Qualitative spatial representation and reasoning: A hierarchical approach

Sanjiang Li[1,2] and Bernhard Nebel[2]
[1] Department of Computer Science and Technology
Tsinghua University, Beijing 100084, China
[2] Institut für Informatik, Albert-Ludwigs-Universität Freiburg
D-79110 Freiburg, Germany

December 14, 2006

Abstract

The ability to reason in space is crucial for agents in order to make informed decisions. Current high-level qualitative approaches to spatial reasoning has serious deficiencies in not reflecting the hierarchical nature of spatial data and human spatial cognition. This paper proposes a framework for hierarchical representation and reasoning about topological information, where a continuous model of space is approximated by a collection of discrete sub-models, and spatial information is hierarchically represented in discrete sub-models in a rough set manner. The work is based on the GRCC theory, where continuous and discrete models of space are coped in a unified way. Reasoning issues such as determining the mereological (part-whole) relations between two rough regions are also discussed. Moreover, we consider an important problem that is closely related to *map generalization* in cartography and Geographical Information Science. Given a spatial configuration at a finer level, we show how to construct a configuration at a coarser level while preserving the mereological relations.

Keywords: Qualitative Spatial Reasoning; hierarchical spatial model; Generalized Region Connection Calculus; resolution; map generalization

1 Introduction

The ability to reason in space is crucial for agents in order to make informed decisions. There are in general two approaches for spatial representation and reasoning. The low-level quantitative approach is based on Euclidean geometry and plays a predominate role in disciplines such as computational geometry [17] and computer vision [11]. The second is the high-level qualitative approach, known as Qualitative Spatial Reasoning (QSR). When describing spatial configurations, qualitative representation and reasoning can be worthwhile if precise, quantitative information is not present or not desirable.

QSR is concerned with the qualitative aspects of representing and reasoning about spatial entities. Among the many aspects of space, topology is perhaps the most fundamental one. The Region Connection Calculus (RCC), initially developed in [18, 19, 4] by the Leeds group, is widely recognized as the major topological formalism for QSR.

1.1 The problem with current high-level approaches

Current high-level approaches to spatial reasoning, RCC in particular, suffer from the following two serious deficiencies:

- they do not relate to quantitative spatial reasoning; and
- they do not reflect the hierarchical nature of spatial data and human spatial cognition.

It has been widely recognized that the high-level, logic-based approaches have little relevance to the quantitative approaches adopted in the acquisition, storage, and manipulation of real-world data. One important reason for this lies in the discrepancy between the continuous space models favored by high-level approaches and the discrete, digital representations used at the low-level [9].

As a matter of fact, one axiom of RCC specifies that each region contains a (non-tangential) proper part, which makes RCC has nothing to do with discrete spaces (in the sense that each region is a union of atomic regions). On the other hand, discrete spaces are evidently important in real-world applications such as digital image processing, manipulation of various kinds of networks in GIS, etc. This discrepancy between qualitative and quantitative approaches to spatial information closely relates to the vector-raster debate in spatial data handling [25].

The second problem is concerned with hierarchy and granularity. A granularity is formed by abstracting away from the world only related information, and a series of abstracting activities result in a hierarchy of granularities. Hierarchy is fundamental to human cognition [12, p.310], and "our ability to conceptualize the world at different granularities and to switch among these granularities is fundamental to our intelligence and flexibility." [10]

In particular, since the spatial environment is infinitely complex, humans typically use hierarchies as the major conceptual tool to structure and reason about the infinite levels of details [30]. This coarse-to-fine approach is usually very efficient since a lot of unrelated information has been discarded, and we need to focus only on a rather restricted domain.

Current high-level formalisms of spatial reasoning, like RCC, consider only ideal, infinite-precision information. It is strongly desirable to extend these theories to support multi-representation and hierarchical reasoning.

The above two problems are closely related. On the one hand, to provide a hierarchical approach to qualitative spatial reasoning, we need to relate discrete representations to the continuous models, which are favored by high-level approaches. In particular we should develop a high-level approach to discrete spaces that is compatible with existing formalisms of QSR. In this context, a discrete model can be taken as an approximation of continuous ones at certain finite-precision. And, on the other hand, a hierarchical structure would be crucial in understanding the relation between discrete and continuous representations of spatial information. In other words, the hierarchical method will bridge the gap between high-level qualitative approaches to spatial information and low-level quantitative ones.

1.2 Related work

In recent years efforts [8, 9, 15, 23] have been devoted to provide a high-level, qualitative account for discrete spaces. These works, however, do not consider problems like "How can a discrete model be linked to a continuous one?" and "Can we construct continuous models step-by-step, using discrete (or even finite) models?" It is in this sense we say that the relationship between discrete models and continuous ones is still unclear, and the connection between high-level (qualitative) and low-level (quantitative) approaches to spatial information handling is still missing.

The importance of hierarchical reasoning has been identified by several authors in the field of Geographical Information Science [30, 29, 2, 32]. Timpf and Frank [30] give a definition of hierarchical spatial reasoning, using hierarchical spatial data structures, which computes increasingly better results in a hierarchical fashion and stops the computation when a 'good enough' result is achieved. Based on this general approach, Winter [32] proposes a hierarchical method for determining mereological (part-whole) relations between two regions represented as quadtrees [24].

Worboys [33] provides a formal framework for treating the notion of resolution and multi-resolution in geographic spaces. He goes further to develop a rough set [16] like approach to reasoning with imprecision about spatial entities and relationships resulting from finite resolution representations. Stell and Worboys [28] also introduce a concept of stratified map space to deal with generalization and vagueness in multi-resolution spatial data handling.

Worboys' approach is to a certain extent orthogonal to Timpf and Frank's hierarchical approach. On the one hand, the multi-resolution representation is more general than the hierarchical representation in the sense that two resolutions may be incomparable, and on the other hand, Worboys does not consider how to reason hierarchically in this framework.

1.3 Our approach

The main objective of this paper is to establish a hierarchical approach to high-level spatial reasoning. Our research is based on the well known RCC theory, which was first proposed by Randell, Cohn, and Cui [18, 19, 4], and studied by many other researchers, see e.g. [21, 26, 6, 20, 13, 14]. In particular, Li and Ying [14] propose a generalization of RCC, termed GRCC, which accommodates both continuous (RCC) and discrete models of space. Several categorical notions such as sub-models and direct limits are introduced in the GRCC theory. Moreover, an approach to constructing RCC models as direct limits of collections of finite models is also given. These notions can be useful for clarifying the connection between discrete and continuous models, and it would be natural to approximately represent space as a family of discrete models which varies over a lattice of levels of details. This paper can be regarded as a continuation of the research line started in Li and Ying [14].

We begin with two definitions of hierarchical spatial model in the GRCC theory. Then we propose an approach to reasoning with imprecision about spatial entities and relationships in a fixed resolution. We represent each object x in a fixed resolution as a pair $(\imath(x), \jmath(x))$ with the constraint that $\imath(x)$ is a (possibly proper) part of $\jmath(x)$. This rough set approach has been used by Cohn and Gotts [3] and Worboys [33]. We further propose several methods to classify the relationships between regions represented in a fixed resolution. Next, we give rules to deduce the possible relation between two objects from the information obtained at the present resolution. Moreover, we consider an important problem that is closely related to *map generalization* in cartography and Geographical Information Science. Given a spatial configuration at a finer level, we show how to construct a configuration at a coarser level while preserving the mereological relations.

1.4 Structure of this paper

The rest of this paper is organized as follows. In Section 2 we recall the generalized RCC theory proposed in [14] and then introduce two definitions of hierarchical spatial models in Section 3. Section 4 concerns the representation of regions and their relationships in a fixed resolution. In Section 5 we give rules to determine the mereological relations between rough regions. Section 6 considers how to generalize information at finer resolution to coarser resolution without changing the relations between any two regions. Conclusions and further work are given in Section 7.

2 Generalized Region Connection Calculus

In this section we recall some basic notions of the generalized RCC model proposed in Li and Ying [14]. The Generalized Region Connection Calculus (GRCC), following RCC, is an axiomatization of space that takes regions as primitive.

Definition 2.1 (GRCC model [14])**.** A GRCC model is a complete Boolean algebra $B = \langle B; 0, 1, +, \cdot, - \rangle$ together with a binary relation $\mathbf{C}$ on B such that $\mathbf{C}$, called a *contact* relation, satisfies the following conditions:

(C1) $\mathbf{C}$ is a reflexive and symmetric relation on $B \setminus \{0\}$.

(C2) For all $x, y, z \in B \setminus \{0, 1\}$, $\mathbf{C}(x + y, z)$ iff either $\mathbf{C}(x, z)$ or $\mathbf{C}(y, z)$ holds.

(C3) For all $x \in B \setminus \{0, 1\}$, $\mathbf{C}(x, -x)$, where $-x$ is the complement of x in B.

Standard GRCC models arise from connected topological spaces.

Example 2.1. Suppose X is a connected topological space, write $\mathsf{RC}(X)$ for the complete Boolean algebra of regular subsets of X. For any two nonempty regular sets $A, B \in \mathsf{RC}(X)$, define $A\mathbf{C}_X B$ if and only if $A \cap B \neq \varnothing$. It is straightforward to show that $\mathbf{C}_X$ is a contact relation on $\mathsf{RC}(X)$. We call $\langle \mathsf{RC}(X), \mathbf{C}_X \rangle$ the standard GRCC model on X.

For a GRCC model $\langle B, \mathbf{C} \rangle$, we often call a non-zero element in B a *region*, and call 1 the *universe*. For two elements a, b in B, we write $a \leq b$ if $a + b = b$, and write $a < b$ if $a \leq b$ but not $b \leq a$. An element a in B is an *atom* if $b < a$ implies $b = 0$ for all $b \in B$. Since elements in a GRCC model are interpreted as regions, we also call $\leq$ the *part-of* relation, and write $\mathbf{P}$ for $\leq$.

In a GRCC model $\langle B, \mathbf{C} \rangle$, using the part-of relation $\mathbf{P}$ and the contact relation $\mathbf{C}$, we can define a collection of relations on B. Table 1 summarizes the definition of these relations. Note that the relations **EQ**,**PO**,**O**,**DR**,**DC**,**EC** are symmetrical, and the relations **P**,**PP**,**TPP**,**NTPP** are non-symmetrical. For a non-symmetrical relation $\mathbf{R}$, we write $\mathbf{R}^\smile$ for its converse. The relations

$$\mathbf{EQ}, \mathbf{DR}, \mathbf{PO}, \mathbf{PP}, \mathbf{PP}^\smile \tag{1}$$

form a jointly exhaustive and pairwise disjoint (JEPD) set of relations, which are known as the RCC5 basic relations. The RCC5 algebra consists of unions (or disjunctions) of RCC5 basic relations, and an RCC5 relation is also known as a *mereological* relation or a *part-whole* relation.

Note that **DR** can be divided into **EC** and **DC**, **PP** ($\mathbf{PP}^\smile$, resp.) can be divided into **TPP** and **NTPP** ($\mathbf{TPP}^\smile$ and $\mathbf{NTPP}^\smile$, resp.). This results in another JEPD set of relations, known as the RCC8 basic relations.

$$\mathbf{EQ}, \mathbf{DC}, \mathbf{EC}, \mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^\smile, \mathbf{NTPP}, \mathbf{NTPP}^\smile \tag{2}$$

The RCC8 algebra consists of unions (or disjunctions) of RCC8 basic relations, and an RCC8 relation is also known as a *topological* relation.

Next, we consider two special kinds of GRCC models.

Definition 2.2 ([14])**.** A GRCC model $\langle B, \mathbf{C} \rangle$ is *continuous* if each region in B contains a non-tangential proper part. We say $\langle B, \mathbf{C} \rangle$ is *discrete* if B is atomic complete, i.e. each region in B is the sum of all atoms it contains.

A continuous GRCC model is precisely an RCC model [19]. Recently, Düntsch and Winter [7] show that every continuous model is isomorphic to a substructure of some standard model on a certain connected space X.

A discrete GRCC model corresponds to a model of Galton [9]. In fact, write $\mathbf{AT}(B)$ for the set of atoms in B, and write $\mathbf{A}$ for the restriction of $\mathbf{C}$ to $\mathbf{AT}(B)$.

Table 1: Relations defined in GRCC

Relation	Interpretation	Definition
EQ	a is *identical with* b	$a = b$
DR	a is *discrete from* b	$a \cdot b = 0$
PP	a is a *proper part of* b	$a < b$
O	a *overlaps* b	$a \cdot b > 0$
PO	a *partially overlaps* b	$(a\mathbf{O}b) \wedge (a \not\leq b) \wedge (a \not\geq b)$
DC	a is *not in contact with* b	$\neg(a\mathbf{C}b)$
EC	a is *in external contact with* b	$(a\mathbf{C}b) \wedge (a\mathbf{DR}b)$
TPP	a is a *tangential proper part of* b	$(a\mathbf{PP}b) \wedge (a\mathbf{EC} - b)$
NTPP	a is a *non-tangential proper part of* b	$a\mathbf{DC} - b$

Then $\mathbf{A}$ is an *adjacency* relation in the sense of Galton [9]. The following lemma links adjacency relations to contact relations.

Lemma 2.1 ([14])**.** *Let B be a discrete GRCC model. Then two regions a, b in B are in contact, i.e. $a\mathbf{C}b$, if and only if we have two atoms $w_1 \leq a$ and $w_2 \leq b$ such that w_1 and w_2 are in contact, i.e. $w_1\mathbf{A}w_2$.*

3 Hierarchical spatial models: definitions and examples

Intuitively, a hierarchical structure arises from a system of resolutions. In this section we introduce two definitions of hierarchical models, and give two examples.

For a set of locations S, Worboys [33] defines a resolution to be a finite partition of S, where a partition is in the set-theoretical sense. This definition may lead to some problems when we interpret elements of a partition as closed regions: adjacent elements share their boundary. This violates the set-theoretical definition of a partition. In this paper we adopt the following lattice theoretical definition of partition.

Definition 3.1 (Partition)**.** Let $B = \langle B; 0, 1, +, \cdot, - \rangle$ be a complete Boolean algebra. For $a \in B$, we say $X = \{d_i : i \in I\} \subset B$ is a *partition* of a if the following condition holds:

$$\Sigma_{i \in I} d_i = a \text{ and } (\forall i \in I) d_i \neq 0 \text{ and } (\forall i, j \in I)[i \neq j \rightarrow d_i \cdot d_j = 0]. \quad (3)$$

The following lemma characterizes the concept of partition in standard GRCC models (see Example 2.1).

Lemma 3.1. *Let X be a connected topological space, and let a be a nonempty regular closed subset of X, i.e. a is a region in $\mathsf{RC}(X)$. A collection of regions $\{a_i\}_{i \in I}$ in $\mathsf{RC}(X)$ is a partition of a if and only if $\overline{\bigcup_{i \in I} a_i} = a$, and for all $i \neq j$ we have $(a_i \cap a_j)^\circ = \varnothing$, where $\overline{b}$ and b° are, respectively, the closure and interior of set b.*

We now define the concept of resolution in the context of GRCC theory.

Definition 3.2 (Resolution)**.** Given a GRCC model $\langle B, \mathbf{C}\rangle$ and a sub-algebra B' of B, write $\mathbf{C}|_{B'}$ for the restriction of $\mathbf{C}$ to $B' \times B'$, and call $\langle B', \mathbf{C}|_{B'}\rangle$ a *sub-model* of $\langle B, \mathbf{C}\rangle$. For a sub-model $\langle B', \mathbf{C}|_{B'}\rangle$, if B' is discrete, then we call $\langle B', \mathbf{C}|_{B'}\rangle$ a *resolution* of B, and call each atom of B' a B'*-granule* or a B'*-pixel* of B.

We note here that if B' is a resolution of B, then the set of atomic regions (i.e. granules) in B' is a partition of the universe 1.

For a GRCC model B, suppose $\mathcal{R}$ is the set of all resolutions of B. There is a natural partial order on $\mathcal{R}$. Given two resolutions B_1 and B_2, we say B_1 is *finer* than B_2, written $B_1 \prec B_2$, if B_2 is a sub-algebra of B_1. In terms of granules, B_1 is finer than B_2 if and only if each B_1-granule is contained in some B_2-granule. In this sense, we say the grain size, or the *granularity*, of B_1 is *finer* than that of B_2.

Two resolutions B_1 and B_2 can be incomparable (cf. [33]). Moreover, we have the following result.

Theorem 3.1. *For a GRCC model B, suppose $\mathcal{R}$ is the set of all resolutions of B. Set $B_1 \sqcup B_2 = B_1 \cap B_2$ and set $B_1 \sqcap B_2$ to be the sub-algebra generated by $B_1 \cup B_2$. Then $\langle \mathcal{R}; \prec; \sqcap, \sqcup\rangle$ is a lattice.*

Proof. This follows from that $B_1 \sqcup B_2$ ($B_1 \sqcap B_2$, resp.) is the finest (the coarsest, resp.) resolution which is coarser (finer, resp.) than both B_1 and B_2. □

We call $\langle \mathcal{R}; \prec; \sqcap, \sqcup\rangle$ the *resolution lattice* of B. Note that this lattice may be *unbounded*. On the one hand, we know $2 = \{0, 1\}$ is the coarsest resolution. But on the other hand, there may be no finest resolution. This is because B may not be discrete.

Given a GRCC model B, a hierarchical spatial model can be obtained by associating a collection of resolutions of B, where each resolution can be taken as an approximation of B at a certain coarse granularity.

Definition 3.3 (Hierarchical spatial model)**.** Let B be a GRCC model, and let $\mathcal{R}$ be its resolution lattice. A hierarchical spatial model over B is a pair $(B, \mathcal{R}_1)$, where $\mathcal{R}_1$ is a sub-lattice of $\mathcal{R}$.

In GIS, hierarchical structures usually are obtained by recursively decomposing a plane region into sub-regions at different levels. This means, in our terms, the set of resolutions we choose (viz. $\mathcal{R}_1$) is in liner order. This observation leads to the following restricted version of a hierarchical model.

Definition 3.4 (Restricted hierarchical spatial model)**.** Let B be an RCC model, and let $\mathcal{R}$ be the resolution lattice of B. A *restricted hierarchical spatial model* over B is a pair $(B, \mathcal{R}_1)$, where

$$\mathcal{R}_1 = \{B_1, B_2, \cdots\}$$

is a sequence of resolutions in $\mathcal{R}$ such that B_{i+1} strictly refines B_i for each $i \geq 1$ and $\bigcup_{i\geq 1} B_i = B$.

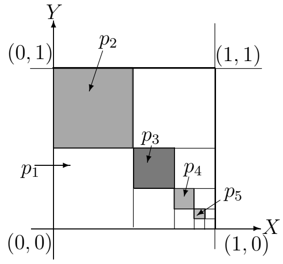


Figure 1: A fragment of a regular hierarchical partition, where p_i is an i-th pixel, and p_{i+1} is a sub-pixel of p_1 for $2 \leq i \leq 5$.

For a restricted hierarchical spatial model B, each region in B can be completely determined in a certain fine enough resolution. This is because B is the union of all resolutions B_i.

The most important example of hierarchical spatial model arises from hierarchical regular partitions of the real plane, where we hierarchically decompose the plane into small squares of equal size, often called *pixel*s (see Figure 1). This means each pixel in the i-th partition, called an i-pixel, can be further decomposed into 4 equally sized pixels of the $(i+1)$-th partition. A region in the i-th partition is defined to be the union of a nonempty set of i-pixels. Write B_i for the set of regions in the i-th partition together with the empty set. Then each B_i is an atomic complete Boolean algebra, which is isomorphic to the powerset algebra of the digital plane $\mathbb{Z}^2$. Moreover, for each $i \geq 1$, B_i is a proper subalgebra of B_{i+1}. Write B for the set of regions in all partitions together with the empty set. Then $B = \bigcup_{i \geq 1} B_i$ and B is an atomless Boolean algebra.

There are, among others, two possible ways of defining the contact relation on B. They are based on, respectively, the well-known notion of 4- and 8-neighborhood in digital topology [22] (see Figure 2). For a fixed i, two i-pixels are *4-adjacent* if they are either identical or 4-neighbors; correspondingly, two regions a, b in B_i are in *4-contact* if there are two 4-adjacent pixels contained respectively in a and b.

Lemma 3.2. *With the 4-contact relation each B_i is a discrete GRCC model.*

Proof. Since each B_i is atomic complete, we need only to show that B_i satisfies the three conditions (C1-3) in Definition 2.1. Take (C3) as an example. For a region $0 \neq x \neq 1$ we show x is in 4-contact with its complement $-x$. There is an i-pixel p contained in x such that x does not contain all 4-neighbors of p. Write q for a 4-neighbor of p such that q is not contained in x. This means q is contained in $-x$. Since p and q are 4-adjacent, we know x and its complement are in 4-contact. □

In general, for two regions a, b in B, recall that $B = \bigcup_{k \geq 1} B_k$, there is some $i \geq 1$ such that $a, b \in B_i$. We define a, b to be in *4-contact* if they are in 4-

Figure 2: The 4 and 8 neighbors of a pixel p.

contact in B_i, i.e. there are two 4-adjacent i-pixels contained in a, b respectively. This is well defined because, if two regions a, b are in 4-contact in B_i, then they are also in 4-contact in B_j for any $j \geq i$.

Write $\mathbf{C}_4$ for this contact relation on B. Then we have the following result.

Theorem 3.2. *For $B = \bigcup_{k\geq 1} B_k$ and $\mathbf{C}_4$ defined as above, $\langle B, \mathbf{C}_4\rangle$ is a continuous GRCC model.*

Proof. By Lemma 3.2 it is clear that $\langle B, \mathbf{C}_4\rangle$ is a GRCC model. To show it is continuous, note that each i-pixel contains a $(i+2)$-pixel as a non-tangential proper part (ntpp for short). For example, p_3 is an ntpp of p_1 in Figure 1. □

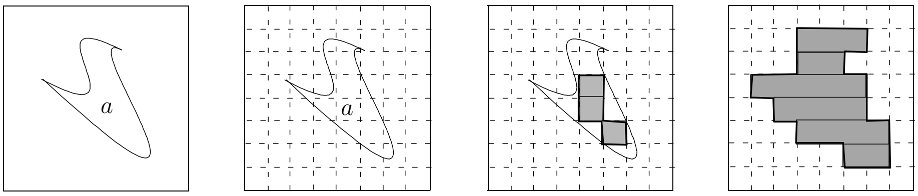
Similar definitions and results apply to $\langle B, \mathbf{C}_8\rangle$.

Theorem 3.3. *For $B = \bigcup_{k\geq 1} B_k$ and $\mathbf{C}_8$, $\langle B, \mathbf{C}_8\rangle$ is a continuous GRCC model.*

It is straightforward to show that two regions in B are in 8-contact if and only if they share a common point. Also note that each region in B is a plane region. This suggests that $\langle B, \mathbf{C}_8\rangle$ is a sub-model of the standard RCC model on the plane (see Example 2.1).

But the situation is different for $\langle B, \mathbf{C}_4\rangle$. Two regions are in 4-contact if they share a segment or a sub-region. Clearly, two 4-contact regions are necessarily 8-contact. The stronger 4-contact relation can be useful in describing situations, say, a worm passing from one region to another without being perceived.

Remark 3.1. One reviewer raised the following interesting question: is it possible each RCC model to be represented as a restricted hierarchical model? The answer is negative, however. Take $X = [0,1]^2$ and consider the standard RCC model $\mathsf{RC}(X)$. Suppose $\mathsf{RC}(X)$ can be represented as a restricted hierarchical model by $\mathsf{RC}(X) = \bigcup_{i\geq 1} B_i$, where $B_1, \cdots, B_n, \cdots$ is a sequence of resolutions of $\mathsf{RC}(X)$ such that B_{i+1} strictly refines B_i for all $i \geq 0$. Note that $\mathsf{RC}(X)$ is a continuous model. We can find a sequence $p_1, p_2, \cdots$ such that $p_k \in \mathbf{AT}(B_{i_k})$ and p_{k+1} is an ntpp of p_k for each $k \geq 1$, where $k \leq i_k < i_{k+1}$ for each $k \geq 1$. Let P be a point in $\bigcap_{k=1}^{\infty} p_k$. Such a point P exists because each p_k is a bounded closed, hence compact, subset of X. Let b be a square contained in X such that P is one of its four endpoints. Clearly, we have $b^\circ \cap p_k^\circ \neq \varnothing$ and $p_k \not\subseteq b$. Moreover, for each atomic region p in B_{i_k}, P is either an interior or exterior point of p. This suggests that b is not a region in B_{i_k}. Therefore $\mathsf{RC}(X) \neq \bigcup_{k\geq 1} B_{i_k}$. But by $k \leq i_k$ and $B_k \subseteq B_{i_k}$ we know $\bigcup_{k\geq 1} B_{i_k} = \bigcup_{i\geq 1} B_i$. This is a contradiction.



a region a in B | a resolution B_1 of B | lower appr.: $\imath(a)$ | upper appr.: $\jmath(a)$

Figure 3: The lower and upper approximation of a region a in the resolution B_1 of a GRCC model B.

4 Spatial representation in a fixed resolution

In this section we discuss how spatial objects and their relation can be represented in a fixed resolution. In what follows, we suppose B is a GRCC model, B_1 is a nontrivial resolution of B, and write $\mathbf{AT}(B_1)$ for its atoms set. This means we require $\{0,1\} \subsetneqq B_1 \subsetneqq B$.

4.1 Rough regions

Since B_1 is a proper subset of B, not all regions in B can have exact representations in B_1. We approximately represent all regions in B at the present resolution in a rough set [16] manner. This method has been adopted by several authors in QSR (see e.g. [3, 5, 33], but also see [1]).

Definition 4.1 (Lower and upper approximation)**.** For a B-region x, we call $\imath(x)$ ($\jmath(x)$, resp.) the *lower approximation* (*upper approximation*, resp.) of x in B_1, where

$$\imath(x) = \sum\{a \in B_1 : a \leq x\}, \tag{4}$$

$$\jmath(x) = \prod\{a \in B_1 : a \geq x\}. \tag{5}$$

The above definition is well-defined since B_1, as a discrete sub-algebra of B, is closed under sums and products.

For any B-region x, $\imath(x)$ is the largest B_1-region contained in x and $\jmath(x)$ is the smallest B_1-region containing x. (see Figure 3) We can interpret $\imath(x)$ as the B_1-interior of x, and $\jmath(x)$ as the B_1-closure of x. We stress that $\imath(x)$ could be 0.

Since B_1 is generated by its atoms set $\mathbf{AT}(B_1)$, we have the following results.

Lemma 4.1. *For any B-region x, we have*

$$\imath(x) = \sum\{w \in \boldsymbol{AT}(B_1) : w \leq x\}, \tag{6}$$

$$\jmath(x) = \sum\{w \in \boldsymbol{AT}(B_1) : w \cdot x \neq 0\}. \tag{7}$$

In this paper we represent each region in $x \in B$ as a pair $(\imath(x), \jmath(x))$ at the present resolution B_1, and call x a *rough* region in B_1. If x happens to be in B_1, we say x is a *crisp* region in B_1, and otherwise call x a *vague* region in B_1. Note that for a region x in B, $x \in B_1$ if and only if $\imath(x) = \jmath(x)$.

Remark 4.1. Our approach is similar to that of Worboys [33] in the way spatial objects are represented in a fixed resolution. But Worboys [33] allows more relaxed representations, where a region x may be represented as any pair $(l(x), u(x))$ with the constraint $l(x) \leq \imath(x) \leq \jmath(x) \leq u(x)$. The *egg-yolk* approach [3], however, is developed directly on a continuous model and does not assume a discrete resolution.

4.2 Rough relations

The RCC5 mereological relations and the RCC8 topological relations are designed for crisp regions. In this section we describe ways of extending these relations to rough regions in a fixed resolution.

Suppose B is a GRCC model and B_1 is a nontrivial resolution of B, i.e. $\{0, 1\} \subset B_1 \subset B$. For a JEPD set of relations $\mathcal{B}$ on crisp regions, we now extend relations in $\mathcal{B}$ to rough regions.

In the remainder of this paper, we require the lower approximation $\imath(x)$ of a rough region $x = (\imath(x), \jmath(x))$ in B_1 to be nonzero. This is natural since, given a fixed resolution, we usually are only interested in objects that can be precisely represented to a certain extent at the present resolution.

Definition 4.2 (Rough relation)**.** Given two regions $x, y \in B$, the rough relation R between x, y is represented as a 4-tuple

$$\langle \rho(\imath(x), \imath(y)), \rho(\jmath(x), \jmath(y)), \rho(\imath(x), \jmath(y)), \rho(\jmath(x), \imath(y)) \rangle \tag{8}$$

where $\rho(u, v)$ is the $\mathcal{B}$-relation between crisp regions u and v in B_1.

Remark 4.2. There are some other methods for approximately representing relations between rough regions, where *some* instead of *all* crisp relations in the above 4-tuple are considered. The lower approximation method (LAM) considers only the relation between $\imath(x)$ and $\imath(y)$, while the upper approximation method (UAM) considers the relation between $\jmath(x)$ and $\jmath(y)$. These two methods are rather imprecise. The lower-upper approximation method (LUAM) combines the previous two methods and consider both the relation between $\imath(x)$ and $\imath(y)$ and the relation between $\jmath(x)$ and $\jmath(y)$.

We fix some notations here. In what follows, we denote rough relations by capital roman fonts R, S, T (and some times M, N), and use corresponding bold fonts, often with subscripts, say $\mathbf{R}_{\imath\jmath}$, to denote the crisp relations between their approximations. For a rough relation R, we write, respectively, $\mathbf{R}_{\imath\imath}, \mathbf{R}_{\jmath\jmath}, \mathbf{R}_{\imath\jmath}, \mathbf{R}_{\jmath\imath}$ for the $\mathcal{B}$-relations between $(\imath(x), \imath(y))$, $(\jmath(x), \jmath(y))$, $(\imath(x), \jmath(y))$, and $(\jmath(x), \imath(y))$. This means $R = \langle \mathbf{R}_{\imath\imath}, \mathbf{R}_{\jmath\jmath}, \mathbf{R}_{\imath\jmath}, \mathbf{R}_{\jmath\imath} \rangle$.

Since each of $\mathbf{R}_{\imath\imath}, \mathbf{R}_{\jmath\jmath}, \mathbf{R}_{\imath\jmath}, \mathbf{R}_{\jmath\imath}$ can be any basic relation in $\mathcal{B}$, there are $|\mathcal{B}|^4$ many possible rough relations between rough regions. A question that arises naturally is to determine whether a 4-tuple R is realizable, i.e. there are two rough regions related by R. When $\mathcal{B}$ contains a large set of relations, the computation work is arduous and error-prone. For example, for the rather small RCC8 basic relations, we should check $8^4 = 4096$ 4-tuples. Fortunately, questions like this can be reformulated as a constraint satisfaction problem.

We take the RCC5 relations as an example, but the approach can be extended to other RCC systems. Our approach is based on the following theorem, which reduces the computation of realizable 4-tuples to a simple consistency checking work.

Theorem 4.1. *An RCC5 4-tuple $\langle \mathbf{R}_{\imath\imath}, \mathbf{R}_{\jmath\jmath}, \mathbf{R}_{\imath\jmath}, \mathbf{R}_{\jmath\imath}\rangle$ is a possible rough relation between two rough regions if and only if the following RCC5 constraint network Θ is consistent.*

$$\Theta = \{\imath(x)\mathbf{P}\jmath(x), \imath(y)\mathbf{P}\jmath(y), \imath(x)\mathbf{R}_{\imath\imath}\imath(y), \jmath(x)\mathbf{R}_{\jmath\jmath}\jmath(y), \imath(x)\mathbf{R}_{\imath\jmath}\jmath(y), \jmath(x)\mathbf{R}_{\jmath\imath}\imath(y)\}$$

where $\mathbf{R}_{\imath\imath}, \mathbf{R}_{\jmath\jmath}, \mathbf{R}_{\imath\jmath}, \mathbf{R}_{\jmath\imath}$ are all RCC5 basic relations.

Proof. Note that relations appeared in Θ are in the set

$$\mathcal{S} = \{\mathbf{P}, \mathbf{EQ}, \mathbf{DR}, \mathbf{PO}, \mathbf{PP}, \mathbf{PP}^{\smile}\}.$$

This set is contained in $\widehat{\mathcal{H}_8}$, one maximal tractable fragment of RCC8 identified by Renz and Nebel [21]. Now since path-consistency decides consistency for any constraint network over $\widehat{\mathcal{H}_8}$, we can apply a path-consistency algorithm for each RCC5 4-tuple to decide whether it is a realizable rough relation. □

There are $5^4 = 625$ RCC5 4-tuples. But only 51 survive the above test. These 51 rough relations are those (46 cases) given in Table 7 and the following

- $\langle \mathbf{EQ}, \mathbf{EQ}, \mathbf{EQ}, \mathbf{EQ}\rangle$ (if x and y are two identical crisp regions),
- $\langle \mathbf{EQ}, \mathbf{PP}, \mathbf{PP}, \mathbf{EQ}\rangle$ (if x is crisp and y is vague, and $x = \imath(y)$),
- $\langle \mathbf{PP}^{\smile}, \mathbf{EQ}, \mathbf{EQ}, \mathbf{PP}^{\smile}\rangle$ (if x is crisp and y is vague, and $x = \jmath(y)$),
- $\langle \mathbf{EQ}, \mathbf{PP}^{\smile}, \mathbf{EQ}, \mathbf{PP}^{\smile}\rangle$ (the converse of the second case),
- $\langle \mathbf{PP}, \mathbf{EQ}, \mathbf{PP}, \mathbf{EQ}\rangle$ (the converse of third case).

Remark 4.3. Similar classifications have been made by Cohn and Gotts [3] and Stell [27] for *egg-yolk* regions, where an egg-yolk region in a continuous model B is a pair (r, s) such that $r, s \in R$ and $r \leq s$. Assuming $r \neq s$ and $r, s > 0$, Cohn and Gotts find 46 possible relations between egg-yolk regions, which are showed in Table 7. On the other hand, Stell shows that there are 85 realizable relations if $0 \leq r \leq s$. Note that r and s may be any region in B. This is different from our approach: We explicitly require a fixed resolution B_1 and only consider egg-yolk regions in B_1 instead of B. Moreover, in our approach each egg-yolk region in B_1 is an approximation of some region in B. Furthermore, we use Theorem 4.1 to determine whether a 4-tuple is realizable. This is different from the approaches of Cohn and Gotts [3] and Stell [27].

Table 2: Relations between a crisp region and a vague region

$(\imath(x), \imath(y)) = (\jmath(x), \imath(y))$	$(\jmath(x), \jmath(y)) = (\imath(x), \jmath(y))$	(x, y)
EQ	**PP**	**PP**
DR	**DR**	**DR**
DR	**PP**	**PO**
DR	**PO**	**PO**
PP	**PP**	**PP**
$\mathbf{PP}^{\smile}$	**EQ**	$\mathbf{PP}^{\smile}$
$\mathbf{PP}^{\smile}$	**PP**	**PO**
$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$
$\mathbf{PP}^{\smile}$	**PO**	**PO**
PO	**PP**	**PO**
PO	**PO**	**PO**

Quite often it is desirable to make further classifications according to whether a and/or b is crisp in B_1. There are 4 cases. Recall that we assume $\imath(x) > 0$ and $\imath(y) > 0$.

- c-c Both x and y are crisp regions. In this case, $\rho(\imath(x), \imath(y)) = \rho(\jmath(x), \jmath(y)) = \rho(\imath(x), \jmath(y)) = \rho(\jmath(x), \imath(y))$, there are only 5 possible relations, viz. the 5 basic RCC5 relations.

- c-v x is crisp and y is vague. In this case, $\rho(\imath(x), \imath(y)) = \rho(\jmath(x), \imath(y))$ and $\rho(\jmath(x), \jmath(y)) = \rho(\imath(x), \jmath(y))$, there are 11 possible relations, which are showed in Table 2.

- v-c x is vague and y is crisp. In this case, $\rho(\imath(x), \imath(y)) = \rho(\imath(x), \jmath(y))$ and $\rho(\jmath(x), \jmath(y)) = \rho(\jmath(x), \imath(y))$, there are also 11 possible relations, which are showed in Table 3.

- v-v Both x and y are vague regions. This case corresponds to the egg-yolk calculus by Cohn and Gotts [3]. There are altogether 46 possible relations in this case (see Table 7).

Remark 4.4. Suppose we adopt the lower-upper approximation method (LUAM), where the relation between x and y is approximately represented by $\rho(\imath(x), \imath(y))$ and $\rho(\jmath(x), \jmath(y))$. There are at most 25 possible different relations. But since $\imath(x) \leq \jmath(x)$ and $\imath(y) \leq \jmath(y)$, the relation pairs $(\mathbf{R}, \mathbf{DR})$ are impossible, where $\mathbf{R}$ can be either of **EQ**, **PO**, **PP**, $\mathbf{PP}^{\smile}$. So using LUAM we obtain 21 possible relations. These are summarized in Table 6.

5 Reasoning in a fixed resolution

In this section we consider how to reason about information represented in a fixed resolution.

Table 3: Relations between a vague region and a crisp region

$(\imath(x), \imath(y)) = (\imath(x), \jmath(y))$	$(\jmath(x), \jmath(y)) = (\jmath(x), \imath(y))$	(x, y)
EQ	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$
DR	**DR**	**DR**
DR	$\mathbf{PP}^{\smile}$	**PO**
DR	**PO**	**PO**
PP	**EQ**	**PP**
PP	**PP**	**PP**
PP	$\mathbf{PP}^{\smile}$	**PO**
PP	**PO**	**PO**
$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$
PO	$\mathbf{PP}^{\smile}$	**PO**
PO	**PO**	**PO**

Suppose B is a GRCC model and B_1 is a resolution of B. Given two rough regions x, y in B_1, suppose we know (or partially know) the rough mereological relation between x and y. Then what about the RCC5 relation between x and y? For example, if we know $\imath(x)\mathbf{DR}\imath(y)$ and $\jmath(x) = \jmath(y)$, then what can be said about the RCC5 relation between x and y. Can they be discrete? Equal?

In what follows, we restrict our discussion to rough regions that have nonzero lower approximation. We have the following basic rules:

(1) If $x \leq y$, then $\imath(x) \leq \imath(y)$ and $\jmath(x) \leq \jmath(y)$;

(2) If $x \geq y$, then $\imath(x) \geq \imath(y)$ and $\jmath(x) \geq \jmath(y)$.

More rules can be obtained by applying the basic rules together with the assumptions that $\imath(x) > 0$ and $\imath(y) > 0$. Two examples are given below.

Lemma 5.1. *If $\imath(x)\mathbf{DR}\imath(y)$, then $x\mathbf{DR}y$ or $x\mathbf{PO}y$.*

Proof. By the basic rules we know $x \not\leq y$ and $x \not\geq y$. The conclusion follows directly. □

Under the assumption that each rough region has a nonzero lower approximation, we have the following lemma.

Lemma 5.2. *Suppose $\jmath(x) = \jmath(y)$. Then x overlaps y.*

Proof. This is because by $\imath(x) > 0$ we have an atomic region $w \in B_1$ such that $w \leq \imath(x)$. Note that w is also contained in $\jmath(y)$ since $\jmath(x) = \jmath(y)$. By Equations 6 and 7, we have $w \leq x$ and $w \cdot y > 0$. Therefore $x \cdot y > 0$, hence $x\mathbf{O}y$. □

We summarize these rules in the following tables. In what follows, we say an RCC5 relation $\mathbf{R}$ is *definite* if it is a basic relation, and say $\mathbf{R}$ is *indefinite* if it is not.

Table 4 asserts which RCC5 relation can hold between x and y if we know the definite RCC5 relation between $\imath(x)$ and $\imath(y)$. Table 5 infers the RCC5

Table 4: Determining the RCC5 relations using lower approximation

$(\imath(x), \imath(y))$	**EQ**	**DR**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
(x, y)	**O**	**PO**,**DR**	**PO**,**PP**	**PO**,$\mathbf{PP}^{\smile}$	**PO**

Table 5: Determining the RCC5 relations using upper approximation

$(\jmath(x), \jmath(y))$	**EQ**	**DR**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
(x, y)	**O**	**DR**	**PO**,**PP**	**PO**,$\mathbf{PP}^{\smile}$	**PO**,**DR**

relation between x and y given the definite RCC5 relation between $\jmath(x)$ and $\jmath(y)$. Table 6 combines the results of Table 4 and Table 5, where there are 8 (out of 21) situations that cannot lead to a definite RCC5 relation.

To obtain more precise information, we need to consider all relations in the 4-tuple $\langle \mathbf{R}_{\imath\imath}, \mathbf{R}_{\jmath\jmath}, \mathbf{R}_{\imath\jmath}, \mathbf{R}_{\jmath\imath} \rangle$. Recall in the last section we have divided the 51 (realizable) rough mereological relations into 4 groups according to whether a and/or b is crisp. There are altogether 73 cases: 5 for crisp-crisp regions, 11 for crisp-vague, and 11 for vague-crisp, and 46 for vague-vague. If a, b are both crisp, then $\mathbf{R}_{\imath\imath} = \mathbf{R}_{\jmath\jmath} = \mathbf{R}_{\imath\jmath} = \mathbf{R}_{\jmath\imath}$ and the RCC5 relation between a, b is definite. Interestingly, if only one of a, b is crisp, then the RCC5 relation between a, b is also definite, see Table 2 and Table 3. As for the 46 vague-vague rough relations, Table 7 shows that 10 situations can lead to indefinite RCC5 relations.

From Tables 2, 3, 7, we can see that altogether nine RCC5 relations can hold between two rough regions. These are

$$\mathbf{EQ}, \mathbf{DR}, \mathbf{PP}, \mathbf{PP}^{\smile}, \mathbf{PO}, \mathbf{O}, \{\mathbf{PO}, \mathbf{PP}\}, \{\mathbf{PO}, \mathbf{PP}^{\smile}\}, \{\mathbf{PO}, \mathbf{DR}\} \tag{9}$$

There are 5 definite and 4 indefinite RCC5 relations. It is natural to classify the 73 rough relations into 9 groups, each of which leads to a (possibly indefinite) RCC5 relation.

Note that a row in Table 7 can be explained as a rule. Take the first row as example. For any two regions $x, y \in B$, suppose $\rho(\imath(x), \imath(y)) = \mathbf{EQ}$, $\rho(\jmath(x), \jmath(y)) = \mathbf{EQ}$, $\rho(\imath(x), \jmath(y)) = \mathbf{PP}$, and $\rho(\jmath(x), \imath(y)) = \mathbf{PP}^{\smile}$. Then Table 7 asserts that the RCC5 relation between x, y is **O**, which is an indefinite RCC5 relation. We also say that **O** is determined by the rough rela-

Table 6: Determining the RCC5 relations using lower-upper approximation

		$(\jmath(x), \jmath(y))$				
		EQ	**DR**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
$(\imath(x), \imath(y))$	**EQ**	**O**	-	**PO**,**PP**	**PO**,$\mathbf{PP}^{\smile}$	**PO**
	DR	**PO**	**DR**	**PO**	**PO**	**DR**,**PO**
	PP	**PP**,**PO**	-	**PP**,**PO**	**PO**	**PO**
	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$,**PO**	-	**PO**	**PO**,$\mathbf{PP}^{\smile}$	**PO**
	PO	**PO**	-	**PO**	**PO**	**PO**

Table 7: Determining the RCC5 relations using rough relations

$(\imath(x),\imath(y))$	$(\jmath(x),\jmath(y))$	$(\imath(x),\jmath(y))$	$(\jmath(x),\imath(y))$	(x,y)
EQ	**EQ**	**PP**	$\mathbf{PP}^{\smile}$	**O**
EQ	**PO**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
EQ	**PP**	**PP**	$\mathbf{PP}^{\smile}$	**PO**,**PP**
EQ	$\mathbf{PP}^{\smile}$	**PP**	$\mathbf{PP}^{\smile}$	**PO**,$\mathbf{PP}^{\smile}$
DR	**EQ**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
DR	**PP**	**PP**	**DR**	**PO**
DR	**PP**	**PP**	**PO**	**PO**
DR	**PP**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
DR	$\mathbf{PP}^{\smile}$	**DR**	$\mathbf{PP}^{\smile}$	**PO**
DR	$\mathbf{PP}^{\smile}$	**PO**	$\mathbf{PP}^{\smile}$	**PO**
DR	$\mathbf{PP}^{\smile}$	**PP**	$\mathbf{PP}^{\smile}$	**PO**
DR	**PO**	**DR**	**DR**	**DR**,**PO**
DR	**PO**	**DR**	**PO**	**PO**
DR	**PO**	**DR**	$\mathbf{PP}^{\smile}$	**PO**
DR	**PO**	**PO**	**DR**	**PO**
DR	**PO**	**PO**	**PO**	**PO**
DR	**PO**	**PO**	$\mathbf{PP}^{\smile}$	**PO**
DR	**PO**	**PP**	**DR**	**PO**
DR	**PO**	**PP**	**PO**	**PO**
DR	**PO**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
DR	**DR**	**DR**	**DR**	**DR**
PP	**PO**	**PP**	**PO**	**PO**
PP	**PO**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
PP	**PP**	**PP**	**PP**	**PP**
PP	**PP**	**PP**	**PO**	**PO**,**PP**
PP	**PP**	**PP**	**EQ**	**PP**
PP	**PP**	**PP**	$\mathbf{PP}^{\smile}$	**PO**,**PP**
PP	$\mathbf{PP}^{\smile}$	**PP**	$\mathbf{PP}^{\smile}$	**PO**
PP	**EQ**	**PP**	$\mathbf{PP}^{\smile}$	**PO**,**PP**
$\mathbf{PP}^{\smile}$	**PO**	**PO**	$\mathbf{PP}^{\smile}$	**PO**
$\mathbf{PP}^{\smile}$	**PO**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$
$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$	**PO**	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$,**PO**
$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$	**EQ**	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$
$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$	**PP**	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$,**PO**
$\mathbf{PP}^{\smile}$	**PP**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
$\mathbf{PP}^{\smile}$	**EQ**	**PP**	$\mathbf{PP}^{\smile}$	$\mathbf{PP}^{\smile}$,**PO**
PO	**PO**	**PO**	**PO**	**PO**
PO	**PO**	**PO**	$\mathbf{PP}^{\smile}$	**PO**
PO	**PO**	**PP**	**PO**	**PO**
PO	**PO**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
PO	**PP**	**PP**	**PO**	**PO**
PO	**PP**	**PP**	$\mathbf{PP}^{\smile}$	**PO**
PO	$\mathbf{PP}^{\smile}$	**PO**	$\mathbf{PP}^{\smile}$	**PO**
PO	$\mathbf{PP}^{\smile}$	**PP**	$\mathbf{PP}^{\smile}$	**PO**
PO	**EQ**	**PP**	$\mathbf{PP}^{\smile}$	**PO**

tion $\langle \mathbf{EQ}, \mathbf{EQ}, \mathbf{PP}, \mathbf{PP}^{\smile} \rangle$. Another example. Row 12 of Table 7 asserts that $\{\mathbf{PO}, \mathbf{DR}\}$ is determined by the rough relation $\langle \mathbf{DR}, \mathbf{PO}, \mathbf{DR}, \mathbf{DR} \rangle$.

Definition 5.1. Let B_1 be a resolution of a GRCC model B, and $\mathbf{R}$ an RCC5 relation. For two regions a, b in B, we say (the fact) $a\mathbf{R}b$ can be determined at resolution B_1 if and only if $\mathbf{R}$ can be determined by the rough relation $\langle \rho(\imath(a), \imath(b)), \rho(\jmath(a), \jmath(b)), \rho(\imath(a), \jmath(b)), \rho(\jmath(a), \imath(b)) \rangle$, where $\rho(x, y)$ is the RCC5 basic relation between B_1 regions x, y.

Given two rough regions $a = (\imath(a), \jmath(a))$, $b = (\imath(b), \jmath(b))$, the rough information are often not sufficient to decide the *definite* RCC5 relation between a, b. For example, suppose we know that $\imath(a) = \imath(b) < \jmath(a) = \jmath(b)$, and this is the only information we have about a, b at the present resolution. Then we do not know whether a is a proper part of b—they may also be equal or partially overlap. But if we luckily know that $\jmath(a) \leq \imath(b)$ and $\imath(a) < \jmath(b)$, then we are sure that a is a proper part of b. Moreover, if either $\jmath(a) \not\leq \imath(b)$ or $\imath(a) \not< \jmath(b)$, then there are two possibilities: (i) a is not a proper part of b; or (ii) a may be (but we are not sure) a proper part of b. In other words, if a is really a proper part of b, then this fact can be ascertained at the current resolution if and only if we know $\jmath(a) \leq \imath(b)$ and $\imath(a) < \jmath(b)$.

The following theorem summarizes when a definite RCC5 relation can be determined.

Theorem 5.1. *Let B be a GRCC model and B_1 be a non-trivial resolution of B. For two rough regions a, b in B_1, we have*

1. *The fact $a\mathbf{EQ}b$ can be determined at resolution B_1 if and only if a, b are identical crisp regions in B_1;*
2. *The fact $a\mathbf{DR}b$ can be determined at resolution B_1 if and only if $\jmath(a)\mathbf{DR}\jmath(b)$;*
3. *The fact $a\mathbf{PP}b$ can be determined at resolution B_1 if and only if $\jmath(a) \leq \imath(b)$ and $\imath(a) < \jmath(b)$;*
4. *The fact $a\mathbf{PP}^{\smile}b$ can be determined at resolution B_1 if and only if $\imath(a) \geq \jmath(b)$ and $\jmath(a) > \imath(b)$;*
5. *The fact $a\mathbf{PO}b$ can be determined at resolution B_1 if and only if the following conditions are satisfied:*
   - $\imath(a) \not\leq \imath(b)$ *or* $\jmath(a) \not\leq \jmath(b)$*; and*
   - $\imath(a) \not\geq \imath(b)$ *or* $\jmath(a) \not\geq \jmath(b)$*; and*
   - $\imath(a) \cdot \jmath(b) > 0$ *or* $\jmath(a) \cdot \imath(b) > 0$.

6 Generalization of a spatial configuration

In the above sections we considered ways to represent and reason with spatial information in a fixed resolution, where a region is represented as a pair in a resolution, and relations between rough regions are characterized by the 4-tuples of relations between their approximations.

In this section we consider, given a spatial configuration of n objects in a higher resolution, how to find another configuration of these objects in a coarser resolution while preserving the mereological relations. By 'spatial configuration' we mean a network of (crisp) regions in a spatial model, say the real plane or one of its resolution, where (mereological) relations among regions can be explicitly determined.

This question relates closely to *map generalization* in cartography and Geographic Information Science [31], where the objective of generalization is to produce maps at coarser levels of detail without changing essential characteristics of underlying geographic information.

We now give a detailed description of the question. Suppose B is a continuous model and, $B_1, B_2, \cdots, B_i, \cdots$ is a collection of finer and finer resolutions of B such that $B = \bigcup_{i\geq 1} B_i$. This means B is a restricted hierarchical spatial model (see Definition 3.4).

Given a spatial configuration $\Omega = \{x_1, x_2, \cdots, x_n\}$ in B, suppose B_λ is the first resolution of B in which all $x_i \in B_\lambda$. Therefore, all x_i can be crisply represented in B_λ, but not all can be crisply represented in any coarser resolution. The question now is: "For a coarser resolution B_h, can we find a reasonable approximation $x_i^* \in B_h$ for each i without changing their mereological relations?" By a reasonable approximation we mean x_i^* should be between the lower and upper approximation of x_i in B_h. Of course, the smaller h is the better.

Definition 6.1. For a spatial configuration $\Omega = \{x_1, x_2, \cdots, x_n\}$ in B, write $\lambda = \lambda(\Omega)$ for the lowest level at which all x_i are crisp. We call a set of n crisp regions $\Omega^* = \{x_1^*, x_2^*, \cdots, x_n^*\}$ at level $h \leq \lambda$ a *generalization* of Ω if the following conditions are satisfied:

(1) The RCC5 relation between x_i^* and x_j^* is the same as that between x_i and x_j $(1 \leq i, j \leq n)$.

(2) $\imath_h(x_i) \leq x_i^* \leq \jmath_h(x_i)$ $(1 \leq i \leq n)$, where $\imath_h(x_i)$ and $\jmath_h(x_i)$ are, respectively, the lower and upper approximation of x_i at level h.

For a spatial configuration Ω, set $\kappa(\Omega)$ to be the lowest level at which all mereological relations between objects in Ω can be determined. We first note that there may be no generalizations for Ω at any level $l \leq \kappa(\Omega)$.

Indeed, consider the hierarchical spatial model $(B, \{B_l\}_{l\geq 1})$ that is generated by hierarchical regular partitions of the unit square, as show in Figure 4, where $B_0 = \{0, 1\}$ is the coarsest level of B, B_1, B_2, B_3 are the first three levels of B, and p_i $(1 \leq i \leq 4)$ is a pixel at level 1, and p_{ij} $(1 \leq i, j \leq 4)$ is a pixel at level 2, and p_{ijk} $(1 \leq i, j, k \leq 4)$ is a pixel at level 3.

Now set

$$x = p_1 + p_{412} + p_{413} + p_{421} + p_{424} \quad (10)$$

$$y = p_4 + p_{13} + p_{142} \quad (11)$$

$$z = p_1 + p_4 + p_{23} + p_{24} + p_{311} + p_{312} + p_{321} + p_{322} \quad (12)$$

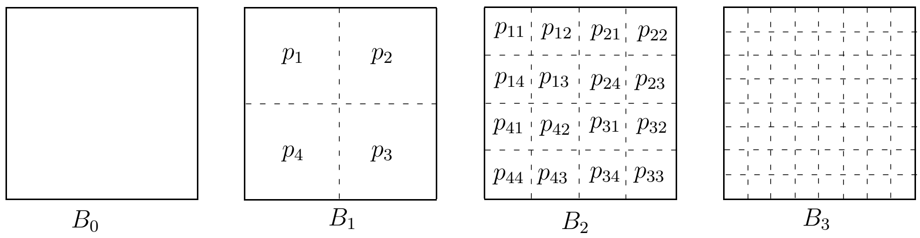


Figure 4: A hierarchical spatial model and its first 3 levels.

See Figure 5 for illustrations. Consider the spatial configuration $\Omega = \{x, y, z\}$. Clearly, x, y, z are vague at levels 1 and 2, and crisp at level 3. Note that $x\mathbf{PO}y$, $x\mathbf{PP}z$, and $y\mathbf{PP}z$. These relations can be determined at level 1, namely $\kappa(\Omega) = 1$. This is because, at level 1,

$$\begin{aligned}(\imath_1(x), \jmath_1(x)) &= (p_1, p_1 + p_4)\\ (\imath_1(y), \jmath_1(y)) &= (p_4, p_1 + p_4)\\ (\imath_1(z), \jmath_1(z)) &= (p_1 + p_2, p_1 + p_2 + p_3 + p_4)\end{aligned}$$

where we add to operators $\imath$ and $\jmath$ a subscript, here 1, to indicate the level at which the region is lower or upper approximated.

We claim that Ω has no generalization at level $\kappa(\Omega) = 1$. To show this, suppose $\{x^*, y^*, z^*\}$ is a generalization of Ω at level 1. Then

$$\begin{aligned}p_1 = \imath_1(x) \leq \quad & x^* \quad \leq \jmath_1(x) = p_1 + p_4\\ p_4 = \imath_1(y) \leq \quad & y^* \quad \leq \jmath_1(y) = p_1 + p_4\\ p_1 + p_4 = \imath_1(z) \leq \quad & z^* \quad \leq \jmath_1(z) = p_1 + p_2 + p_3 + p_4\end{aligned}$$

Since x^*, y^*, z^* are crisp regions in B_1, x^* and y^* must be either p_1 or $p_1 + p_4$. But if $x^* = p_1$, then $x^* \leq y^*$; if $x^* = p_1 + p_4$, then $x^* \geq y^*$. Both cases contradict $x^*\mathbf{PO}y^*$, which follows from the assumption that $\{x^*, y^*, z^*\}$ is a generalization of Ω.

Although generalizations cannot be found for all spatial configuration Ω at level $\kappa(\Omega)$, we can certainly find a generalization at level $\kappa(\Omega)+1$. To prove this result, we need the following lemma, where two regions x, y are k-equivalent, written $x \sim_k y$, if they have the same lower and upper approximation at level k, i.e. $\imath_k(x) = \imath_k(y)$ and $\jmath_k(x) = \jmath_k(y)$.

Lemma 6.1. *Suppose $(B, \{B_l\}_{l\geq 1})$ is a restricted hierarchical spatial model, where each pixel at level l contains more than one sub-pixels at level $l + 1$. Then, for any region $x \in B$ and for any resolution B_k, we can find a crisp region x^* in B_{k+1} such that $x^* \sim_k x$.*

Proof. To construct a region x^* which is k-equivalent to x, our idea is, roughly speaking, reserving all $(k+1)$-pixels that are contained in x, and deleting at least

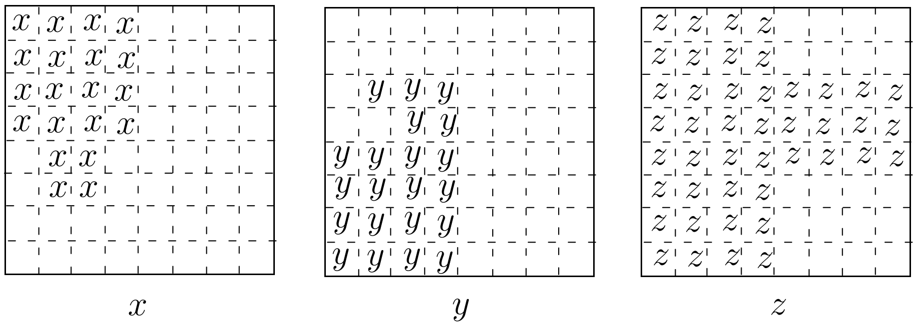


Figure 5: A spatial configuration $\Omega = \{x, y, z\}$.

one $(k+1)$-pixel from a k-pixel that is at the 'boundary' of x. More precisely, we choose for each k-pixel w a set, written X_w, of its sub-pixels at level $k+1$. Write $S(w)$ for the set of sub-pixels of w at level $k+1$. Then

- if $w \leq x$, set $X_w = S(w)$;
- if $w \cdot x = 0$, set $X_w = \varnothing$;
- if $0 < w \cdot x < w$, set X_w to be a nonempty proper subset of $S(w)$ which contains all sub-pixels w' that is contained in x.

The last case is possible since there is more than one sub-pixel in $S(w)$ and not all sub-pixels are contained in x.

Now sum up all sub-pixels in $\bigcup\{X_w : w \text{ is a } k\text{-pixel}\}$. The result x^* is a crisp region in B_{k+1} which is k-equivalent to x. □

Also note that each x^* constructed as above is also between the lower and upper approximations of x at level $k+1$, i.e. $\imath_{k+1}(x) \leq x^* \leq \jmath_{k+1}(x)$. We now arrive at the main result of this section:

Theorem 6.1. *Suppose* $(B, \{B_l\}_{l \geq 1})$ *is a restricted hierarchical spatial model, where each pixel at level* l *contains more than one sub-pixel at level* $l+1$*. Then for any spatial configuration* Ω*, we have a generalization of* Ω *at any level* $k \geq \kappa(\Omega)+1$*, where* $\kappa(\Omega)$ *is the coarsest level at which relations among regions in* Ω *can be determined.*

Proof. Suppose $\Omega = \{x_1, x_2, \cdots, x_n\}$. For each level $k \geq \kappa(\Omega)+1$, we now construct a realization of Ω at level k. For each x_i, by Lemma 6.1, we can find a crisp region x_i^* at level k which is $(k-1)$-equivalent to x_i. We claim that $\{x_1^*, x_2^*, \cdots, x_n^*\}$ is a realization of Ω at level k. To show this, we only need to prove that the RCC5 relation between x_i^* and x_j^* is the same as that between x_i and x_j. Recall that the relation of x_i and x_j can be determined at level $\kappa(\Omega)$. Now, since the rough relation of x_i^* and x_j^* at level k is the same as that of x_i and x_j, the relation between x_i^* and x_j^* can also be determined at level $\kappa(\Omega)$, which can only be the relation of x_i and x_j. □

Obviously, computing the answer is of lower polynomial complexity.

Now return to the example described in Figure 5. Set $x^* = p_1 + p_{41}$, $y^* = p_4 + p_{13}$, $z^* = p_1 + p_4 + p_{23} + p_{24} + p_{31}$. Then $\{x^*, y^*, z^*\}$ is a generalization of Ω at level $2 = \kappa(\Omega) + 1$.

7 Conclusions and further work

In this paper we have proposed a framework for hierarchical representation and reasoning about topological information. In our hierarchical spatial model, each region is represented as rough regions at various levels of details. Mereological relations between rough regions are characterized by the RCC5 relation between lower and upper approximations. We gave rules (Table 7) for determining the RCC5 relation between two rough regions using simply the rough relation between the rough regions. We also gave a method for constructing mereological generalizations of spatial configuration at coarser levels of details.

Although only RCC5 relations were discussed in this paper, our approach also applies to RCC8 and other RCC systems of relations. We take the view that, for rough regions, mereological relations are more important than finer topological distinctions. This is partially because rough RCC8 relations rarely provide information for us to differ **TPP** from **NTPP**, and to differ from **EC** from **DC**, where a rough RCC8 relation between two rough regions a, b is a 4-tuple $\langle \rho(\imath(a), \imath(b)), \rho(\jmath(a), \jmath(b)), \rho(\imath(a), \jmath(b)), \rho(\jmath(a), \imath(b)) \rangle$ with $\rho(x, y)$ the RCC8 basic relation between x and y. In our opinion, it is not necessary to make finer topological distinctions until we know the definite mereological relation.

There are several directions for further work.

An important reasoning problem we have not discussed is the problem of deciding the consistency of a rough relation network. The solution to this problem can be based either on the RCC5 composition table or on a rough RCC5 composition table.

Winter [32] proposes a method for hierarchically determining the mereological relation between two quadtree regions. It is our future work to extend this and other hierarchical reasoning techniques to general hierarchical models.

Acknowledgements

This work was partially supported by the Alexander von Humboldt Foundation, the NSFC (60673105, 60321002, 60496321), a DFG grant (No. 623/1-3), and the DFG project Logospace as part of the collaborative research center SGB/TR-8 “Spatial Cognition”. We also gratefully acknowledge the invaluable suggestions of three anonymous referees.

References

[1] T. Bittner and J.G. Stell. A boundary-sensitive approach to qualitative location. *Annals of Mathematics and Artificial Intelligence*, 24:93–114, 1998.

[2] A. Car. Hierarchical spatial reasoning: a geocomputation method. In *Proceedings of the 3rd International Conference on GeoComputation*, 1998. Available via: `http://www.geocomputation.org/1998/18/gc_18.htm`.

[3] A.G. Cohn and N.M. Gotts. The 'egg-yolk' representation of regions with indeterminate boundaries. In P. Burrough and A.M. Frank, editors, *Proceedings, GISDATA Specialist Meeting on Geographical Objects with Undetermined Boundaries,*. Francis Taylor, 1996.

[4] Z. Cui, A.G. Cohn, and D.A. Randell. Qualitative and topological relationships in spatial databases. In D. Abel and B.C. Ooi, editors, *Advances in Spatial Databases, LNCS 692*, pages 293–315. Springer Verlag, Berlin, 1993.

[5] I. Düntsch, E. Orłowska, and H. Wang. Algebras of approximating regions. *Fundamenta Informaticae*, 46:71–82, 2001.

[6] I. Düntsch, H. Wang, and S. McCloskey. A relation-algebraic approach to the Region Connection Calculus. *Theoretical Computer Science*, 255:63–83, 2001.

[7] I. Düntsch and M. Winter. A representation theorem for Boolean contact algebras. *Theoretical Computer Science (B)*, 347:498–512, 2005.

[8] M.J. Egenhofer and J. Sharma. Topological relations between regions in $\mathbb{R}^2$ and $\mathbb{Z}^2$;. In *SSD'93: Proceedings of the Third International Symposium on Advances in Spatial Databases*, pages 316–336, London, UK, 1993. Springer-Verlag.

[9] A. Galton. The mereotopology of discrete space. In C. Freksa and D. Mark, editors, *Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science, International Conference COSIT'99*, pages 251–266, Berlin, 1999. Springer.

[10] J.R. Hobbs. Granularity. In A.K. Joshi, editor, *Proceedings of the 9th International Joint Conference on Artificial Intelligence*, pages 432–435. Morgan Kaufmann, 1985.

[11] B. Jahne, H. Haussecker, and P. Geissler, editors. *Handbook of Computer Vision and Applications*. Academic Press, 1999.

[12] R.W. Langacker. *Foundations of Cognitive Grammar, Vol. 1 Theoretical Prerequisites*. Stanford University Press., Stanford, CA, 1987.

[13] S. Li and M. Ying. Region Connection Calculus: Its models and composition table. *Artificial Intelligence*, 145(1-2):121–146, 2003.

[14] S. Li and M. Ying. Generalized Region Connection Calculus. *Artificial Intelligence*, 160(1-2):1–34, 2004.

[15] C. Masolo and L. Vieu. Atomicity vs. infinite divisibility of space. In C. Freksa and D. Mark, editors, *Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science, International Conference COSIT'99*, pages 235–250, Berlin, 1999. Springer.

[16] Z. Pawlak. *Rough sets : theoretical aspects of reasoning about data*. Kluwer Academic Publishers, Dordrecht, 1991.

[17] F.P. Preparata and M.I. Shamos. *Computational Geometry: An introduction*. Springer, 1993.

[18] D.A. Randell and A.G. Cohn. Modelling topological and metrical properties of physical processes. In R.J. Brachman, H.J. Levesque, and R. Reiter, editors, *First International Conference on the Principles of Knowledge Representation and Reasoning*, pages 55–66, Los Altos, 1989. Morgan Kaufmann.

[19] D.A. Randell, Z. Cui, and A.G. Cohn. A spatial logic based on regions and connection. In B. Nebel, W. Swartout, and C. Rich, editors, *Proceedings of the 3rd International Conference on Knowledge Representation and Reasoning*, pages 165–176, Cambridge, MA, 1992. Morgan Kaufmann.

[20] J. Renz. *Qualitative spatial reasoning with topological information*, volume 2293 of *LNAI*. Springer-Verlag, Berlin, Germany, 2002.

[21] J. Renz and B. Nebel. On the complexity of qualitative spatial reasoning: A maximal tractable fragment of the Region Connection Calculus. *Artificial Intelligence*, 108:69–123, 1999.

[22] A. Rosenfeld. Digital topology. *American Mathematical Monthly*, 86:621–630, 1979.

[23] A.J. Roy and J.G. Stell. A qualitative account of discrete space. In M.J. Egenhofer and D.M. Mark, editors, *Proceedings of Second International Conference of Geographic Information Science, GIScience 2002, LNCS 2478*, pages 276–290, Berlin, 2002. Springer.

[24] H. Samet. *The Design and Analysis of Sptaial Data Structures*. Addison-Wesley, Reading, Massachusetts, 1990.

[25] M. Schneider. Design and implementation of finite resolution crisp and fuzzy spatial objects. *Data & Knowledge Engineering*, 44(1):81–108, 2003.

[26] J.G. Stell. Boolean connection algebras: A new approach to the Region-Connection Calculus. *Artificial Intelligence*, 122:111–136, 2000.

[27] J.G. Stell. Part and complement: Fundamental concepts in spatial relations. *Annals of Mathematics and Artificial Intelligence*, 41(1):1–17, 2004.

[28] J.G. Stell and M.F. Worboys. Stratified map spaces: A formal basis for multi-resolution spatial databases. In T.K. Poiker and N. Chrisman, editors, *Proceedings of the 8th International Symposium on Spatial Data Handling*, pages 180–189, 1998.

[29] S. Timpf. Abstraction, levels of detail, and hierarchies in map series. In C. Freksa and D. Mark, editors, *Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science, International Conference COSIT'99*, pages 125–139, Berlin, 1999. Springer.

[30] S. Timpf and A.U. Frank. Using hierarchical spatial data structures for hierarchical spatial reasoning. In S.C. Hirtle and A.U. Frank, editors, *COSIT*, volume 1329 of *Lecture Notes in Computer Science*, pages 69–83. Springer, 1997.

[31] R. Weibel. Map generalization in the context of digital systems. *Cartography and Geographic Information Systems*, 22:259–263, 1995.

[32] S. Winter. Topological relations in hierarchial partitions. In C. Freksa and D. Mark, editors, *Spatial Information Theory. Cognitive and Computational Foundations of Geographic information Science, International Conference COSIT'99*, pages 141–155, Berlin, 1999. Springer.

[33] M. Worboys. Imprecision in finite resolution spatial data. *GeoInformatica*, 2:257–279, 1998.