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Optimal Actuator/Sensor Selection Through Dynamic Output Feedback

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Abstract—This paper is devoted to the problem of optimal selection of a subset of available actuators/sensors through a multi-channel \mathcal{H}_2 dynamic output feedback controller for continuous linear time invariant systems. Incorporating two extra terms for penalizing the number of actuators and sensors into the optimization objective function, we develop an iterative process to identify the favorable row/column-wise sparse DOF gains. Employing the identified structure, we solve the constructed row/column structured multi-channel \mathcal{H}_2 DOF problem in order to derive a gain that exploits optimum number of sensors/actuators by which the closed-loop stability is maintained and the performance degradation of the closed-loop system is restricted. Through an example we demonstrate the remarkable performance and broad applicability of the proposed approach.

Index Terms—Simultaneous actuator/sensor selection, multi-channel \mathcal{H}_2 row-column-sparse dynamic output feedback (DOF) problem, linear matrix inequality.

I. INTRODUCTION

The number of components (actuators or sensors) in modern control systems can be very large, and hence, it is often not very feasible to manually find a subset of all available components to meet a specific control objective. Hence, the problem of selecting a configuration of actuators (sensors) from the set of all available actuators (sensors), while the control performance remains in an acceptable level compared to the non-sparse performance, is a well-known problem in the literature of control theory; see e.g. [1]–[6]. This problem can equivalently be considered as the design of a row (column) sparse feedback gain for the underling system. This paper aims to develop a unified framework to systematically design a sparse row-wise and column-wise dynamic output feedback (DOF) gain via convex optimization, while satisfying multi-channel \mathcal{H}_2 performance specifications. One immediate application of this issue will be in the over-actuated (over-sensed) systems [7]. It is known that one method for reconfiguration strategy of fault tolerant control is usually to build an over-actuated (over-sensed) system first and then design a nominal controller using some of the available components. Hence whenever a fault happens in the system, the configuration of the control system is changed by using some of the redundant components in order to attain the nominal control objective [8].

This problem leads to a difficult combinatorial optimization problem. There are a large number of investigations in

the literature in this field, but most of them do not consider dynamical systems. The paper [1] provides a convex sensor selection formulation for a problem with linear measurements. In [2] a method is developed in order to identify sparse subsets of sensors while it minimizes the Cramer-Rao bound of a class of nonlinear measurement models. A genetic algorithm-based method is proposed in [3] for the problem of actuator selection. In [4], a convex semi-definite programming (SDP) characterization of the problem formulation is considered by incorporating certain forms of sparsity in the feedback gain.

Most of the sparse control design approaches in the literature have focused on the problem of sparse static output/state feedback design. This paper instead utilizes DOF scheme to control the system using the measured system outputs. Moreover, to the best knowledge of authors, concurrent actuator and sensor selection problem through DOF has not been researched thoroughly. Further, it should be noted that the sparse row-column-wise feedback design using DOF is a totally different problem compared to the case of row (column) sparse static output (state) feedback design, as the sparsification procedure should be applied only to a certain part of the controller.

A well-developed method to identify sparsity patterns of feedback gains in the literature (see e.g. [9] and [10]) is to solve e.g. the \mathcal{H}_2 (or the \mathcal{H}_∞) problem, by incorporating a sparsity promoting penalty function to the objective function. Basically, the so-called reweighted ℓ_1 (REL1) norm, which is known as a convex relaxation of the ℓ_0 -norm, is usually exploited to make a convex problem rather than the original combinatorial optimization problem. The weights (entries of the weighting matrix) are then updated at each step inversely proportional to the strength of individual entries of feedback gain in the previous step. This scheme successively applies to the applications that the sparsity is required to be achieved at the entry-wise level, i.e. minimizing the number of communication links in distributed control networks exploiting the so-called *bilateral communication scheme* [11]. However, in the cases that the sparsity is noted at a *group* (e.g. row or column) level, the strength of groups of variables (entries of feedback gain) should be considered. We consider $\|\cdot\|_{\text{row-}\ell_0}$ ($\|\cdot\|_{\text{col-}\ell_0}$), that counts the number of nonzero rows (columns) of a matrix, as the row (column) sparsity promoting penalty function and then propose its reweighted convex relaxation. This paper then develop an iterative algorithm, using the relaxed row/column sparsity promoting penalty functions, which is able to penalize the number of sensors and actuators employed in the control system simultaneously.

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Notation: $[\Phi_{ij}]_{r \times r}$ is a (block) matrix with (block) entries Φ_{ij} , $i = 1, \dots, r$, $j = 1, \dots, r$. $\text{diag}[\Phi_i]_{i=1}^r$ is a (block) diagonal matrix with (block) entries Φ_i , $i = 1, \dots, r$. Moreover, $\text{col}(v_i(t))_{i=1}^r$ denotes a (block) vector with (block) entries $v_i(t)$, $i = 1, \dots, r$. $\text{herm}(F)$, where F is a square matrix, stands for $F + F^*$.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Problem statement

Consider the following LTI system,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_2u(t) + B_1w(t) \\ z(t) &= C_2x(t) + D_2u(t) \\ y(t) &= Cx(t), \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $z \in \mathbb{R}^q$ are the state vector, control input vector, output vector and \mathcal{H}_2 performance output vector of system, respectively. The matrices in (1) are constant and of appropriate dimensions. It is also assumed that (A, B_2) is stabilizable and (A, C) is detectable. $w(t)$ is the external disturbance of the system. The main goal of this paper is to synthesise dynamic output feedback (DOF) gains, utilizing optimum number of actuators and sensors, for the system in (1), while satisfying multi-channel \mathcal{H}_2 performance specifications. In doing so, we need primarily consider the multichannel \mathcal{H}_2 DOF synthesis. In the following of this section, to avoid the conservatism introduced by the so-called *quadratic* approach for the design of feedback gains with respect to multichannel \mathcal{H}_2 (or mixed $\mathcal{H}_2/\mathcal{H}_\infty$) performance specifications, we develop a novel LMI method for the multichannel \mathcal{H}_2 control problems. Before doing so, let us present some definitions which we will make use of later in this paper.

Definition 1: A matrix is said to be *structure* matrix if its elements are either 0 or 1. The structure matrix of a block matrix $Y = [Y_{ij}]_{m \times n}$ with $Y_{ij} \in \mathbb{R}^{r_i \times s_j}$ is $S(Y) \triangleq [s_{ij}]_{m \times n}$ with

$$s_{ij} = \begin{cases} 0 & \text{if } Y_{ij} = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2: Two matrices Y_1 and Y_2 are said to have the same structure if $S(Y_1) = S(Y_2)$.

Definition 3: The matrix Y_1 with $S(Y_1) \triangleq [s_{ij}^1]_{m \times n}$ is said to be structurally subset of Y_2 with $S(Y_2) \triangleq [s_{ij}^2]_{m \times n}$ while $s_{ij}^2 - s_{ij}^1 \geq 0$. We denote this as $S(Y_1) \subseteq S(Y_2)$.

B. Controller dynamic

We represent the DOF controller $\mathcal{K}(s)$ in state-space as:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t), \end{aligned} \quad (2)$$

where $x_c \in \mathbb{R}^n$ is the state vector of the controller and A_c , B_c , C_c and D_c are of appropriate dimensions. Let (A_{cl}, B_{cl}, C_{cl}) denotes realization of $T_{wz} := C_{cl}(sI - A_{cl})^{-1}B_{cl}$ (the closed loop transfer function from $w(t)$ to $z(t)$), where

$$\begin{aligned} A_{cl} &:= \begin{bmatrix} A + B_2 D_c C & B_2 C_c \\ B_c C & A_c \end{bmatrix}, \quad B_{cl} := \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ C_{cl} &:= \begin{bmatrix} C_2 + D_2 D_c C & D_2 C_c \end{bmatrix}. \end{aligned} \quad (3)$$

C. LMI characterization

1) \mathcal{H}_2 LMI characterization:

Lemma 1: Let (A_{cl}, B_{cl}, C_{cl}) denotes realization of T_{wz} . The following three statements, involving symmetric matrix variables X , Z and the general matrix variable G are equivalent.

- i) A_{cl} is stable and $\|C_{cl}(sI - A_{cl})^{-1}B_{cl}\|_2^2 < \gamma$.
- ii) $\exists X > 0$ and $Z > 0$ such that

$$\begin{aligned} \begin{bmatrix} A_{cl}X + XA_{cl}^T & \star \\ C_{cl}X & -\gamma I \end{bmatrix} &< 0, \\ \begin{bmatrix} -Z & \star \\ B_{cl} & -X \end{bmatrix} &< 0, \\ \text{trace}(Z) &< 1. \end{aligned}$$

- iii) $\exists X > 0$, $Z > 0$ and G such that

$$\begin{bmatrix} -(G + G^T) & \star & \star \\ A_{cl}G + X + G & -2X & \star \\ C_{cl}G & 0 & -\gamma I \end{bmatrix} < 0, \quad (4)$$

$$\begin{bmatrix} -Z & \star \\ B_{cl} & -X \end{bmatrix} < 0, \quad (5)$$

$$\text{trace}(Z) < 1. \quad (6)$$

Proof: Refer to Appendix. \blacksquare

Remark 1: The advantage of the item *iii)* over the LMI condition derived in [12] lies within the smaller dimension of the first LMI in *iii)* than that of in [12].

It should be emphasized that the specific LMI characterization in (4) enables us to utilize different Lyapunov matrices (X_h) for each of the involved LMI constraints in the problem. Moreover, the advantage of the LMI (4) lies within this fact that the product terms between the matrix A_{cl} and the Lyapunov matrices (X_h) is disappeared. More importantly, the control gain does not depend on the Lyapunov matrix, but the instrumental matrix variable G . This feature has a significant implication in the design of controllers satisfying multiple objectives such as the multi-channel \mathcal{H}_2 (or even mixed multi-channel $\mathcal{H}_2/\mathcal{H}_\infty$) specifications. It is also worth mentioning that, usually, the projection lemma is used, in the field of robust control, to eliminate the variable which contains the controller state-space data, and thus, deal only with a set of LMIs which include less number of decision variables. However, as seen, the proposed methodology here utilizes the projection lemma in the opposite direction, that is, introducing an additional matrix variable G to the problem.

2) *Multi-channel \mathcal{H}_2 synthesis using improved LMI characterizations:* Our target is primarily to compute a full-order DOF $\mathcal{K}(s)$ presented in (2) which meets several performance specifications of the form

$$\begin{aligned} &\text{minimize} \quad \|T_{w_k z_k}\|_2 \\ &\text{subject to} \quad \|T_{w_1 z_1}\|_2^2 < \gamma_1, \dots, \|T_{w_{k-1} z_{k-1}}\|_2^2 < \gamma_{k-1}, \\ &\quad \|T_{w_{k+1} z_{k+1}}\|_2^2 < \gamma_{k+1}, \dots, \|T_{w_{\mathcal{N}} z_{\mathcal{N}}}\|_2^2 < \gamma_{\mathcal{N}}, \end{aligned} \quad (7)$$

where $\|T_{w_i z_i}\|_2 := \|L_i T_{wz} R_i\|_2$, in which L_i and R_i are selection matrices that specify which channel is involved in the associated constraint, and notice that \mathcal{N} hereafter denotes the

number of channels. Furthermore, a realization of the closed loop system $T_{w_i z_i}$ will be obtained by replacing matrices B_1 , C_2 and D_2 by $B_{1,i}$, $C_{2,i}$ and $D_{2,i}$, $i = 1, \dots, \mathcal{N}$, respectively, in (1). Notice that, in such a case, the closed-loop performance is ensured by constraining (minimizing) the \mathcal{H}_2 norm of the closed-loop transfer functions associated with the (input/output) signals $w_i = R_i w$ and $z_i = L_i z$; see [13], [14]. Letting each channel to be associated with the LMI constraints in (4), (5), and (6), the desired characterization with multi-channel specifications can be obtained by introducing a different Lyapunov variable X_l for every channel and exploiting common variable G for all channels. Hence, it is readily verified that using the part *iii*) of Lemma 1, the LMI characterization for each channel l can be written as:

$$\begin{bmatrix} -(G + G^T) & \star & \star \\ A_{cl}G + X_l + G & -2X_l & \star \\ C_{cl,l}G & 0 & -\gamma_l I \end{bmatrix} < 0, \quad (8)$$

$$\begin{bmatrix} -Z_l & \star \\ B_{cl,l} & -X_l \end{bmatrix} < 0, \quad (9)$$

$$\text{trace}(Z_l) < 1, \quad (10)$$

where $X_l > 0$, $Z_l > 0$ and G are LMI variables. Now the multi-objective problem described in (7) can be set as

$$\begin{aligned} & \text{minimize} && \gamma_k && \text{(MCH2)} \\ & \text{subject to} && (8), (9), \text{ and } (10) \text{ for } k\text{-th channel,} \\ & && (8), (9), \text{ and } (10) \text{ for } j\text{-th channel,} \\ & && \text{with given } \gamma_l, l = 1, \dots, \mathcal{N}, l \neq k. \end{aligned}$$

III. ROW-COLUMN-SPARSE MULTI-CHANNEL \mathcal{H}_2 DOF

The aim is to design a feedback gain such that it

- ensures the \mathcal{H}_2 performances with respect to channels $j = 1, \dots, \mathcal{N}$, $l \neq k$ which means that for a prescribed closed loop \mathcal{H}_2 performance $\gamma_l > 0$, we have $\|T_{w_l z_l}\|_2^2 < \gamma_l$;
- minimizes the \mathcal{H}_2 performance of the k -th channel subject to the above constraints and $S(\mathcal{K}_a) \subseteq \Gamma_a$, $S(\mathcal{K}_s) \subseteq \Gamma_s$, where Γ_a and Γ_s are given row and column sparse structures, respectively, and $\mathcal{K}_a \triangleq [c_c \ d_c]$, $\mathcal{K}_s \triangleq [B_c^T \ D_c^T]^T$.

Remark 2: It is worth mentioning that each nonzero row of \mathcal{K}_a corresponds to an actuator used by the DOF controller, and likewise, each nonzero column of \mathcal{K}_s corresponds to a sensor employed by the DOF controller.

This problem can be formulated through an optimization program in decision variables $X_l > 0$, $Z_l > 0$, $l = 1, \dots, \mathcal{N}$, G , A_c , B_c , C_c , D_c and $\gamma_k > 0$:

$$\begin{aligned} & \text{minimize} && \gamma_k && \text{subject to} && (11) \\ & && (8), (9), \text{ and } (10), && \text{for } k\text{-th channel,} \\ & && (8), (9), \text{ and } (10), && \text{for } l\text{-th channel,} \\ & && \text{with given } \gamma_l, l = 1, \dots, \mathcal{N}, l \neq k, \\ & && S(\mathcal{K}_a) \subseteq \Gamma_a \text{ and } S(\mathcal{K}_s) \subseteq \Gamma_s, \end{aligned}$$

where Γ_a and Γ_s are preset row and column sparse structure matrices respectively. A difficulty in (11) is that it involves

nonlinear terms. The nonlinearities can fortunately be eliminated by some appropriate change of controller variables. The change of variables here is performed inspired by the one introduced in [12]. We let the instrumental variable G and $V = G^{-1}$ be of the form:

$$G := \begin{bmatrix} G_1 & G_3 \\ G_2 & G_4 \end{bmatrix}, \quad V := \begin{bmatrix} V_1 & V_3 \\ V_2 & V_4 \end{bmatrix}, \quad (12)$$

where $\dim G_1 = \dim V_1 = \dim A$. As stated in e.g. [12] and [15], without loss of generality it can be assumed that G_2 and V_2 are invertible. Now consider the following invertible matrices:

$$\mathsf{T}_G := \begin{bmatrix} G_1 & I \\ G_2 & 0 \end{bmatrix}, \quad \mathsf{T}_V := \begin{bmatrix} I & V_1 \\ 0 & V_2 \end{bmatrix}. \quad (13)$$

It can readily be deduced that

$$V \mathsf{T}_G = \mathsf{T}_V, \quad G \mathsf{T}_V = \mathsf{T}_G.$$

Now, performing congruence transformations $\text{diag}(\mathsf{T}_V, \mathsf{T}_V, I)$, $\text{diag}(I, \mathsf{T}_V)$, and $\text{diag}(I_\xi \otimes \mathsf{T}_V, I_\xi \otimes \mathsf{T}_V, I_\xi \otimes \mathsf{T}_V)$ in (8) and (9) respectively, along with the following change of variables:

$$\begin{aligned} \tilde{A}_c &:= V_1^T A G_1 + V_1^T B_2 D_c C G_1 + V_2^T B_c C G_1 \\ &\quad + V_1^T B_2 C_c G_2 + V_2^T A_c G_2, \end{aligned} \quad (14)$$

$$\tilde{B}_c := V_1^T B_2 D_c + V_2^T B_c, \quad (15)$$

$$\tilde{C}_c := D_c C G_1 + C_c G_2, \quad (16)$$

$$\tilde{D}_c := D_c, \quad (17)$$

$$\tilde{X}_l := \mathsf{T}_V^T X_l \mathsf{T}_V,$$

$$H := G_1^T V_1 + G_2^T V_2,$$

lead to the LMIs in (18) and (19). Therefore, the sparse row/column-wise multi-channel \mathcal{H}_2 DOF problem can be recast as an optimization program in decision variables $\tilde{X}_l = \begin{bmatrix} \tilde{X}_{l,1} & \tilde{X}_{l,2} \\ \tilde{X}_{l,2}^T & \tilde{X}_{l,3} \end{bmatrix} > 0$, $Z_l > 0$, $l = 1, \dots, \mathcal{N}$, V_1 , G_1 , H , \tilde{A}_c , \tilde{B}_c , \tilde{C}_c , \tilde{D}_c and $\gamma_k > 0$:

$$\begin{aligned} & \text{minimize} && \gamma_k && \text{subject to} && \text{(SMH2)} \\ & && (18), (19), \text{ and } (10), && \text{for } k\text{-th channel,} \\ & && (18), (19), \text{ and } (10), && \text{for } l\text{-th channel,} \\ & && \text{with given } \gamma_l, l = 1, \dots, \mathcal{N}, l \neq k, \\ & && S(\tilde{\mathcal{K}}_a) \subseteq \Gamma_a \text{ and } S(\tilde{\mathcal{K}}_s) \subseteq \Gamma_s, \end{aligned}$$

where $\tilde{\mathcal{K}}_a := [\tilde{c}_c \ \tilde{d}_c]$ and $\tilde{\mathcal{K}}_s := [\tilde{b}_c \ \tilde{d}_c]$.

Remark 3: It is readily derived that

$$\begin{bmatrix} \tilde{A}_c & \tilde{B}_c \\ \tilde{C}_c & \tilde{D}_c \end{bmatrix} = \begin{bmatrix} V_2^T & V_1^T B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} G_2 & 0 \\ C G_1 & I \end{bmatrix} + \begin{bmatrix} V_1^T A G_1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (20)$$

As a result, it can be seen

$$\tilde{\mathcal{K}}_a = \mathcal{K}_a \begin{bmatrix} G_2 & 0 \\ C G_1 & I \end{bmatrix}.$$

$$\begin{bmatrix} -(G_1 + G_1^T) & * & * & * & * \\ -H^T - I & -(V_1 + V_1^T) & * & * & * \\ AG_1 + B_2\tilde{C}_c + \tilde{X}_{l,1} + G_1 & A + B_2\tilde{D}_cC + \tilde{X}_{l,2} + I & -2\tilde{X}_{l,1} & * & * \\ \tilde{A}_c + \tilde{X}_{l,2}^T + H^T & V_1^T A + \tilde{B}_cC + \tilde{X}_{l,3} + V_1^T & -2\tilde{X}_{l,2} & -2\tilde{X}_{l,3} & * \\ C_2G_1 + D_2\tilde{C}_c & C_2 + D_2\tilde{D}_cC & 0 & 0 & -\gamma I \end{bmatrix} < 0 \quad (18)$$

$$\begin{bmatrix} -Z & * & * \\ B_1 & -\tilde{X}_{l,1} & * \\ V_1^T B_1 & -\tilde{X}_{l,2}^T & -\tilde{X}_{l,3} \end{bmatrix} < 0 \quad (19)$$

Thus, since post-multiplication retains the row-wise sparsity, it can be stated that \mathcal{K}_a is row-sparse if and only if $\tilde{\mathcal{K}}_a$ is row-sparse. Moreover, as

$$\tilde{\mathcal{K}}_s = \begin{bmatrix} V_2^T & V_1^T B_2 \\ 0 & I \end{bmatrix} \mathcal{K}_s,$$

and since pre-multiplication retains the column-wise sparsity, then \mathcal{K}_s is column-sparse if and only if $\tilde{\mathcal{K}}_s$ is column sparse. Obtaining a solution from (SMH2), the DOF controller can readily be found from:

- 1) computing a full rank factorization $G_2^T V_2$ of $H - G_1^T V_1$ and thus invertible G_2 and V_2 ;
- 2) solving the equations (14)-(17) for controller matrices A_c , B_c , C_c and D_c .

IV. ACTUATOR/SENSOR SELECTION

While the previous section considers the design of a multi-channel \mathcal{H}_2 DOF with a priori specified set of actuators and sensors, this section explores favorable row and column sparse DOF gains (selection of actuators and sensors). To do so, an optimization framework, in which the sparsity of the feedback gain is directly incorporated into the objective function, is considered here. This problem can be formulated as:

$$\text{minimize } \gamma_k + \eta_a \|\tilde{\mathcal{K}}_a\|_{\text{row}-\ell_0} + \eta_s \|\tilde{\mathcal{K}}_s\|_{\text{col}-\ell_0}, \quad (\text{RC0})$$

subject to the constraints given in (SMH2) except the structural constraint on $\tilde{\mathcal{K}}_a$ and $\tilde{\mathcal{K}}_s$, where the row- ℓ_0 (col- ℓ_0) is a quasi-norm that counts the number of non-zero rows (columns) of $\tilde{\mathcal{K}}_a$ ($\tilde{\mathcal{K}}_s$), and $\eta_a > 0$ ($\eta_s > 0$) is the regularization parameter that implies the emphasis on the row-sparsity (column-sparsity) of $\tilde{\mathcal{K}}_a$ ($\tilde{\mathcal{K}}_s$); i.e. a larger η_a (η_s) will result in a more row-sparse (column-sparse) $\tilde{\mathcal{K}}_a$ ($\tilde{\mathcal{K}}_s$). Clearly the optimization problem (RC0) is a combinatorial one and broadly speaking impossible to solve, as an intractable combinatorial search is required to address this problem. In the literature several alternatives such as ℓ_1 -norm or weighted ℓ_1 -norm are proposed as convex relaxations of the ℓ_0 -quasi-norm [16]. Besides, [16] proposes an iterative scheme, called reweighted ℓ_1 (REL1) minimization, that solves a sequence of weighted minimization problems, which the weights are updated according to the previous iteration's solution. The REL1 algorithm has recently been used by a number of researchers (e.g. see [17], [18]) for the design of sparse controllers for the distributed systems. Notice that the variable selection in the aforementioned papers typically

amounts to the selection of important individual variables (elements in the feedback gain) rather than the important groups of variables (rows or columns).

Let us now recast the optimization problem (RC0) as

$$\text{minimize } \gamma_k + \eta_a f(\tilde{\mathcal{K}}_a) + \eta_s g(\tilde{\mathcal{K}}_s) \quad (\text{SAS})$$

subject to (18), (19), and (10), for k-th channel,

(18), (19), and (10), for l-th channel,

with given γ_l , $l = 1, \dots, \mathcal{N}$, $l \neq k$,

where $f(\cdot)$ ($g(\cdot)$) denotes the relaxed row-sparsity (column-sparsity) promoting function for which different choices will be proposed in the following.

A. Row (column) sparsity promoting penalty function

Let us consider a relaxed row-sparsity promoting function as

$$f(\tilde{\mathcal{K}}_a) = \sum_{i,j} W_{a,i} |\tilde{\mathcal{K}}_{a,ij}|, \quad (21)$$

where $W_{a,i}$ denotes the weight corresponds to the i -th row of $\tilde{\mathcal{K}}_a$. One may now resort to update the weights inversely proportional to the ℓ_1 -norm of its corresponding row in $\tilde{\mathcal{K}}_a$ obtained at the previous iteration:

$$W_{a,i}^l = \frac{1}{\sum_j |\tilde{\mathcal{K}}_{a,ij}^{(l-1)}| + \epsilon}, \quad (22)$$

and form the weighting matrix as $W_a = \text{diag}[W_{a,i}]_{i=1}^m$. One can also imagine a variety of possible norms in place of (22), e.g. ℓ_2 -norm and ℓ_∞ -norm. Therefore the update rule in (22) can be revised to:

$$W_{a,i}^l = \frac{1}{\sqrt{\sum_j |\tilde{\mathcal{K}}_{a,ij}^{(l-1)}|^2 + \epsilon}}, \quad (23)$$

for ℓ_2 -norm, and

$$W_{a,i}^l = \frac{1}{\max_j (|\tilde{\mathcal{K}}_{a,ij}^{(l-1)}|) + \epsilon}, \quad (24)$$

for ℓ_∞ -norm. Although we have found the one in (22) to perform well in a wide range of experiments, using ℓ_2 -norm or ℓ_∞ -norm can sometimes outperform ℓ_1 -norm. Notice that these norms only indicate the method that we update the weighting matrix W_a , while the sparsity term in the objective function is considered as the one in (21). For the sensor

selection problem, we can similarly replace column- ℓ_0 by a relaxed column-sparsity promoting function as

$$g(\tilde{\mathcal{K}}_s) = \sum_{i,j} |\tilde{\mathcal{K}}_{s,ij}| W_{s,j}, \quad (25)$$

while the update rule e.g. in (22) can be converted to:

$$W_{s,j}^l = \frac{1}{\sum_i |\tilde{\mathcal{K}}_{s,ij}^{(l-1)}| + \varepsilon}, \quad (26)$$

and the weighting matrix is similarly formed as $W_s = \text{diag}[W_{s,j}]_{j=1}^p$. Algorithm 1 is presented in Appendix to identify a row-sparse $\tilde{\mathcal{K}}_a$ and a column-sparse $\tilde{\mathcal{K}}_s$. We denote the obtained structure of the minimization problem (SAS), as $S(\tilde{\mathcal{K}}_a) \triangleq \Gamma_a$ and $S(\tilde{\mathcal{K}}_s) \triangleq \Gamma_s$. Eventually, in order to find the multi-channel \mathcal{H}_2 DOF associated with the achieved Γ_a and Γ_s , we turn to the minimization problem in (SMH2).

Remark 4: It should be pointed out that if due to some reasons (e.g. high execution cost or faulty situation), some actuators (sensors) are required to be by-passed, one can penalize those actuators (sensors) by a large initial weight (e.g. $1/\varepsilon$) in the proposed iterative algorithm.

V. NUMERICAL EXAMPLES

A numerical example is presented here in order to evaluate the effectiveness of the proposed method for the problem of actuator-sensor selection via DOF approach. All the LMI optimization problems are solved by YALMIP [19] as the interface and SDPT3 [20] as the solver.

Let us consider the problem HE3 from COMpleib [21]. This problem is related to the eight order linearized state space model of the dynamics of the Bell201A-1 helicopter that has four inputs and six outputs. The system matrices can be seen in [21]. Besides, we let $C_1 = \begin{bmatrix} C \\ 0_{4 \times 8} \end{bmatrix}$, $D_1 = \begin{bmatrix} 0_{6 \times 4} \\ I_4 \end{bmatrix}$, $C_2 = \begin{bmatrix} \tilde{C} \\ 0_{4 \times 8} \end{bmatrix}$ and $D_2 = \begin{bmatrix} 0_{2 \times 4} \\ I_4 \end{bmatrix}$, where

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The objective is firstly to identify the minimum number of actuators and sensors by which the closed-loop stability is maintained and the performance degradation of the closed-loop system is restricted. Solving the optimization problem in (SMH2), with $\mathcal{N} = 2$, $\gamma_2 < 10$, $\Gamma_a = \mathbf{1}_{m \times (n+p)}$, $\Gamma_s = \mathbf{1}_{(n+m) \times p}$, results in a true H_2 cost of 0.8117 for channel 1 and 0.4608 for channel 2.

Remark 5: Notice that the value of the \mathcal{H}_2 cost obtained from (SMH2) is not the true one, due to the conservatism introduced by employing common G , A_c , B_c , C_c and D_c . Nevertheless, the true value can be computed by solving the following Lyapunov equation

$$X_{i,true} A_{cl} + A_{cl}^T X_{i,true} + C_{cl,i}^T C_{cl,i} = 0. \quad (27)$$

One can then find the \mathcal{H}_2 cost as $\sqrt{\text{trace}(B_{cl,i}^T X_{i,true} B_{cl,i})}$. Now we solve the optimization problem in (SAS) by utilizing Algorithm 1 in Appendix with $\beta = 0.1$, $\varepsilon = 0.0001$ and exploiting the row-sparsity promoting function (21) and the column-sparsity promoting function (25) in addition to

TABLE I
SUMMARY OF THE OBTAINED RESULTS FOR DIFFERENT VALUES OF η_s
AND η_a

η_a	η_s	\mathcal{H}_2 cost (Ch 1)	Sensors	Actuators
0	0	0.8117	All	All
1	0.5	0.9946	1&2&3	2&3
5	10	0.9946	2&3	2&3

the update rule in (23) and its counterpart for the sensor selection, respectively. As increases η_a and η_s the underlying \mathcal{K}_a and \mathcal{K}_s gradually become row-sparse and column-sparse, respectively. For instance, with $\eta_a = 1$ and $\eta_s = 0.5$ the algorithm suggests exploiting only the control inputs 2 and 3, while the controller does not receive information of sensors 4, 5 and 6, but at the expense of about 23% H_2 performance degradation in channel 1 compared to the non-sparse \mathcal{K} . Moreover, letting $\eta_a = 5$ and $\eta_s = 10$, Algorithm 1 proposes exploiting sensors 2 and 3 and actuators 2 and 3 ($\|\mathcal{K}_a\|_{\text{row}-\ell_0} = 2$ and $\|\mathcal{K}_s\|_{\text{col}-\ell_0} = 2$). About 23% H_2 performance degradation happens in channel 1, in this case. The obtained results for different values of η_s and η_a is summarized in Table I.

VI. CONCLUSIONS

This paper develops a framework for addressing the issue of selecting an optimal set of actuators (sensors) for dynamical systems, satisfying several performance constraints. Firstly, an LMI-based framework for the design of multi-channel \mathcal{H}_2 DOF gain has been proposed. Besides, this framework is capable of incorporating additional structural constraints as well as, e.g., regional pole placement constraints on the feedback gain matrix. Then a procedure has been developed which includes two stages; the first stage identifies the desirable row-column-sparsity pattern for necessary parts of dynamic feedback gain via iterative processes and then the second one solves the multi-channel \mathcal{H}_2 problem, augmented by structural constraints. The simulation results illustrated the effectiveness of our proposed approach.

APPENDIX

A. Proof of Lemma 1

The equivalence between the first two statements is a standard \mathcal{H}_2 state feedback synthesis and can be seen e.g. in [22]. We just show the equivalence between the statements *ii*) and *iii*). Exploiting the well-known Schur complement, one can show that the first LMI in *iii*) can be reformulated as

$$\begin{bmatrix} -(G+G^T) + \gamma^{-1}(C_{cl}G)^T(C_{cl}G) & \star \\ A_{cl}G + X + G & -2X \end{bmatrix} < 0.$$

Note that as $G^T + G > 0$, G is invertible. By performing congruence transformation $\begin{bmatrix} G^{-T} & 0 \\ 0 & X^{-1} \end{bmatrix}$ in the above inequality, we obtain

$$\begin{bmatrix} -(\tilde{G} + \tilde{G}^T) + \gamma^{-1}C_{cl}^T C_{cl} & \star \\ \tilde{X}A_{cl} + \tilde{X} + \tilde{G} & -2\tilde{X} \end{bmatrix} < 0.$$

where $\tilde{G} = G^{-1}$ and $\tilde{X} = X^{-1}$. The above inequality can be written as

$$\begin{bmatrix} \gamma^{-1}C_{cl}^TC_{cl} + \star & \\ \tilde{X}A_{cl} + \tilde{X} & -2\tilde{X} \end{bmatrix} + \text{herm} \left(\begin{bmatrix} -I \\ I \end{bmatrix} \tilde{G} \begin{bmatrix} I & 0 \end{bmatrix} \right) < 0.$$

According to the well-known Projection lemma, the above inequality holds if and only if the following two projection inequalities with respect to \tilde{G} are satisfied:

$$\begin{bmatrix} I \\ I \end{bmatrix}^T \begin{bmatrix} \gamma^{-1}C_{cl}^TC_{cl} & \star \\ \tilde{X}A_{cl} + \tilde{X} & -2\tilde{X} \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} < 0, \quad (28)$$

$$\begin{bmatrix} 0 \\ I \end{bmatrix}^T \begin{bmatrix} \gamma^{-1}C_{cl}^TC_{cl} & \star \\ \tilde{X}A_{cl} + \tilde{X} & -2\tilde{X} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} < 0. \quad (29)$$

As seen the inequality (29) indeed becomes the trivial inequality $-\tilde{X} < 0$ and (28) is equivalent to

$$\tilde{X}A_{cl} + A_{cl}^T\tilde{X} + \gamma^{-1}C_{cl}^TC_{cl} < 0,$$

which with pre- and post-multiplying by $X = \tilde{X}^{-1}$ leads to

$$A_{cl}X + XA_{cl}^T + \gamma^{-1}XC_{cl}^TC_{cl}X < 0,$$

The above inequality can simply be written as item *ii*), using the Schur complement.

B. Iterative algorithm for identifying row-column-sparsity patterns through DOF

Define the matrices $R_a = \begin{bmatrix} 0_{(n+p) \times n} & \mathbf{1}_{(n+p) \times m} \end{bmatrix}$ and $R_s = \begin{bmatrix} 0_{(n+m) \times n} & \mathbf{1}_{(n+m) \times p} \end{bmatrix}^T$. The optimization problem in (SAS), by letting $f(\mathcal{K}_a)$ and $g(\mathcal{K}_s)$ as (21) and (25) respectively, is equivalent to

$$\begin{aligned} & \text{minimize } \gamma_k + \eta_a \text{trace}(R_a \mathcal{W}) + \eta_s \text{trace}(\mathcal{W} R_s) \\ & \text{subject to (18), (19), and (10), for } k\text{-th channel,} \\ & \quad (18), (19), \text{ and (10), for } l\text{-th channel,} \\ & \quad \text{with given } \gamma_l, l \neq k, l = 1, \dots, \mathcal{N}, \\ & \quad -\mathcal{W} \leq \text{diag}(I, W_a) \cdot \tilde{\mathcal{K}} \cdot \text{diag}(I, W_s) \leq \mathcal{W}, \quad (30) \end{aligned}$$

where W_a and W_s denote the weighting matrices and the last inequality is element-wise with $\mathcal{W} \in \mathbb{R}^{(n+m) \times (n+p)}$ whose entries are nonnegative. Besides, to solve the above optimization problem, the following algorithm is utilized:

- Algorithm 1:* 1) With given $\varepsilon > 0$, $\beta > 0$, $\eta_a > 0$ and $\eta_s > 0$, initialize $W_a = I_m$, $W_s = I_p$, $l = 1$ and $\tilde{\mathcal{K}}^l = 0$.
- 2) Solve the minimization problem (30) to obtain $\tilde{\mathcal{K}}^*$ and the associated $\tilde{\mathcal{K}}_a^* = [\tilde{c}_c^* \tilde{d}_c^*]$ and $\tilde{\mathcal{K}}_s^* = [\tilde{b}_c^* \tilde{d}_c^*]^T$.
- 3) Update $W_{a,i}^l$ using the update rule in (22) (or (23) or (24)) and its counterpart $W_{s,i}^l$ for sensor selection e.g. in (26), form $W_a^l = \text{diag}[W_{a,i}^l]_{i=1}^m$ and $W_s^l = \text{diag}[W_{s,j}^l]_{j=1}^p$.
- 5) If $\|\tilde{\mathcal{K}}^* - \tilde{\mathcal{K}}^l\| \leq \beta$ go to Step 6, else $\tilde{\mathcal{K}}^l = \tilde{\mathcal{K}}^*$, $l = l + 1$ and return to Step 2.
- 6) Let the unnecessary rows of $\tilde{\mathcal{K}}_a^*$ and columns of $\tilde{\mathcal{K}}_s^*$ be zero and return $\Gamma_a^* = S(\tilde{\mathcal{K}}_a^*)$ and $\Gamma_s^* = S(\tilde{\mathcal{K}}_s^*)$.

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