1, \ldots, N-1$, are defined by (16)–(17). Taking $n \geq 2$ and $N \geq 1$, it follows from (18) and (19) that there exists a positive number, $0 < \eta < \frac{1}{2}$, such that

$$\left|R_A^{(N)}(t)\right| \leq ce^{-\eta at}, \qquad \left|R_B^{(N)}(t)\right| \leq ce^{-\eta at}.$$

Hence, we have proved that as $t \longrightarrow +\infty$

$$2i\widehat{u_1}(t) = e^{-at}\left[A\cos(\sigma t) + B\sin(\sigma t)\right] + O(e^{-(1+\eta)at}),$$

$$A = 2i\sum_{N=0}^{\infty} \varepsilon^{N+1} A^{(N)}, B = 2i\sum_{N=0}^{\infty} \varepsilon^{N+1} B^{(N)}, \qquad (24)$$

where $0 < \eta < \frac{1}{2}$, $A^{(N)}$ and $B^{(N)}$ are defined by (23) and the series above converge absolutely and uniformly for $\varepsilon \in [0, \varepsilon_0]$.

Now, we can represent the solution by

$$u(x,t) = 2i\widehat{u_1}(t) + R_u(x,t),$$

$$R_u(x,t) = 2i\sum_{n=2}^{\infty} \sin nx \sum_{N=0}^{\infty} \widehat{\xi_n}^{(N)}(t). \qquad (25)$$

Using (18), we deduce that

$$| R_u(x,t) | \leq \exp[-(1+\eta)at] \sum_{n=0}^{\infty} c^N \varepsilon^{N+1} (N+1)^{-2} \sum_{N=2}^{\infty} n^{-6}$$

$$\leq c\exp[-(1+\eta)at], \qquad (26)$$

$$u(x,t) = e^{-at}[(A\cos\sigma t + B\sin\sigma t)\sin x] + O(\mathrm{e}^{-(1+\eta)at}]),$$
$$0 < \eta < \frac{1}{2}. \qquad (27)$$

It follows from (27) that the equality (6) holds.

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