OXFORD UNIVERSITY PRESS Cambridge Journal of Economics

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Journal:	*Cambridge Journal of Economics*
Manuscript ID:	Draft
Manuscript Type:	Article
CJE Keywords:	risk, uncertainty, de Finetti, Keynes, Knight
JEL Classification:	B21, D81

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Abstract

The well-known Knightian distinction between quantifiable risk and unquantifiable uncertainty is at odds with the dominant subjectivist conception of probability associated with de Finetti, Ramsey and Savage. Risk and uncertainty are rendered indistinguishable on the subjectivist approach insofar as an individual's subjective estimate of the probability of any event can be elicited from the odds at which she would be prepared to bet for or against that event. The risk/uncertainty distinction has however never quite gone away and is currently under renewed theoretical scrutiny. The purpose of this paper is to show that de Finetti's understanding of the distinction is more nuanced than is usually admitted. Relying on usually overlooked excerpts of de Finetti's works commenting on Keynes, Knight, and interval valued probabilities, we argue that de Finetti suggested a relevant theoretical case for uncertainty to hold even when individuals are endowed with subjective probabilities. Indeed, de Finetti admitted that the distinction between risk and uncertainty is relevant when different people sensibly disagree about the probability of the occurrence of an event. We conclude that the received interpretation of de Finetti's understanding of subjective probability is inaccurate and shows that de Finetti's peculiar notion of uncertainty pervades the current literature on source uncertainty.

Probability is our *guide*
when we *think* and *act* in conditions of *uncertainty*,
and uncertainty is *ubiquitous*

Bruno de Finetti, 1976, p. 293

1. Introduction

The idea that there are uncertainties that cannot be reduced to numerically definite probabilities, once regularly denied in the mainstream economics literature dominated by the standard model of decision theory, has become quite commonplace. The subjective expected utility model represented by Savage's (1954) axiomatization of the subjectivist interpretation of probability championed by Ramsey (1931) and de Finetti (1937) is now widely questioned, partly due to experimental findings in the wake of Kahneman and Tversky's (1979) pathbreaking research, but also to theoretical concerns about subjectivists' denials that the quality of information available to decision makers may affect their ability to arrive at numerical probabilities (Schmeidler 1989). Knightian uncertainty — a term denied of any significance on normative grounds since Arrow (1951) and kept alive only in the writings of heterodox scholars (Lawson 1985) — is now back in vogue, even in technical papers on other topics including financial markets (Epstein and Wang 1994), incomplete contracts (Mukerjii 1998) and general equilibrium theory (Rigotti and Shannon 2005). In wake of this, mentions of uncertainty as distinct from risk have become routine in dictionaries, review articles and graduate textbooks (Eichberger and Kelsey 2007; Wakker 2008; Gilboa 2009).

Thanks to the experimental literature, it is now widely accepted that peoples' beliefs about uncertain events are typically vague and ill defined, and that theoretical models of risk behaviour in simple gambling situations are a poor guide to what people do when assessing uncertainty in real world tasks (Camerer and Weber 1992). Experimental work on insurance markets, much of which alludes to Knight's distinction between risk and uncertainty, has been unequivocal. Contrary to anecdotal evidence about Lloyd's of London being ready to offer insurance against the event of the Loch Ness Monster being captured – supposedly proving the subjectivist contention that any event can be attributed a probability (Borch 1976) – experiments have shown that insurance premia are

higher when there is ambiguity about loss probabilities (Hogarth and Kunreuther 1989; Cabantous 2007). In a similar vein, field surveys of actuaries asked to set premia for specific scenarios show that they anchor their choices on expected values, but then adjust the recommended price upward, in an apparently idiosyncratic manner, to confront the perceived ambiguity in probabilities (Hogarth *et al*. 1995). The well-known reluctance of insurance firms to insure 'new' risks like environmental catastrophes and pandemic diseases, for which no historical frequencies are available, is usually justified in term of uncertainties that are not risks (Swiss RE 2005), and reports of insurance firms confirm that 'unknown probabilities' are a crucial issue among practitioners (Taylor and Shipley 2009). Mostly as a result of this overwhelming empirical evidence, the investigation of the relevance of uncertainty in economics, long considered commonplace in economics discourse, but dismissed in theoretical investigations, is now back on the agenda.

Our aim in this paper is to assess the role in this story of the Italian mathematician Bruno de Finetti, who, together with Frank Ramsey, is credited as one of the two independent founders of the subjectivist approach to probability (de Finetti 1930, 1931a, 1931b). Working in Italian, de Finetti showed how an individual's degrees of belief about uncertain events can be rendered numerically determinate and to conform to the conventional rules of the probability calculus, by imposing the requirement of 'coherence' — better known as the Dutch book argument — for the consistency of subjective degrees of belief. In addition to this, he replaced the objectivist notion of independent events with that of 'exchangeable' events (de Finetti 1937), which provided the bridge between the subjective theory of probability and statistical inference and without which the subjective approach may have remained 'pretty much a philosophical curiosity' (Kyburg and Smokler 1964, p. 12). De Finetti's role in the development of the subjectivist-Bayesian perspective, relatively unknown for years outside the circles of philosopher statisticians, became universally accepted after Savage's acknowledgement that his own theory of the foundations of statistics 'derived mainly from the work of Bruno de Finetti' (Savage 1954, pp. 3-4).

De Finetti is famous within the subjectivist-Bayesian camp for the strictness of his views.

Part of this reputation stems from his strong denial of any role for objective elements in both the theory and the practice of statistics (Jeffrey 1989, Mongin 1997).[1] As is well-known, his main treatise, *Theory of Probability,* opens with the iconic claim 'probability does not exist' (de Finetti 1975, p. x), by which he meant that probability exists only subjectively in the minds of individuals, a point made from the very beginning of his investigation on the philosophy of probability (de Finetti 1931b, p. 295). But his reputation for strictness is also connected with his operational definition of probability, and it is this aspect of his work that will concern us here. On de Finetti's theory probability is defined in terms of betting quotients, derived as the ratio between the sum of money an individual would be ready to bet on the occurrence of a certain event in exchange for a given prize and the prize itself. Further, this notion of probability is intended as an instrument for behaving under uncertainty, to be applied to choices related to any kind of event, from the tossing of a coin to the score of a soccer match. The connection between probability and decision theory, and the implication that one can make a decision in any instance, was later made precise by Savage (1954). Savage provided the axiomatic structure for the joint consideration of probability and utility, and derived the representation theorem that extended von Neumann-Morgenstern's expected utility to subjective probabilities. Savage's model went to become the standard in mainstream economic theory, which, at least until recently, led to the once received view that the distinction between risk and uncertainty had been rendered redundant (Hirshleifer and Riley 1992).

The principal point we make in this paper is that de Finetti's position on the notion of uncertainty is rather less strict than is commonly supposed. On the basis of generally overlooked parts of his vast contribution to economics and statistics, all of it published in Italian and most of it yet to be translated, we show that he considered the possibility of distinguishing between uncertainty and risk. In a recent paper examining de Finetti's understanding of insurance, we (BLINDED) show that, although he rejected Knight's sharp distinction between risk and uncertainty, he accepted that uncertainty may come in degrees and that there may be cases in which

[1] The International Conference commemorating the 100th anniversary of his birthday in 2006 was entitled 'Bruno de Finetti: Radical Probabilist' (Galavotti 2009).

insurability is not guaranteed. Our aim in the present paper is to go a step further and show that this position pervades much of de Finetti's work. In particular we will argue that it is reflected in a long-standing interest in a qualitative notion of probability that was already apparent in his discussion of Keynes's non-numerical probabilities in 1938 and elaborated in his 1960s comments on introducing of interval valued probabilities in a Bayesian context. It is our contention that these aspects of his work show that the traditional interpretation of de Finetti as the champion of a strictly subjective approach must be qualified. Further, we will argue that de Finetti's theoretical case for a distinction between risk and uncertainty within the subjectivist approach is relevant to a key issue currently under discussion in the modern literature on ambiguity and uncertainty, namely the effect of disagreement among experts (Abdellaoui *et al*. 2010, and Cabantous *et al*. 2010).

The paper is organized as follows. Section 2 begins with an overview of the risk-uncertainty distinction as reflected in the writings of Knight and Keynes. While these two authors represent different philosophical traditions, we argue that they distinguish between decision-making under risk and decision-making under uncertainty in a broadly similar way, and that it is therefore legitimate to speak of Knightian and Keynesian uncertainty in the same breath. Section 3 explains the rationale for de Finetti's probabilistic conception and shows why his work is often invoked in support of claims to the effect that the distinction between risk and uncertainty is theoretically meaningless. Section 4 argues that, contrary to the received view, de Finetti's commentary on what Knight and Keynes had to say about uncertainty reveals a theoretical case for uncertainty within the subjectivist approach. We make this case both on the basis of de Finetti's early commentary on Keynes's notion of non-numerical probability (de Finetti 1938) and his observations on the significance of Knight's distinction in insurance markets (de Finetti 1967a). Section 5 discusses the notion of interval probabilities from a subjectivist perspective, a theme de Finetti tackled in a long joint essay with Savage written in Italian (de Finetti and Savage 1962), and shows that de Finetti's taxonomy of the subjective approaches to probability includes more permissive variations on the strict version he is usually associated with. Section 6 concludes with some observations on the

relevance of de Finetti's understanding of uncertainty for modern decision theory.

2. Knight and Keynes on uncertainty and probability

Frank Knight and J.M. Keynes are the two economists most famous for putting the notion of uncertainty at the heart of their respective schemes. We begin with Knight because he is probably best remembered for providing a reasonably unambiguous and widely accepted distinction between risk and uncertainty. Keynes's *A Treatise on Probability* is also of major interest, however, as it presents the rationale for his insistence on uncertainty in his subsequent works in economics. The *Treatise* provides a sound probabilistic perspective on the issue at stake, something not attempted by Knight. Our main aim in this section is to point out that, notwithstanding two different philosophical approaches to probabilities, Knight and Keynes had remarkably similar views about when numerically definite probabilities can be determined and when they cannot.

In his *Risk, Uncertainty and Profit*, Knight (1921, pp. 224-225) distinguishes between risk and uncertainty on the basis of a taxonomy of 'probability situations'. Situations of 'risk' are ones in which it is possible to calculate numerically definite probabilities. According to Knight there are two possibilities here. The first is where probabilities can be calculated on the basis of general principles, namely by starting out with equal probabilities as in fair games of chance ('*a priori* probability'). The second is where it is possible to identify classes of more or less homogeneous trials on the basis of which relative frequencies can be determined empirically ('statistical' probability). Situations of 'uncertainty' are ones in which neither of these two avenues is available. In these cases, according to Knight, only 'estimates' can be formulated. Estimates are described as a 'third type of probability judgement' or 'somewhat like a probability judgement' (Knight 1921, p. 233), but they are not made the subject matter of accurate probabilistic inquiry, since 'the processes of intuition or judgement, being unconscious, are inaccessible to study' (Knight 1921, p. 230).

In the process of explaining his distinction Knight discusses insurance and maintains that private insurance markets fail to cover 'uncertain' contingencies. Indeed his discussion is so closely

related to the notion of insurability that it is sometimes argued that his distinction between risk and uncertainty is actually between situations in which insurance markets exist and situations in which they do not (Schumpeter 1954). In any event, it is in his analysis of insurance that he makes the point about uncertainty concerning events for which no (objective) probability is available. According to Knight, only 'an uncertainty which can by any method be reduced to an objective, quantitatively determinate probability, can be reduced to complete certainty by grouping cases' (Knight 1921, pp. 231-32). Insurance activity is, in fact, 'an illustration of the principle of eliminating uncertainty by dealing with groups of cases instead of individual cases' (Knight 1921, p. 245). But insurance strictly depends 'upon the measurement of probability on the basis of a fairly accurate grouping into classes', and it is thus impossible to provide insurance when the events to be insured against 'are far too unique, generally speaking, for any sort of statistical tabulation to have any value for guidance' (Knight 1921, p. 246). Knight notices that the 'unusual forms of policies issued by some of the Lloyd's underwriters' when insuring the loss of ships at sea or the destruction of crops by storm, do not conform to his definition. He argues that insurance is offered in these cases possibly on the grounds of a 'certain vague grouping of cases on the basis of intuition or judgement', but concludes that, in extreme cases like the insurance offered to a business for whatever reason 'concerned that a royal coronation will take place as scheduled', almost 'pure guesswork' substitutes for '"scientific" rate-making' (Knight 1921, p. 250).[2]

In summary, Knight's distinction between risk and uncertainty boils down to a distinction between situations in which probability calculus can be applied and situations in which it cannot. In the latter case individual agents formulate estimates, which do not have the status of a probability measure, as there is nothing 'objective' in their determination. Obviously, the subjectivist might respond by saying that estimates can indeed conform to probability calculus (LeRoy and Singell 1987), since probability is not an objective attribute of external events, but a property of the way individuals think about them. But as we will see, it can equally be argued that Knight's estimates

[2] Knight (1921, p. 250) argues that 'it is notorious that such policies costly much more that they should' and that 'insurance does not take care of the whole risk'.

do not necessarily coincide with subjective probabilities even from an epistemic viewpoint. This brings us to the alternative approach to epistemic probability put forward by Keynes.

Keynes's own understanding of uncertainty is informed by his specialist contribution to the foundations of probability *A Treatise on Probability* first published in 1921.[3] The book is explicitly aimed at reviving the epistemic approach, and it is for this reason that Keynes takes pains to interpret probability as something different from chance or frequency.[4] Instead, Keynes (1921, p. 4) treats probability as a measure of the strength of the logical relation between two propositions, namely a conclusion and the evidence for it: 'the terms *certain* and *probable* describe the various degrees of rational belief about a proposition which different amount of knowledge authorise us to entertain'. [5] The subject matter of the theory of probability, as a result, is defined as the relation between two sets of propositions in virtue of which 'if we know the first, we can attach to the latter some degree of rational belief' (Keynes 1921, pp. 6-7).

The peculiarity of Keynes's epistemic approach is that the probability of a conclusion given certain evidence corresponds to the degree of belief that is 'rational' to hold, and that this probability is 'objective' insofar as it corresponds to what can be logically deduced from that evidence. Keynes was particularly interested in identifying principles of inductive rationality entailing that different individuals sharing the same evidence are constrained by these principles to agree on definite probability judgements. This aspect of the *Treatise* was harshly criticized by Ramsey (1931) and, after the emergence of the subjectivist approach, became a minority viewpoint

[3] While this work has sometimes been taken to be devoid of clear cut implications for decision theory (Braithwaite 1973), it does contain a chapter on the 'application of probability to conduct' in which Keynes provides a critique of the 'Benthamite calculus' and which informs his later work in economics (Runde 1994).

[4] Keynes (1921, p. 103) argues that 'the identification of probability with statistical frequency is a very grave departure from the established use of words; for it clearly excludes a great number of judgements which are generally believed to deal with probability'. As we shall see later, de Finetti attributed much importance to this aspect of Keynes's approach. On the historical evolution of the epistemic approach see Hacking (1975).

[5] If H is the conclusion of an argument and E is a set of premises, then $p = H/E$ represents the degree of rational belief that the probability relation between H and E justifies. If H is a logical consequence of E, that is, if the conclusion follows directly from the premises, then $p = 1$. If not-H is a logical consequence of E, then $p = 0$. In cases that fall in between these two extremes, where E provides some but not conclusive grounds for believing (or disbelieving) H, p lies somewhere between 0 and 1.

in discussions of the philosophical foundations of probability (Kyburg and Smokler 1964). Nevertheless, Keynes's theory remains relevant for its discussion of the potential of the epistemic approach as opposed to the frequency approach. Of particular relevance for our purposes is Keynes's rejection of the idea that probabilities can always be given a numerical representation, an idea that is implied by the definition of frequency probability as the ratio of favourable to total number of cases, and to which subjectivists are committed through the Dutch book argument. Before tackling the formal logic of his own system of probability, Keynes (1921, p. 21) is therefore keen to restrict the range of applicability of probability theory, and argues against the generally accepted opinion that 'a numerical comparison between the degrees of any pair of probabilities is not only conceivable but it is actually within our power'.[6]

However, Keynes's (1921, p. 29) point that in most instances 'no exercise of the practical judgement is possible, by which a numerical value can actually be given to the probability of every argument' does not imply that probabilistic reasoning is of no use in such cases. He offers an ordinal notion of probability here, and allows that 'non-numerical' or 'numerically indeterminate' probabilities are sometimes comparable in qualitative terms. He suggests there may be cases in which one can argue only that the degree of rational belief in a proposition is either equal to, greater than, or less than that in another proposition. But he also believes there may be cases in which pairs of probability relations may not even be comparable in qualitative terms. He concludes that a numerical comparison of probability is not the general case as contended by frequentists, but a special one.[7]

It will be evident that these ideas are strongly reminiscent of Knight's distinction between risk and uncertainty. Situations in which numerical probabilities are available would be ones of risk, while situations in which at best a non-numerical representation is available would be ones of uncertainty. Like Knight, Keynes thinks of numerical probabilities as restricted to chance-based set-

[6] Keynes held that, in general, degrees of belief can be measured numerically only when it is possible to apply the principle of indifference and proceed on the basis of equal probabilities, or when it is possible to estimate statistical frequencies.

[7] For a presentation of Keynes's probability theory see Gillies (2000).

ups, that is, where they can be calculated on the basis of equal probabilities as in fair games of chance, or when it is possible to determine statistical frequencies on the basis of more or less homogenous trials (Runde 2001). Outside these instances either there is no probability or probability cannot be numerically determined.

However, unlike Knight, Keynes attempted to provide a mathematical structure for these non-numerical probability values. In the second part of the *Treatise* Keynes tries to give a meaning to a numerical measure of a relation of probability through 'numerical approximation, that is to say, the relating of probabilities, which are not themselves numerical, to probabilities, which are numerical':

> many probabilities, which are incapable of numerical measurement, can be placed nevertheless *between* numerical limits. And by taking particular non-numerical probabilities as standards a great number of numerical comparison or appropriate measurements become possible' (Keynes 1921, p.176).

Keynes clearly points to inexact numerical comparison rather than simply to the impossibility of attributing cardinal numbers and deriving probability comparisons (Brady 1993; Basili and Zappia 2009). His attempt to develop what he called a 'systematic method of approximation' was later taken up by Koopman (1940), who provided an axiomatization of Keynes's ideas by introducing interval-valued probabilities. While the modern treatment of imprecise probabilities that followed (Good 1952; Smith 1961) did not adopt the logical interpretation of probability, the mathematical models developed draw heavily on Keynes's intuition (Walley 1991).

Keynes's position on non-numerical probabilities is therefore highly reminiscent of Knight. And like Knight, he refers to the practice of underwriters and maintains that the practice of Lloyd's of London does not imply that 'underwriters are actually willing … to name a numerical measure in every case, and to back their opinion with money', only 'that many probabilities are greater or less than some numerical measure, not that they themselves are numerically definite' (Keynes 1921, 23). That there is no rational basis for naming a premium attached to an idiosyncratic risk is made clear,

in Keynes's view, by the fact that different brokers usually offer different premia even on the basis of the same evidence, and that terms offered on a policy usually vary in relation to the number of applicants. Risks that are properly insurable, either because probabilities can be known or because it is possible to make a book which covers all possibilities, are distinguished by underwriters themselves from risks which cannot be dealt in the same way and 'cannot form the basis of a regular business of insurance, – although an occasional gamble may be indulged in' (Keynes 1921, p. 25). Keynes concludes that 'the practice of underwriters weakens rather than supports the contention that all probabilities can be measured and estimated numerically'.

3. De Finetti and the standard subjectivist viewpoint

This section introduces the subjective interpretation of probability, starting from de Finetti's 1937 definition, and explains the standard rationale for rejecting the distinction between risk and uncertainty as meaningless. As both Knight and Keynes referred to the practice of underwriters to support their arguments, we also show how de Finetti's approach provides the theoretical justification for the willingness of insurance firms to insure against unique events.

The main argument supporting the subjectivist's contention that a numerical probability can be derived in every instance is best illustrated by de Finetti's claim that the probability of an event, or proposition, simply represents an individual's degree of belief in that event or proposition.[8] De Finetti (1937, p. 111) argues that the subjective notion of probability is very close to that of 'the man in the street', that is, the one that is usually applied 'every day in practical judgements'. Subjective probability theory simply makes mathematically precise the 'trivial and obvious idea that the degree of probability attributed by an individual to a given event is revealed by the conditions under which he would be disposed to bet on that event' (de Finetti, 1937, p. 101). Examining the behaviour of a decision-maker who is 'obliged to evaluate' the exchange between a certain amount

[8] Ramsey's 1926 talk on 'Truth and Probability', published only after his death in 1931, introduces an assessment of probabilities that is similar to de Finetti's in both the derivation of degrees of belief from betting behaviour and the introduction of the notion of coherent behaviour. However de Finetti developed his subjectivist viewpoint in a series of lecture notes and papers in Italian in the late 1920s and early 1930s without any knowledge of Ramsey's work (in particular see de Finetti 1930, 1931a, and 1931b).

of money dependent on the occurrence of an event and a certain amount for sure, her personal belief p in the occurrence of a certain event can be measured by finding the highest (lowest) utility value P that she is willing to pay (accept) for a ticket that pays a positive (negative) amount S, if the event occurs, and 0, if it does not. A decision-maker who chooses a certain amount P signals her reservation price for the bet, that is, she must be indifferent between P and the gamble. Then from $P = pS$ ($-P = -pS$) the probability p can be elicited as $p = P/S$ ($p = -P/-S$).[9]

Degrees of belief are thus an expression of the decision-maker's disposition to make certain kind of choices in precisely defined choice situations. The suggested procedure of measurement makes it possible 'to transform our degree of uncertainty into a determination of a number' (de Finetti 1931b, p. 302), and even in cases in which the given event is unique. Indeed de Finetti (1937, p. 102) is explicit that 'an event is always a singular fact', and that the allegedly similar trials referred to in classical probability theory are in fact each distinct from a subjectivist viewpoint.

Since the probability is subjective to the decision-maker and not a property of the event to which it is assigned, it may well be that different individuals assign different subjective probabilities to the same event. However, de Finetti emphasises that subjective does not mean arbitrary, since subjective probabilities have to be 'coherent' in the sense that they are such that they would not make it possible to engage in bets with the decision-maker that would result in a certain gain. De Finetti's main contribution (1937, pp. 103-109) is to prove that the condition of coherence 'constitutes the sole principle from which one can deduce the whole calculus of probability'. He shows that coherence constitutes a necessary and sufficient condition for subjective probabilities to conform to the rules of probability calculus, that is, the fundamental theorems of total and compound probability of Kolmogorov's theory of additive and non-negative functions of events. In de Finetti's (1931b, p. 298) words from his early paper on the topic: 'the theory of probability is then nothing but the mathematical theory teaching one to be coherent'.

[9] De Finetti defined subjective probabilities in terms of the rates at which individuals are willing to bet money on events, and argued that he preferred 'considering sufficiently small stakes, rather than to build up a complex theory to deal with [expected utilities]'. For a discussion of this issue, see Nau (2001).

As the probability of an event can be numerically determined in every instance, using the method described above, the actuarial mechanism necessary for insurance works in respect of any event. Insurance and gambling may differ in their social and economic functions, but not in their technical features (Feduzi and Runde 2011). In a strictly subjective probability perspective, then, the comparison between the decision-makers' utilities provides a general and coherent criterion to explain the insurance mechanism, since it is the difference between individuals' risk attitudes that explains the mutual benefits of exchanging an insurance contract. On de Finetti's view, the insurer always evaluates the option to offer insurance cover by considering an 'isolated event'; a set of insurance contracts is nothing but the sum of operations relative to 'isolated events'. The fact that there are numerous events does not modify the profitability of each operation *per se*; it simply increases the total amount of profits.[10] The activity of insurance firms like the Lloyd's of London, and in particular their willingness to insure against idiosyncratic risks such as the possible existence of the Loch Ness Monster, proves the practical irrelevance of the distinction between risk and uncertainty (de Finetti 1967a, p. 34; Borch 1976).

On this view, Knight's distinction is meaningless both theoretically and in practice. Since it is always possible to assign numerical probabilities to unique events all uncertainties can be reduced to measurable risks, and von Neumann and Morgenstern's maximization of expected utility constitutes the appropriate decision rule also in uncertain contexts. Already in 1951, Arrow's assessment of contemporary theories of risk and uncertainty, examining the 'dramatic break' in continuity constituted by von Neumann and Morgenstern's axiomatic treatment of the 'theory of choice under uncertainty', pointed out the potential of the new approach to definitely settle the issue of uncertainty as distinct from risk (Arrow 1951, p. 405). Commenting on Knight's argument that 'if the individual does not have the ability to repeat the experiment indefinitely often, the

[10] It is apparent that this explanation is entirely different from the usual one in the insurance literature, that is, the existence of a group of units that are subject to the same peril in respect to which it is possible to talk about the frequency of an event (de Finetti 1967a, p. 259). The homogeneity of events is irrelevant and eventually negative since homogeneous events are likely to be correlated; the concept of 'compensation' is thus more likely to work in the case of heterogeneous rather than homogenous events (de Finetti 1967a, p. 28).

probabilities, being essentially long-run frequency ratios, are irrelevant to his conduct', Arrow (1951, p. 415) concluded that 'this argument would obviously have no validity in the degree-of-belief theory of probability'.[11] Savage's 1954 axiomatization made de Finetti's point amenable to decision theory proper by introducing utility in the subjective set up, and a new subjectivist consensus came to be established.[12]

In summary, the received view of de Finetti's approach can be reduced to the three pillars of mainstream decision theory. First, the decision-makers' degrees of belief can be always represented by probability distributions that conform to the axioms of the probability calculus. Second, decision-makers are always prepared to take both sides of the bet at the subjectively appropriate prizes. Third, different decision-makers, even on the basis of the same information, can hold different degrees of belief about the occurrence of the same event.

4. De Finetti on uncertainty in Knight and Keynes

This section is devoted to an exegesis of two excerpts from de Finetti's vast contribution to economics and statistics originally published in Italian. The first, taken from a 1938 review article on the logical approach to probability, is relatively well known since it was translated into English in 1985. This piece offers de Finetti's early thoughts on Keynes's probability theory and the notion of non-numerical probability. The second comes from a 1967 textbook on the economics of insurance, never translated into English. This piece comments on Knight's notion of uncertainty and its links with the issue of insurability. It is our contention in this section that these two excerpts

[11] See also Marschak (1950).

[12] The following passage from Hirshleifer's popularization of the state-preference approach is highly illustrative of what the new consensus would imply in terms of subsuming uncertainty issues to risky ones: 'One surprising aspect of the time-and-state preference model is that it leads to a theory of decision under uncertainty while entirely excluding the "vagueness" we usually associate with uncertainty. Uncertainty in this model takes the form not of vagueness but rather of completely precise beliefs as to endowments, productive opportunities, etc., just as in the case of certainty – the only difference being that the beliefs span alternative possible states of the world as well as successive time periods. … So far as vagueness is concerned, we have already in our simplest timeless and riskless models assumed a precision in preference (as when we draw maps of indifference between shoes and apples) that can scarcely be regarded as closely descriptive of mental states. A similarly "unrealistic" or "depsychologized" portraying of uncertainty may really be what is required for comparably fruitful results in our analysis of risky choice' (Hirshleifer 1965, p. 534).

show an attitude towards the issue of uncertainty and its justification in the theory of probability that does not conform to the traditional representation of de Finetti as the champion of a strictly subjective approach. As we shall argue in the following section, this assessment shows that de Finetti's view of the foundations of probability, while subjective, was also pluralistic, and one that brings him closer to the broader subjective perspective more commonly associated with the likes of Irvin Good (1952) and Cedric Smith (1961).

In his 1938 Italian review of the works of the 'Cambridge probability theorists', Keynes's *Treatise on Probability* and Jeffreys's *Scientific Discovery*, de Finetti praises the renewed interest in the epistemic perspective in the foundational studies on probability. This perspective, conceiving probability as a branch of logic, is ascribed to Hume and the English philosophical tradition, that de Finetti views as having been ignored in Continental Europe in favour of what he terms 'Kant's lucubrations.' De Finetti (1938, p. 81) compares Keynes's approach favourably with that of frequentist interpreters of probability and argues in favour of Keynes's idea of interpreting probability theory as 'the logic of thinking about evaluations of probability, possibility and likelihood,' that is, 'an absolutely new (kind of) logic' dealing with a field previously ignored by ordinary logic, that determines the 'degree of uncertainty [of propositions] at a given time when there is not enough information to judge them true or false'.

Although ultimately critical of the objective perspective implicit in the logic of probability endorsed by Keynes in his foundational study, and explicitly evidencing the differences with his own subjective interpretation (de Finetti 1938, pp. 83-84), de Finetti offers support to what he sees as a revival of an epistemic approach to probability blurred by the empiricist perspective of frequency probability.[13] In particular, he makes clear that 'disagreement regarding the subjective or objective character of probability does not rule out, except for different shades of meaning and points of view consequently entailed, a substantial identity of views on a deeper philosophical

[13] As noted by Gillies and Ietto-Gillies (1987, p. 203) the real contrast is between the degree of belief theories of de Finetti and Keynes on the one hand, and von Mises's frequency theory on the other. De Finetti (1938, p. 90 fn.) regrets not giving the same prominence to Keynes in his 1931 essay 'Probabilism', attributing this to an inability to comprehend the English text adequately, and arguing that his new understanding is based on the German translation of the *Treatise*.

problem: that of induction'. As regards induction, he concludes:

> the positions are clear: there are two kinds of conceptions, each susceptible to infinite variations, but clearly different from each other: the *logical* approach to probability theory as instrument and law of reasoning and the *empirical* approach as alleged description of certain categories of facts (de Finetti 1938, p. 85).

To be sure, the historical context is important to understand de Finetti's supportive attitude. The subjective approach to probability he was pursuing since the early 1930s was to remain in the wilderness until Savage's *Foundations*, and a minority viewpoint in the statistical literature until even later (Fienberg 2006). Even by 1950, in his presentation at the second Berkeley Symposium on Mathematical Statistics and Probability, de Finetti was still acknowledging that the subjective theory was considered an extreme position (de Finetti 1951, p. 217). It therefore seems reasonable to conclude that in 1938, apparently still unaware of Ramsey's contribution, de Finetti may have been keen to build bridges with other representatives of the epistemic approach.[14]

However, de Finetti's broadly favourable attitude towards Keynes is not limited to their shared commitment to epistemic probability. In a comment on how Keynes's logical theory relates to the probability calculus, de Finetti (1938, p. 88) observes that Keynes rejects the postulates that every probability corresponds to a number between 0 and 1, and that two probabilities are always comparable one with the other (since 'for Keynes there also exist (in addition [to numerically measurable probabilities]) probabilities which cannot be expressed as numbers'). Viewed from de Finetti's subjectivist viewpoint, Keynes's position appears 'certainly not suited to the development of a mathematical probability theory,' and 'hardly in keeping with the intuitive idea of probability'. However, despite all this, de Finetti concedes that Keynes's viewpoint deserves consideration with respect to at least one specific aspect:

[14] The contrast with Ramsey's own rather harsh assessment of the *Treatise* is striking. On Ramsey's critique and Keynes's reaction to it, see Runde (1994a).

> without denying that for each individual the probabilities for two events must be comparable, it may be that, based on certain assumptions shared by all, certain inequalities already have a determinate sense which is common to everyone's opinion, whereas others vary from individual to individual.

And he concludes that, outside a strictly subjective interpretation of probability, 'it would not be at all irrational to interpret this in agreement with Keynes as an absence of comparability' (de Finetti 1938, p. 88).[15]

To the best of our knowledge this is the first instance in which de Finetti acknowledges that subjective interpretation of probability confronts a variety of problems when applied across different individuals, that the assumption of coherence makes it possible to avoid when referring to a single individual. De Finetti's considerations are also reminiscent of the problem of interval-valued probabilities that will be discussed in the next section. As detailed above, on Keynes's viewpoint a single individual can face a situation that makes her unable to formulate a point probability estimate, which, on certain occasions, can be made subject of numerical approximation and represented through a probability interval. These issues resurface in de Finetti's comment on Knight.

As we have seen in Section 3, strictly applied, de Finetti's subjective approach denies that Knight's distinction between risk and uncertainty should be relevant to decision-making. But his analysis of insurance suggests a somewhat different interpretation. In a contribution to the 1967 volume *Economia delle Assicurazioni* (*Economics of Insurance*),[16] de Finetti points out that Knight's distinction is not linked to the actual possibility of transferring an individual risk to an

[15] De Finetti (1938, p.88) provides the following example: 'one can assume, for example, the equal probability of certain events which are in a certain sense symmetrical, e.g., of a slightly oblong die, one may say that two square faces are equally probable and also that the four oblong faces are equally probable but more probable than the square sides. In that case, we must admit that this probability will fall between 1/6 and 1/4 (and $4a + 2b = 1$ with $0 < b < a$), but it is not determined which values between 1/6 and 1/4 will obtain and, based on the only assumption made, we may expect that each individual will evaluate that probability differently.'

[16] While the 1967 volume is co-authored with Filippo Emanuelli, de Finetti was uniquely responsible for the draft of Part I, titled *L'Incertezza nell'Economia* (*Economics of Uncertainty*). As we quote only from Part I, we refer to it simply as de Finetti 1967a. The following quotations from de Finetti's 1967 volume are translated from the original Italian by the authors.

insurance company, something which depends also on institutional and contingent factors, and thus has no interesting conceptual and general meaning. As for the theoretical issue, de Finetti reaffirms that the distinction cannot concern the particular features of a risk that might make it theoretically uninsurable, if this consideration is independent of the fact that the private market actually insures that risk (de Finetti 1967a, pp. 33-34). As a matter of fact, de Finetti contends, every risk can be insured as long as there is someone willing to accept it.

However, de Finetti ends his analysis with an observation that hints at a different interpretation of Knight's distinction. Specifically, he suggests that 'from this [subjectivist] perspective, what Knight would refer to as 'risks' are cases in which one finds minor discrepancies in valuations made by different individuals, or by different insurers. This is what renders them insurable' (de Finetti 1967a, p. 36). He continues as follows:

> the individual appreciation of the various risks translates (more or less explicitly) into a subjective valuation of the probability which, depending on whether the conditions are favourable, will be roughly uniform amongst the various individuals. This could even lead to the creation of an insurance market in which the valuations that are more or less accepted constitute the foundation for the setting of premiums (de Finetti 1967a, p. 37).

De Finetti maintains that uniformity of judgements is likely to occur with respect to games of chance or where statistical historical data are available (for example, accidents or fires, where individuals' judgements can be based on the frequency observed in a group of instances). Outside these cases, however, he suggests that the degree of difference between individuals' subjective probabilities will depend on the particular circumstances under which these judgements are elicited, and may well constitute an obstacle to insurability. He therefore regards it a mistake to think that it is possible to clearly distinguish between cases in which there is perfect uniformity of judgements and cases in which there is not. Knight's insistence on a sharp distinction between 'risk' and 'uncertainty' was accordingly misplaced in his view, since it gives the impression that the distinction is clear-cut and fundamental rather than fuzzy and secondary. It is for this reason that de

Finetti (1967a, p. 37) refused to adopt that terminology. However the conclusion remains that uniformity of judgements is always a matter of degree, and something that may affect the functioning of the insurance market.

At least two considerations follow from de Finetti's discussion of Knight's work. The first is that, although he rejects Knight's distinction, he nevertheless hints at an interpretation that justifies it. Following de Finetti's line of reasoning, a situation of 'uncertainty' may be interpreted as one in which individuals' opinions about the probability of the occurrence of a given event sensibly differ, so that, even though people are regarded as always attaching sharp numerical probabilities to events, insurance markets may nevertheless fail. In this case a market for insuring risks conditioned on a specific event may not exist, as much as in Knight. It therefore seems possible to speak of Knightian uncertainty even from de Finetti's subjectivist viewpoint.[17] The second consideration is that while it is obvious that de Finetti's understanding of a case for uncertainty is consistent with the 'orthodox' subjective view that individuals should always be regarded as attaching sharp numerical probabilities to events, he was keen to discuss the idea that different individuals may attach different probabilities even on the basis of the same information, a point which we have just seen he discussed for the first time in his review of Keynes. In this respect, de Finetti's version of Knightian uncertainty provides a critique of how subjective probabilities are used in economic theory.

It may be worth noting that, before becoming full professor of Financial Mathematics at the University of Trieste in 1947, de Finetti had worked as an actuary at *Assicurazioni Generali* for 15 years, and that he continued to collaborate with the company afterwards as an external advisor. It should therefore not be surprising that, as a practitioner, he was aware that the adoption of subjectivist probabilities does not guarantee complete markets as postulated by the standard economic model of risk exchange. However, de Finetti's caution about the practical application of an economic principle does not simply reflect a descriptively motivated concern. His understanding of the key elements at the basis of Knight's and Keynes's analyses of uncertainty, as we have shown

[17] Recent experimental studies on insurance markets point exactly to the issue of divergent subjective probabilities among operatives to justify the unwillingness of insurers to fix premia in the manner predicted by the orthodox model (Cabantous 2007).

in this section, is also of normative significance. Even where critical, he showed a clear interest in the rationale for alternative views, and his observations are not as peripheral as they may seem. Although long hidden in his works in Italian, they are indeed part of a general attitude displayed since his early studies and maintained even in his major later works.

5. Interval-valued probabilities

As we have seen, de Finetti made some concessions to a qualitative notion of probability in his review of Keynes's theory, admitting that there are cases in which the probability of an event, though numerically determined, may remain indeterminate in the subjective understanding of individuals. This attitude does not conform to the conventional view of de Finetti as the strictest of subjectivist thinkers, but should not come as a surprise from an author who insisted on a philosophical perspective on probability from his very first investigations.

Since his first essays after graduating in 1927 (see in particular de Finetti 1930, and 1931b), and later in his 1937 acclaimed contribution, de Finetti emphasised the qualitative nature of probability, and argued that it is 'from a purely qualitative system of axioms' that one arrives at a 'quantitative measure of probability' (de Finetti 1937, pp. 100-101). De Finetti insisted that, as many of our ordinary ideas are expressed in qualitative terms, in most practical situations the intuitive notion of probability simply concerns comparability of more, equal or less than. He interpreted the process of deriving numerical probabilities from choices as a form of second step in his investigation, a step intended to show that subjective probability judgements, when satisfying coherence, comply to the standard axioms of probability. As seen earlier, in conjunction with his peculiar notion of exchangeability, this aspect constitutes the building block of de Finetti's long-standing contribution to the theory of probability. But his methodological viewpoint did not hinge on a quantitative notion of probability. Measurement is important to prove that the notion of probability he wanted to introduce can be made subject of a mathematical theory, but constitutes 'only the precise expression of the rules of logic of the probable which are applied in an

unconscious manner, qualitatively if not numerically, by all men' (de Finetti 1937, p. 111). In the Italian essay introducing the axiomatic structure first published in his 1937 essay, de Finetti (1931b, p. 296) had argued:

> substantially, one uses a quantitative formulation to immediately reach quantitative results; it is instead possible to start from a qualitative formulation leading to qualitative results, and then to demonstrate that these results can be expressed in a quantitative form.

De Finetti was accordingly friendly to thinkers who chose not to follow him in the quantitative elaboration of subjective probability and preferred to concentrate instead on its qualitative aspects. Indeed, he regarded the works of Bernard Koopman as part of the foundational corpus of the subjectivist approach (de Finetti 1972, p. 186).[18]

De Finetti's position on non-numerical probability is also reflected in a joint paper with Savage called 'Sul modo di scegliere le probabilità iniziali' (How to choose the initial probabilities) published in Italian in 1962. This paper, which grew out of a prolonged discussion over the course of a sabbatical Savage spent in Rome in 1959, has never been translated into English. It is nevertheless referenced in the statistical literature, possibly because Savage (1962) provided an English summary of it. Introducing the subjective approach to probability, de Finetti and Savage (1962, p. 84-85) argue that it is 'unrealistic and impractical' to confine the use of probability to those 'limiting situations' in which either frequencies or impressions of symmetry are available. The subjectivist viewpoint assumes instead that in the 'general case', that is, when probabilities must be based on 'vague, complex, uncertain and fragmentary information', the initial probabilities are the 'arbitrary' opinions of a person expressing a judgement.[19] The primary objective of the paper is to clarify the meaning of fixing the initial probabilities arbitrarily from a subjectivist perspective.

Among the foundational issues concerning initial probabilities, the question of whether

[18] On this aspect of de Finetti's methodology see in particular Galavotti (2001).

[19] These and the following quotations from de Finetti and Savage (1962) are translated from the original Italian into English by the authors.

'inexactly determined' and 'fuzzy' initial opinions can be expressed through an exact probability value is dealt with. Here the tension between the strictly subjective interpretation and de Finetti's further elaboration with Savage is apparent. On the one hand, the idea that there can be no exact knowledge of initial probabilities is claimed to be 'meaningless' (de Finetti and Savage 1962, p. 94). Indeed, attempts to argue that the exact probabilities are non-existent, or that they must be replaced by interval probabilities, are claimed in the English summary to pose 'more severe problems that they are intended to resolve', as they entail the identification of precise upper and lower values, and rejected on the grounds that when the decision-maker makes a decision she implicitly resolves her earlier indecision (de Finetti and Savage 1962, p. 150).[20] On the other hand, it is conceded that 'it is often practically impossible to anyone to state that … the probability which he can attribute to a certain event has a precise value' (de Finetti and Savage 1962, p. 95), and that imprecision can constitute an 'actual epistemic state' of the individual facing uncertainty whose nature is 'difficult to be made precise in a convincing manner' (de Finetti and Savage 1962, p. 134).

This ambivalent attitude is discernible also in a long final section of the essay in which the theme of imprecise probabilities is dealt with in deeper detail. This section first reproduces some correspondence between the two authors ensuing from a preliminary draft of the paper, with each author speaking for himself, and then provides an examination of the literature on upper and lower probabilities, with specific regard to Cedric Smith's then recent elaboration.[21] Savage claims that the fact that 'we argue [in the preliminary draft] that imprecision in probability judgements can be always removed … is not in harmony with my experience and introspection', and that he finds it difficult to insist on numerical precision when for instance discussing the probability of a war in a

[20] Later, de Finetti (1975, p. 334) made his point as follows: 'That an evaluation of probability often appears to us more or less vague cannot be denied; it seems even more imprecise, however (as well as devoid of any real meaning), to specify the limits of this uncertainty'.

[21] This section was in fact precipitated by Smith's 1961 essay and the ensuing discussion published in the *Journal of the Royal Statistical Society*, including comments by G. A. Barnard, D. R. Cox, I. Good, and D. Lindley among others. De Finetti clarifies in an introductory footnote that the last part of the essay was added in September 1961, after a further meeting between the authors, to take into account the insights provided by Smith and the ensuing exchange de Finetti himself had with Smith (de Finetti and Savage 1962, p. 82). It is worth noting that Savage's English summary does not cover this additional scrutiny of interval probabilities.

near future (de Finetti and Savage 1962, p. 130).[22] De Finetti agrees that 'we do not really claim that we can seriously attribute a precise value to every probability … but only that it can be done with adequate approximation ... and most of all that if this is not enough it cannot be for any other method as well' (de Finetti and Savage 1962, p. 131). As a result, the main issue is how to make imprecision precise, or, from de Finetti's operational perspective, how to justify that certain specific values delimiting a zone of imprecision can be elicited from choice.

The relevance of Smith's argument is readily apparent here. Smith is the first author working in the tradition of interval probabilities – a tradition de Finetti himself identified with the work of Keynes, Koopman and Good (de Finetti and Savage 1962, p. 133) and considered rejuvenated by authors like Ellsberg and Dempster (de Finetti 1975, p. 368) – who followed Ramsey and de Finetti himself in the adoption of an operational perspective. Smith (1961) presented his work as a generalization of the subjective approach to admit imprecision, and measured 'imprecise' beliefs by means of betting quotients showing that a person refusing to bet on either an event or its complement can do so consistently. Smith's argument is that to be prepared to bet for an event at a certain maximum price does not mean to be prepared to bet against it at an infinitesimally higher one. Personal betting quotes can be interpreted as upper and lower probabilities and the person be attributed an interval of initial probabilities.[23] Smith then showed how the fundamental principles of avoiding sure loss and coherence could be applied to the axiomatic context of interval probabilities of Koopman (1940) and Good (1952).[24]

De Finetti and Savage admit that Smith's approach provides a precise criterion to determine

[22] Savage's attitude towards this issue is interesting in its own right. Both in the *Foundations of Statistics* (Savage 1954, pp. 58-59) and in later works (Savage 1971, p. 795) he distinguishes between the theoretical issue of maxims of behaviour and the practical one of confronting vagueness of opinion in day-to-day instances.

[23] Smith (1965, p. 478) argued as follows: 'if I am willing to bet 2 to 1 on sun against rain, and 1 to 4 on rain against sun, this means that I regard sun as between 2 and 4 times as probable as rain; and I do not need to be more precise than this'. As a result the elicitation of probabilities from choices entails that 'probabilities and utilities are no longer uniquely defined, but, in accordance with human vagueness and imprecision, they are only determined within a certain range'.

[24] Smith (1961; 1965) also provided an extension of interval probabilities to statistical inference and decision making, showing that coherent lower probabilities can be seen as lower envelopes of precise probability measures. For an analysis of Smith's role in the development of imprecise probability see Walley (1991).

the two limiting probability values p_* and p^* , where $p^* > p_*$. Surprisingly enough, the 'fact' that there may be a 'non-betting zone' is taken for granted. When the decision-maker is not assumed to be a 'stat-rat' under scrutiny in a controlled environment, there emerge a number of reasons for justifying a reluctance to bet, similar to when an insurance firm specialises in certain insurance fields and rejects to insure others; or when an individual rejects to bet on fields in which she feels herself incompetent (de Finetti and Savage 1962, pp. 136-139).[25] What is questioned, however, is not the descriptive validity of Smith's interval probabilities, but its normative value: in principle, consistency is guaranteed only by 'stat-rat' behaviour and any progress towards a 'more realistic' set-up is deemed to be theoretically irrelevant.

But even from a normative viewpoint there is an important case in which Smith's argument is accepted by de Finetti and Savage. In what reads as an involute sentence typical of de Finetti's Italian style, Smith's considerations are said to 'express what can be said of a certain behaviour when one has an incomplete knowledge of the opinions justifying a decision'. And while 'an opinion implying an indeterminate probability concerning a certain event is not admissible', it may be possible 'to know imperfectly an opinion, and thus to be capable of identifying only partially the preferences which the opinion implies (in a complete manner) among alternative possible decisions' (de Finetti and Savage 1962, pp. 141-142). Then implication is that Smith's approach may be of help in cases in which one has 'partial knowledge' of a preference so that the probability of an event can be said to be indeterminate. Among these cases two are singled out as particularly relevant. The first is the case of a group of decision-makers who have to make a collective decision, and the second, the case of a single individual who experiences a 'kind of personality dissociation' (de Finetti and Savage 1962, p. 142).

De Finetti and Savage's discussion of interval probabilities thus has clear links with the two excerpts discussing Knight and Keynes that we examined in the previous sections. As a matter of

[25] De Finetti and Savage (1962, p. 137) regret not being clear enough in previous work that they were not claiming that a 'reasonable and coherent' man should always be ready to bet on any occasion, and state that they do not personally aim to bet in any instance! De Finetti distanced himself from any descriptive interpretation of his theory in an often-quoted editorial footnote to the 1964 English translation of his 1937 paper (de Finetti 1937, p. 111, fn.).

fact, the first case in which Smith's approach is admitted to be sound, that concerning group decision-making, precedes de Finetti's 1967 justification of Knight's notion of uncertainty. The second case, the one suggesting the existence of multiple probabilities, coincides with de Finetti's understanding of Keynes's non-numerical probabilities.

To be sure, de Finetti never endorsed the approach of interval probabilities followed by Smith, but we take this as an indication that both de Finetti's early support of Keynes's probability theory and his late recognition of the significance of Knight's distinction are part of a long-standing interest in a generalised subjectivist perspective that is usually disregarded by critics.[26] This also explains the rationale of de Finetti's (1967b) taxonomy of philosophical interpretations of probability, an assessment of the state of the art published in 1967 as entry contributed to the *International Encyclopedia of Social Sciences*. Here de Finetti (1967b, pp. 499-501) classifies subjective theories under three headings: psychological, consistent and rational behaviour under uncertainty. While it is clear that the consistent subjective probability theory includes Ramsey, Savage and de Finetti himself, and that the rational one refers to the logical approach, psychological theories are not linked explicitly to any author. A psychological subjective theory of probability emerges, de Finetti argues, from recent experimental studies investigating the actual behaviour of individuals under uncertainty. But the existence of a certain actual behaviour diverging from coherent behaviour cannot be used to object to normative theories, since normative theories like the subjective and the rational approaches are intended to state 'what behaviour is god or bad', irrespective of actual behaviour (de Finetti 1967b, p. 500).

What is interesting here is that de Finetti introduces his classification of subjective theories into three kinds as 'a matter of convenience', and acknowledges that some theories do not fit his scheme. In particular, theories in which 'probabilities may be noncomparable, and hence nonnumerical' can be classified only as 'variants' of the main three ones. Unsurprisingly, Keynes's theory is named as a 'variant with noncomparability' of the subjective rational theory. Ellsberg

[26] For a notable exception see Suppes and Zanotti (1989).

(1961) is quoted, possibly for the first time in de Finetti's works, as providing a noncomparability variant of subjective psychological theories, and is as such dismissed as a possible counterexample to a normative theory. Smith's approach, however, is awarded the status of a noncomparability variant of the subjective consistent kind, testifying to the normative appeal of upper and lower probabilities.

6. Concluding remarks

De Finetti's notion of probability grew out of a perspective emphasizing the operational character of the theoretical concept. Betting odds offer a device for the measurement of numerical probabilities intended as degrees of belief. However de Finetti's more fundamental theory was based purely on qualitative notions, intended to capture the practical judgements of everyday life, which are usually expressed in term of a loose, partial ordering.

We have attempted to show that this approach to probability is reflected in de Finetti's treatment of uncertainty and that his axiomatic theory of qualitative probability was in principle much more open to a representation of uncertainty issues than usually admitted. In particular, Knight's notion of uncertainty is not simply rejected, but reinterpreted as relevant for those instances in which peoples' opinions about the probability of a given event sensibly differ. This notion of uncertainty is relevant in current research on source uncertainty and the causes of aversion to ambiguity, and testifies to the modernity of de Finetti's thought.

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