When catalysis is useful for probabilistic entanglement transformation

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We determine all 2×2 quantum states that can serve as useful catalysts for a given probabilistic entanglement transformation, in the sense that they can increase the maximal transformation probability. When higher-dimensional catalysts are considered, a sufficient and necessary condition is derived under which a certain probabilistic transformation has useful catalysts.

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In the field of quantum information theory, entanglement plays an essential role in quantum information processing such as quantum cryptography [1], quantum superdense coding [2] and quantum teleportation [3]. When entanglement is treated as a type of resource, the study of transformations between different forms of entanglement becomes very crucial. It is well known that the entanglement quantity shared among separate parties cannot be increased only using local operations on the separate subsystems and classical communication between them (or LOCC for short). The restriction on the possibility of entanglement transformations that can be realized by LOCC is, however, beyond this. Nielsen proved in his brilliant work [4] that a pure bipartite entangled quantum state $|\psi_1\rangle$ can be transformed into another pure bipartite entangled state $|\psi_2\rangle$ by LOCC if and only if $\lambda_{\psi_1} \prec \lambda_{\psi_2}$, where the probability vectors λ_{ψ_1} and λ_{ψ_2} denote the Schmidt coefficient vectors of $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. Here the symbol $\prec$ stands for the "majorization relation". An n-dimensional probability vector x is said to be majorized by another n-dimensional probability vector y, denoted by $x \prec y$, if the following relations hold

$$\sum_{i=1}^{l} x_i^{\downarrow} \leq \sum_{i=1}^{l} y_i^{\downarrow} \quad \text{for any} \quad 1 \leq l < n,$$

where $x^{\downarrow}$ denotes the vector obtained by rearranging the components of x in nonincreasing order.

What Nielsen has done indeed gives a connection between the theory of majorization in linear algebra [5] and the entanglement transformation. Furthermore, since the sufficient and necessary condition is very easy to check, it is extremely useful to decide whether one pure bipartite entangled state can be transformed into another pure bipartite state by LOCC. There exist, however, incomparable states in the sense that any one cannot be transformed into another only using LOCC. To cope with the transformation between incomparable states, Vidal [6] generalized Nielsen's work with a probabilistic manner. He found that although a deterministic transformation cannot be realized between incomparable states, a probabilistic one is always possible. Furthermore, he gave an explicit expression of the maximal probability of transforming one state to another. To be more specific, let $P(|\psi\rangle \to |\phi\rangle)$ denote the maximal transformation probability of transforming $|\psi\rangle$ into $|\phi\rangle$ by LOCC; then

$$P(|\psi\rangle \to |\phi\rangle) = \min_{1 \leq l \leq n} \frac{E_l(\lambda_\psi)}{E_l(\lambda_\phi)},$$

where n is the maximum of the Schmidt numbers of $|\psi\rangle$ and $|\phi\rangle$, and $E_l(x)$ denotes the abbreviation of $\sum_{i=l}^{n} x_i^{\downarrow}$ for probability vector x.

Another interesting phenomenon was discovered by Jonathan and Plenio [7] that sometimes an entangled state can help in making impossible entanglement transformations into possible without being consumed at all. That is, there exist quantum states $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\phi\rangle$ such that $|\psi_1\rangle \nrightarrow |\psi_2\rangle$ but $|\psi_1\rangle \otimes |\phi\rangle \to |\psi_2\rangle \otimes |\phi\rangle$. In this transformation, the role of the state $|\phi\rangle$ is just like a catalyst in a chemical process. They found by examining an example that in some cases an appropriately chosen catalyst can increase the maximal transformation probability of incomparable states. It was also shown that enhancement of the maximal transformation probability is not always possible. However, there were no further results about such an interesting field in their paper.

In this paper, we examine the ability of catalysts in a probabilistic entanglement transformation. We first consider the simple case of when a given probabilistic transformation has useful 2×2 catalysts and determine all of them. Then a sufficient and necessary condition is derive which can decide whether or not a certain transformation has (not necessarily 2×2) useful catalysts.

For simplicity, in what follows we denote a quantum state by the probability vector of its Schmidt coefficients. This will not cause any confusion because it is well known that the fundamental properties of a bipartite quantum state under LOCC are completely determined by its Schmidt coefficients. Therefore, from now on, we consider only probability vectors instead of quantum states and always identify a probability vector with the quantum state represented by it.

Suppose x, y are two n-dimensional probability vectors and the components are arranged nonincreasingly. It is well known that if $P(x \to y) = E_n(x)/E_n(y)$ or $P(x \to y) = E_1(x)/E_1(y)(= 1)$, then the maximal prob-

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ability of transforming x into y cannot be increased by any catalyst. That is, for any probability vector c, we have $P(x \otimes c \to y \otimes c) = P(x \to y)$. Thus, in what follows, we assume

$$P(x \to y) < \frac{E_n(x)}{E_n(y)} \quad , \quad P(x \to y) < \frac{E_1(x)}{E_1(y)}. \tag{1}$$

Without loss of generality, we concentrate on catalysts with nonzero components, since c and $c \oplus 0$ have the same catalysis ability for any probability vector c in the sense that in any situation, if one serves as a partial catalyst for some transformation, so does the other for the same transformation. Let

$$L = \{l : 1 < l < n \ \text{ and } \ P(x \to y) = \frac{E_l(x)}{E_l(y)}\}.$$

In what follows, we derive a sufficient and necessary condition when a two-dimensional catalyst can increase the maximal transformation probability from x to y. In order to state the theorem compactly, we first denote

$$m_{r_1}^{r_2} = \min\{\frac{x_{r_1-1}}{x_{r_2}}, \frac{y_{r_1-1}}{y_{r_2}}\} \tag{2}$$

and

$$M_{r_1}^{r_2} = \max\{\frac{x_{r_1}}{x_{r_2-1}}, \frac{y_{r_1}}{y_{r_2-1}}\} \tag{3}$$

for any $r_1, r_2 \in L$. Furthermore, we let $M_{n+1}^{r_2} = 0$.

Theorem 1. The maximal probability of transforming x into y can be increased by a 2-dimensional catalyst if and only if the set

$$S = \bigcap \{(0, M_{r_1}^{r_2}) \cup (m_{r_1}^{r_2}, 1)\} \tag{4}$$

is not empty, where the intersection is taken over all pairs of r_1, r_2 such that $r_1, r_2 \in L \cup \{n+1\}$, $r_1 \geq r_2$, and $r_2 < n+1$. In fact, any two-dimensional probability vector (c_1, c_2) with $c_1 \geq c_2$ can serve as a useful catalyst for transforming x into y if and only if $c_2/c_1 \in S$.

Proof. Suppose $c = (c_1, c_2)$ is a 2-dimensional probability vector and $c_1 \geq c_2$. We need only show that c cannot serve as a useful catalyst for transforming x into y, that is,

$$P(x \otimes c \to y \otimes c) = P(x \to y)$$

if and only if there exist $r_1, r_2 \in L \cup \{n+1\}$, $r_1 \geq r_2$, and $r_2 < n+1$, such that

$$M_{r_1}^{r_2} \leq \frac{c_2}{c_1} \leq m_{r_1}^{r_2}, \tag{5}$$

where any constraint having meaningless terms is satisfied automatically.

For an arbitrarily fixed integer l satisfying $1 < l \leq 2n$, we can arrange the summands in $E_l(x \otimes c)$ as

$$E_l(x \otimes c) = c_1 \sum_{i=r_1}^{n} x_i + c_2 \sum_{i=r_2}^{n} x_i. \tag{6}$$

Here r_i, $1 \leq r_i \leq n+1$, denotes the smallest index of the components of x in the summands of $E_l(x \otimes c)$ that have the form $c_i x_j$, where $1 \leq j \leq n$. The case $r_i = n+1$ denotes that any term that has the form $c_i x_j$ does not occur. In the case of repeated values of components of $x \otimes c$, we regard terms with larger i, and larger j if they have the same i, to be included in the sum first.

From these assumptions, we can show that $r_1 \geq r_2$. Otherwise, $r_1 \leq r_2 - 1$, and from the fact $c_1 x_{r_1}$ is in the summands of $E_l(x \otimes c)$ while $c_2 x_{r_2-1}$ is not, we can deduce that $c_1 x_{r_1} \leq c_2 x_{r_2-1}$. But on the other hand, we have $c_1 \geq c_2$ and $x_{r_1} \geq x_{r_2-1}$. So it follows that $c_1 = c_2$ and $x_{r_1} = x_{r_2-1}$. Especially, $c_1 x_{r_2-1} = c_2 x_{r_2-1}$ which contradicts our assumption that the term with larger i is included in $E_l(x \otimes c)$ first, since the former is while the latter is not included in s_x. Furthermore, from Eq. (6) we have $r_1 + r_2 = l+1$; then, $r_2 < n+1$ since $1 < l \leq 2n$.

Now, by the definition of $E_l(y \otimes c)$ and $P(x \to y)$, the following inequality is easy to check:

$$\begin{aligned} \frac{E_l(x \otimes c)}{E_l(y \otimes c)} &\geq \frac{c_1 \sum_{i=r_1}^{n} x_i + c_2 \sum_{i=r_2}^{n} x_i}{c_1 \sum_{i=r_1}^{n} y_i + c_2 \sum_{i=r_2}^{n} y_i} \\ &\geq \frac{P(x \to y)(c_1 \sum_{i=r_1}^{n} y_i + c_2 \sum_{i=r_2}^{n} y_i)}{c_1 \sum_{i=r_1}^{n} y_i + c_2 \sum_{i=r_2}^{n} y_i} \\ &= P(x \to y). \end{aligned} \tag{7}$$

The first equality holds if and only if $E_l(y \otimes c) = c_1 \sum_{i=r_1}^{n} y_i + c_2 \sum_{i=r_2}^{n} y_i$ while the second equality holds if and only if r_1 and r_2 are both included in L or, $r_2 \in L$ and $r_1 = n+1$. Notice that l and r_i can be uniquely determined by each other from Eq. (6); it follows that the sufficient and necessary condition of when $P(x \otimes c \to y \otimes c) = P(x \to y)$ is there exist $r_1, r_2 \in L \cup \{n+1\}$ satisfying $r_1 \geq r_2$ and $r_2 < n+1$, such that

$$E_l(x \otimes c) = c_1 \sum_{i=r_1}^{n} x_i + c_2 \sum_{i=r_2}^{n} x_i \tag{8}$$

and

$$E_l(y \otimes c) = c_1 \sum_{i=r_1}^{n} y_i + c_2 \sum_{i=r_2}^{n} y_i. \tag{9}$$

In what follows, we derive the conditions presented in Eq. (5) from Eqs.(8) and (9). In fact, what Eq. (8) says is simply that $c_1 x_{r_1-1} \geq c_2 x_{r_2}$ and $c_2 x_{r_2-1} \geq c_1 x_{r_1}$ or, equivalently,

$$\frac{x_{r_1}}{x_{r_2-1}} \leq \frac{c_2}{c_1} \leq \frac{x_{r_1-1}}{x_{r_2}}. \tag{10}$$

The special case when r_1 takes value $n+1$ can be included in Eq. (10) simply by assuming that the constraints in Eq. (10) containing meaningless terms are automatically satisfied. Analogously, we can show that Eq. (9) is equivalent to

$$\frac{y_{r_1}}{y_{r_2-1}} \le \frac{c_2}{c_1} \le \frac{y_{r_1-1}}{y_{r_2}}. \quad (11)$$

Combining Eqs.(10) and (11) together and noticing the denotations in Eqs.(2) and (3), we derive the sufficient and necessary condition for two-dimensional probability vector c such that $P(x \otimes c \to y \otimes c) = P(x \to y)$ is just what Eq. (5) presents. That completes our proof. □

A special and perhaps more interesting case of the above theorem is when the number of elements in L is 1, that is $L = \{l\}$ for some $1 < l < n$. In this case, the possible values of the pair (r_1, r_2) are just (l, l) and $(n+1, l)$. So the set S in Eq. (4) is simply $(0, M_l^l) \cap (m_{n+1}^l, 1)$ and the sufficient and necessary condition of when two-dimensional catalysts exist which can increase the maximal probability of transforming x into y is $m_{n+1}^l < M_l^l$, that is,

$$\min\{\frac{x_n}{x_l}, \frac{y_n}{y_l}\} < \max\{\frac{x_l}{x_{l-1}}, \frac{y_l}{y_{l-1}}\}. \quad (12)$$

Furthermore, any two-dimensional probability vector c, $c_1 \ge c_2$, which satisfies

$$\min\{\frac{x_n}{x_l}, \frac{y_n}{y_l}\} < \frac{c_2}{c_1} < \max\{\frac{x_l}{x_{l-1}}, \frac{y_l}{y_{l-1}}\}, \quad (13)$$

can be a useful catalyst for this transformation. In the simplest case of $n = 3$ (notice that when $n = 2$, any entanglement transformation has no catalyst), the set L must be $\{2\}$ and furthermore, from the assumption Eq. (1) we have $x_3/x_2 > y_3/y_2$ and $x_2/x_1 < y_2/y_1$. It follows that when x and y are both three-dimensional, the sufficient and necessary condition for them to have a useful two-dimensional catalyst is

$$\frac{y_3}{y_2} < \frac{y_2}{y_1} \quad (14)$$

and any $c = (c_1, c_2)$ with $c_1 \ge c_2$ and

$$\frac{y_3}{y_2} < \frac{c_2}{c_1} < \frac{y_2}{y_1} \quad (15)$$

can increase the maximal transformation probability. Note that these conditions are all irrelevant to x.

To illustrate the utility of the above theorem, let us give some simple examples.

Example 1. This example is given by Jonathan and Plenio in [7]. Let $x = (0.6, 0.2, 0.2)$ and $y = (0.5, 0.4, 0.1)$, we have $y_3/y_2 = 0.25$ and $y_2/y_1 = 0.8$. So from Eq. (15), any state $c = (c_1, c_2)$, $c_1 \ge c_2$, can serve as a useful catalyst for transforming x into y, provided that $0.25 < c_2/c_1 < 0.8$ or, equivalently, $5/9 < c_1 < 4/5$. Especially, when choosing $c_1 = 0.65$, we get $c = (0.65, 0.35)$, which is the one given in [7].

Suppose x is just as above while $y = (0.5, 0.3, 0.2)$; then, $y_3/y_2 = 2/3$ and $y_2/y_1 = 0.6$. Since $0.6 < 2/3$, we deduce that any two-dimensional state cannot serve as a useful catalyst for the probabilistic transformation from x to y in the sense that it cannot increase the maximal transformation probability.

Example 2. This well-known example is exactly the original one that Jonathan and Plenio used to demonstrate entanglement catalysis [7]. Let $x = (0.4, 0.4, 0.1, 0.1)$ and $y = (0.5, 0.25, 0.25, 0)$; then $L = \{3\}$, and from (12) we have

$$\min\{\frac{x_4}{x_3}, \frac{y_4}{y_3}\} = \min\{1, 0\} = 0$$

and

$$\max\{\frac{x_3}{x_2}, \frac{y_3}{y_2}\} = \max\{0.25, 1\} = 1.$$

It follows from Eq. (13) that any state $c = (c_1, c_2)$, $0 < c_2/c_1 < 1$, can serve as a useful catalyst. That is, any two-dimensional nonpure and nonuniform state can increase the maximal transformation probability from x to y.

We have examined when there exists a two-dimensional catalyst which is useful for probabilistic transformation. In what follows, we consider the case of higher-dimensional catalysts and derive a sufficient and necessary condition for a certain probabilistic transformation to have a useful (not necessarily two-dimensional) catalyst. More important, the proof process indeed constructs an appropriate catalyst explicitly. Some techniques in the proof are from Lemma 4 in [8].

Theorem 2. Suppose x and y are two n-dimensional probability vectors with the components ordered nonincreasingly. Then there exists a probability vector c such that $P(x \otimes c \to y \otimes c) > P(x \to y)$ if and only if

$$P(x \to y) < \min\{x_n/y_n, 1\}.$$

Proof. The "only if" part is easy and we omit the details here. The proof of "if" is as follows.

We denote $P(x \to y)$ as P for simplicity in this proof. Let h, $1 \le h < n$, be the smallest index of the components such that $x_h/y_h \ne P$ and α be a positive real number such that $Py_n/x_n < \alpha < 1$. Furthermore, if $P > x_h/y_h$, then assume $\alpha > x_h/(Py_h)$; otherwise, assume $\alpha > Py_h/x_h$. Let k be a positive integer such that $x_n > x_h\alpha^{k-1}$ and

$$c = (1, \alpha, \alpha^2, \dots, \alpha^{k-1}).$$

We omit the normalization of c here. In what follows, we show that for any $1 < l < nk$, $E_l(x \otimes c) > PE_l(y \otimes c)$; then, the catalyst we constructed above indeed increases the maximal transformation probability.

Fix l as an arbitrary integer that satisfies $1 < l < nk$ and denote $s_x = E_l(x \otimes c)$. It is obvious that we can arrange the summands in s_x as

$$s_x = \sum_{i=1}^{n} \sum_{j=r_i}^{k-1} x_i \alpha^j.$$

Here r_i, $0 \le r_i \le k$, denotes the minimal power of α of the terms in the summands of s_x that have the form $x_i\alpha^j$, where $1 \le i \le n$ and $0 \le j \le k-1$. The case $r_i = k$ denotes that any term that has the form $x_i\alpha^j$ does not occur. Again, we regard terms with larger i to be included in the sum first in the case of repeated values of components of $x \otimes c$. Consider the sum

$$s_y = P \sum_{i=1}^{n} \sum_{j=r_i}^{k-1} y_i \alpha^j.$$

It is obvious that $s_y \ge PE_l(y \otimes c)$ by definition. On the other hand, we can rearrange the summands of s_x and s_y ,respectively, as

$$s_x = \sum_{j=1}^{k-1} \alpha^j \sum_{i=t_j}^{n} x_i \quad \text{and} \quad s_y = \sum_{j=1}^{k-1} \alpha^j P \sum_{i=t_j}^{n} y_i,$$

where $1 \le t_j \le n+1$. Since

$$\sum_{i=t_j}^{n} x_i = E_{t_j}(x) \ge PE_{t_j}(y) = P \sum_{i=t_j}^{n} y_i$$

by the definition of P, we have $s_x \ge s_y$. Now, if $s_x > s_y$, then $s_x > PE_l(y \otimes c)$. In the case of $s_x = s_y$, let m_y denote the minimum of the components of $Py \otimes c$ not included in s_y and M_y denote the maximum included in s_y. If we can prove $m_y < M_y$, then by swapping m_y and M_y (that is, including m_y to and excluding M_y from s_y), we can show $s_x > PE_l(y \otimes c)$ again and that will complete the proof of this theorem.

In what follows, we prove that in the assumption of $s_x = s_y$, $m_y < M_y$ holds. Suppose on the contrary $m_y \ge M_y$; then, $s_y = PE_l(y \otimes c)$. It is then not difficult to show $m_x \ge m_y$ and $M_x \le M_y$, where m_x and M_x are defined analogously to m_y and M_y, since $m_x < m_y$ leads to

$$E_{l-1}(x \otimes c) = s_x + m_x < s_y + m_y = PE_{l-1}(y \otimes c)$$

and $M_x > M_y$ leads to

$$E_{l+1}(x \otimes c) = s_x - M_x < s_y - M_y = PE_{l+1}(y \otimes c),$$

which contradict the well-known fact that $P(x \otimes c \to y \otimes c) \ge P(x \to y)$.

Now, we show that a contradiction will arise by considering the following two cases.

Case 1: $r_n > 0$. Since $1 < l < nk$, we have $r_n < k$. Then

$$M_y \ge M_x \ge x_n \alpha^{r_n} > P y_n \alpha^{r_n - 1} \ge m_y$$

since $Py_n/x_n < \alpha$. Thus $M_y > m_y$, which is a contradiction.

Case 2: $r_n = 0$. In this case, $x_n \le M_x$ since x_n is included in s_x. By definition of α we have $x_h\alpha^{k-1} < M_x$ and so $0 \le r_h \le k-1$. Furthermore, we can prove that $r_h > 0$ since otherwise x_h is in the summands of s_x and any terms not included are of the form $x_i\alpha^j$ where $1 \le i < h$. By the definition of h, we have $x_i\alpha^j = Py_i\alpha^j$ for $1 \le i < h$. So the sum of the components of $x \otimes c$ not included in s_x is equal to the sum of the components of $Py \otimes c$ not included in s_y. This fact, together with the assumption that $s_x = x_y$ will lead to a contradiction that

$$1 = E_1(x \otimes c) = PE_1(y \otimes c) = P.$$

Now, if $P > x_h/y_h$, then

$$m_y \le m_x \le x_h \alpha^{r_h - 1} < P y_h \alpha^{r_h} \le M_y$$

from our assumption that $\alpha > x_h/(Py_h)$, and if $P < x_h/y_h$, then

$$M_y \ge M_x \ge x_h \alpha^{r_h} > P y_h \alpha^{r_h - 1} \ge m_y$$

from our assumption that $\alpha > Py_h/x_h$. Again, $m_y \ge M_y$ is contradicted. That completes our proof. □

Recall that in Example 1, when $x = (0.6, 0.2, 0.2)$ and $y = (0.5, 0.3, 0.2)$, there exists no two-dimensional useful catalyst for the probabilistic transformation from x to y. We show here how to construct a higher-dimensional one by the above theorem. It is easy to check that

$$P(x \to y) = 0.8 < \frac{0.6}{0.5} = \frac{x_1}{y_1},$$

so $h = 1$ and we need only take a real α such that $Py_3/x_3 = 0.8 < \alpha < 1$ and $\alpha > Py_1/x_1 = 2/3$, that is, $0.8 < \alpha < 1$. In order not to make k too large, we should take α as small as possible. For example, $\alpha = 0.801$. Then, from the constraint $x_3 > x_1\alpha^{k-1}$ in the theorem, we have $k \ge 6$. Thus the state

$$c = (1, \alpha, \ldots, \alpha^5)$$

can increase the maximal transformation probability of x into y.

The above theorem gives us a sufficient and necessary condition under which the transformation x into y has a catalyst which can increase the maximal probability transformation. Furthermore, the proof process constructs a real catalyst vector. What we should like to point out here is, however, that the catalyst presented in the proof is not very economical in the sense that it is usually not the minimally dimensional one among all states which can serve as a useful catalyst. How to find a most economical one remains for further study.

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