APPROXIMATION OF PRICES FOR
AVERAGE-TYPE OPTIONS VIA BOUNDS

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 Declaration of authorship

I certify that the work in this thesis has not previously been submitted for a degree nor been submitted as part of requirements for a degree. I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Scott Alexander
17th February, 2017
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Dedication

For my mother Denise, father John and brother Tim.
Abstract

The problem of pricing multi-dimensional arithmetic average-type options is a complex problem both analytically and numerically, analytically because the distribution of the average on which the payoff function depends is unknown in closed-form and numerically because of the high dimensionality of the problem. In this thesis we develop methods that avoid these issues by providing price approximations in the form of lower and upper bounds. We do so by approximating the event that the option finishes in-the-money with a closely related proxy and use an optimisation procedure to tighten up the bounds. The result is a pricing tool that provides accurate approximations without suffering from the exponential curse of dimensionality associated with other techniques. The method is developed for underlying assets modelled as exponential Lévy processes and numerical examples are provided under a variety of these models.
# Table of contents

1 Introduction

2 Mathematical Background
   2.1 Measure and integration ........................................ 5
      2.1.1 Measure .................................................... 5
      2.1.2 Lebesgue integration ....................................... 9
      2.1.3 Other results from integration theory .................... 13
      2.1.4 Some useful inequalities ................................. 16
   2.2 Elements of Fourier analysis .................................. 17
   2.3 Probability and stochastic processes ......................... 19
      2.3.1 Probability ................................................ 19
      2.3.2 Stochastic processes and integration .................... 26
   2.4 Risk-neutral pricing and European options .................. 33
      2.4.1 Risk-neutral pricing ...................................... 33
      2.4.2 Examples of European options ............................ 37

3 Average-type options ............................................. 39
   3.1 Average-type options defined ................................. 39
      3.1.1 Continuous monitoring ................................... 44
      3.1.2 Discrete monitoring ...................................... 46
# TABLE OF CONTENTS

3.1.3 The difficulty in pricing average-type options . . . . . . . . 48
3.2 Examples of average-type options . . . . . . . . . . . . . . . . . 50
3.3 Pricing in two special cases . . . . . . . . . . . . . . . . . . . . . 54
   3.3.1 In-the-money options . . . . . . . . . . . . . . . . . . . . . 55
   3.3.2 Out-of-the-money options . . . . . . . . . . . . . . . . . . 57
3.4 Alternative price representations . . . . . . . . . . . . . . . . . 59
3.5 Asset Price Process . . . . . . . . . . . . . . . . . . . . . . . . . 62

4 Literature Review 66

5 Lower Bound 73
   5.1 Developing the lower bound . . . . . . . . . . . . . . . . . . . 73
   5.2 Joint characteristic function . . . . . . . . . . . . . . . . . . . 78
      5.2.1 Continuous monitoring . . . . . . . . . . . . . . . . . . . 78
      5.2.2 Discrete monitoring . . . . . . . . . . . . . . . . . . . . . 83
      5.2.3 Other characteristic functions . . . . . . . . . . . . . . . 89
      5.2.4 Distribution and density functions . . . . . . . . . . . . . 90
   5.3 Step 1 - solving the optimisation step . . . . . . . . . . . . . . 94
   5.4 Step 2 - evaluating the lower bound I . . . . . . . . . . . . . . 102
      5.4.1 An approximation for delta . . . . . . . . . . . . . . . . 113
   5.5 Step 2 - evaluating the lower bound II . . . . . . . . . . . . . . 117
      5.5.1 Another derivation of the Step 1 procedure . . . . . . . . . 126
      5.5.2 An approximation for delta . . . . . . . . . . . . . . . . 130

6 Upper Bound 133
   6.1 Developing the upper bound . . . . . . . . . . . . . . . . . . . 133
   6.2 Joint characteristic function . . . . . . . . . . . . . . . . . . . 137
      6.2.1 Distribution and density functions . . . . . . . . . . . . . 139
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.3 Evaluating the upper bound</td>
<td>140</td>
</tr>
<tr>
<td>7 Numerical Examples</td>
<td>155</td>
</tr>
<tr>
<td>7.1 Geometric Brownian motion</td>
<td>157</td>
</tr>
<tr>
<td>7.2 Geometric Merton jump-diffusion</td>
<td>161</td>
</tr>
<tr>
<td>7.3 Geometric normal inverse Gaussian</td>
<td>164</td>
</tr>
<tr>
<td>7.4 Geometric variance gamma</td>
<td>165</td>
</tr>
<tr>
<td>7.5 Observations</td>
<td>166</td>
</tr>
<tr>
<td>8 Conclusions</td>
<td>176</td>
</tr>
<tr>
<td>Bibliography</td>
<td>178</td>
</tr>
</tbody>
</table>
List of Tables

7.1 General parameter settings .......................... 158
7.2 Example 1 (GBM) .................................. 159
7.3 Example 2 (GBM) .................................. 160
7.4 Example 3 (GBM) .................................. 161
7.5 Example 4 (GBM) .................................. 162
7.6 Example 5 (GBM) .................................. 169
7.7 Example 1 (GMJD) ................................ 170
7.8 Example 2 (GMJD) ................................ 170
7.9 Example 3 (GMJD) ................................ 171
7.10 Example 4 (GMJD) ............................... 171
7.11 Example 1 (GNIG) ............................... 172
7.12 Example 2 (GNIG) ............................... 172
7.13 Example 3 (GNIG) ............................... 173
7.14 Example 4 (GNIG) ............................... 173
7.15 Example 1 (GVG) ............................... 174
7.16 Example 2 (GVG) ............................... 174
7.17 Example 3 (GVG) ............................... 175
7.18 Example 4 (GVG) ............................... 175
List of Figures

7.1 Plots for fixed-strike Asian basket under GBM . . . . . . . . . . . 160
7.2 Plots for fixed-strike Asian basket under Merton jump diffusion . 163
7.3 Plots for fixed-strike Asian basket under GNIG . . . . . . . . . . 165
7.4 Plots for fixed-strike Asian basket under GVG . . . . . . . . . . . 166