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Fuzzy Multi-objective Bilevel Decision Making by an Approximation *K*th-Best Approach

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Many industrial decisions problems are decentralized in which decision makers are arranged at two levels, called bilevel decision problems. Bilevel decision making may involve uncertain parameters which appear either in the objective functions or constraints of the leader or the follower or both. Furthermore, the leader and the follower may have multiple conflict decision objectives that should be optimized simultaneously. This study proposes an approximation *K*th-best approach to solve the fuzzy multi-objective bilevel problem. Two case based examples further illustrate how to use the approach to solve industrial decision problems.

Keywords: Bilevel programming, Fuzzy sets, Optimization, Multi-objective decision making, Fuzzy programming, *K*th-best approach.

1 INTRODUCTION

Bilevel programming (BP) is a special case of multilevel programming with a two level structure to model bilevel decision problems. In a BP problem, decision makers are arranged at two levels and both try to make decision successively. When the leader at the upper level attempts to optimize his/her objective(s), the follower at the lower level tries to find an optimized strategy according to each of possible decisions made by the leader [3,4]. Here, although each decision maker (the leader or the follower) tries to optimize his/her own objective functions with partially or without considering the objectives of the other level, the decision of each level affects the objective optimization of the other level [16]. The Stackelberg solution [33] has been employed as a solution concept to bilevel programming problems, and a considerable number of approaches for obtaining the solution have been developed [1,2,5–13,15,18–20,34].

To solve a real BP problem, a BP model needs to be established first. The parameters in the objective functions and constraints of the leader and the follower are required to be fixed at some values in an experimental and/or subjective manner through the experts' understanding of the nature of the parameters in the problem-formulation process. It has been observed that, in most real-world situations, the possible values of these parameters are often only imprecisely or ambiguously known to the experts, such as planning of land-use, transportation and water resource. With this observation, it would be certainly more appropriate to interpret the experts' understanding of the parameters of a BP problem as fuzzy numbers [35]. Many researchers, such as Sakawa *et al*. [22–27], have formulated BP problems with fuzzy parameters and propose fuzzy programming methods for fuzzy bilevel programming problems. Our recent research work has extended Kuhn-Tucher, Kth-best and branch-and-bound approaches to solve BP problems with fuzzy parameters.

Another issue in bilevel decision practice is that multiple conflicting objectives often need to be considered simultaneously by the leader, and/or the follower. For example, a coordinator of a multi-division firm considers three objectives in making an aggregate production plan: maximise net profits, maximise quality of products, and maximise worker satisfaction. The three objectives could be in conflict with each other, but must be considered simultaneously. Any improvement in one objective may be achieved only at the expense of others. The normal multi-objective decision-making problem has been well researched by many researchers such as Hwang and Masud [14]. But in a bilevel model, the selection of a satisfactory solution for the leader is imparted by his/her follower's optimal reaction. Therefore, how to find an optimal solution for the leader which has multiple objectives under the consideration of both its constraints and its followers needs to be explored.

Following our previous research results shown in [17,28–32,37–42], this study aims at developing an approach to solve fuzzy multi-objective linear bilevel programming (FMOLBP) problems. It first transforms a FMOLBP problem into a non-fuzzy multi-objective linear bilevel programming (MOLBP) problem. Based on the definition and related theorems [29,41], it then solve the FMOLBP problem by solving the associated MOLBP problem. As this paper focuses a linear bilevel problem, so BP means linear BP in the paper.

Following the introduction, Section 2 reviews related definitions, theorems and properties of fuzzy numbers and a FMOLBP model [41]. A general fuzzy number based approximation Kth-best approach for solving FMOLBP problems is presented in Section 3. Two case based examples are shown in Section 4 for illustrating the proposed model and approach. Conclusions and further study are discussed in Section 5.

2 PRELIMINARIES

In this section, we present some basic concepts, definitions and theorems that are to be used in the subsequent sections. The work presented in this section also can be found from our recent papers in [36,41].

2.1 Fuzzy numbers

Let R be the set of all real numbers, R^n be n-dimensional Euclidean space, and $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ be any two vectors, where $x_i, y_i \in R, i = 1, 2, \ldots, n$ and T denotes the transpose of the vector. Then we denote the inner product of x and y by $\langle x, y\rangle$. For any two vectors $x, y \in R^n$, we write $x \geqq y$ iff $x_i \geq y_i, \forall i = 1, 2, \ldots, n$; $x \geq y$ iff $x \geqq y$ and $x \neq y$; $x > y$ iff $x_i > y_i, \forall i = 1, 2, \ldots, n$.

Definition 2.1. A fuzzy number $\tilde{a}$ is defined as a fuzzy set on R, whose membership function $\mu_{\tilde{a}}$ satisfies the following conditions:

1. $\mu_{\tilde{a}}$ is a mapping from R to the closed interval [0,1];
2. it is normal, i.e., there exists $x \in R$ such that $\mu_{\tilde{a}}(x) = 1$;
3. for any $\lambda \in (0, 1]$, $a_\lambda = \{x; \mu_{\tilde{a}}(x) \geq \lambda\}$ is a closed interval, denoted by $[a^L_\lambda, a^R_\lambda]$.

Let *F(R)* be the set of all fuzzy numbers. By the decomposition theorem of fuzzy sets, we have

$$\tilde{a} = \bigcup_{\lambda\in[0,1]} \lambda[a^L_\lambda, a^R_\lambda],$$

for every $\tilde{a} \in F(R)$.

Let $F^*(R)$ be the set of all finite fuzzy numbers on R.

Theorem 2.1. *Let $\tilde{a}$ be a fuzzy set on R, then $\tilde{a} \in F(R)$ if and only if $\mu_{\tilde{a}}$ satisfies*

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 & x \in [m, n] \\ L(x) & x < m \\ R(x) & x > n \end{cases},$$

where $L(x)$ is the right-continuous monotone increasing function, $0 \leqq L(x) < 1$ and $\lim_{x\to-\infty} L(x) = 0$, $R(x)$ is the left-continuous monotone decreasing function, $0 \leqq R(x) < 1$ and $\lim_{x\to\infty} R(x) = 0$.

Corollary 2.1. *For every $\tilde{a} \in F(R)$ and $\lambda_1, \lambda_2 \in [0, 1]$, if $\lambda_1 \leqq \lambda_2$, then $a_{\lambda_2} \subset a_{\lambda_1}$.*

Definition 2.2. For any $\tilde{a}, \tilde{b} \in F(R)$ and $0 \leqq \lambda \in R$, the sum of $\tilde{a}$ and $\tilde{b}$ and the scalar product of λ and $\tilde{a}$ are defined by the membership functions

$$\mu_{\tilde{a}+\tilde{b}}(t) = \sup \min_{t=u+v} \{\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)\},$$

$$\mu_{\tilde{a}-\tilde{b}}(t) = \sup \min_{t=u-v} \{\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)\},$$

$$\mu_{\lambda\tilde{a}}(t) = \sup_{t=\lambda u} \mu_{\tilde{a}}(u).$$

Theorem 2.2. *For any* $\tilde{a}, \tilde{b} \in F(R)$ *and* $0 \leqq \alpha \in R$,

$$\tilde{a} + \tilde{b} = \bigcup_{\lambda\in[0,1]} \lambda[a_\lambda^L + b_\lambda^L, a_\lambda^R + b_\lambda^R],$$

$$\tilde{a} - \tilde{b} = \tilde{a} + (-\tilde{b}) = \bigcup_{\lambda\in[0,1]} \lambda[a_\lambda^L - b_\lambda^R, a_\lambda^R - b_\lambda^L],$$

$$\alpha\tilde{a} = \bigcup_{\lambda\in[0,1]} \lambda[\alpha a_\lambda^L, \alpha a_\lambda^R].$$

Definition 2.3. Let $\tilde{a}_i \in F(R), i = 1, 2, \ldots, n$. We define $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$

$$\mu_{\tilde{a}} : R^n \to [0, 1]$$

$$x \mapsto \bigwedge_{i=1}^{n} \mu_{\tilde{a}_i}(x_i),$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, and $\tilde{a}$ is called an n-dimensional fuzzy number on R^n. If $\tilde{a}_i \in F^*(R), i = 1, 2, \ldots, n, \tilde{a}$ is called an n-dimensional finite fuzzy number on R^n.

Let $F(R^n)$ and $F^*(R^n)$ be the set of all n-dimensional fuzzy numbers and the set of all n-dimensional finite fuzzy numbers on R^n respectively.

Proposition 2.1. *For every* $\tilde{a} \in F(R^n)$, $\tilde{a}$ *is normal.*

Proposition 2.2. *For every* $\tilde{a} \in F(R^n)$, *the* λ*-section of* $\tilde{a}$ *is an* n*-dimensional closed rectangular region for any* $\lambda \in (0, 1]$.

Proposition 2.3. *For every* $\tilde{a} \in F(R^n)$ *and* $\lambda_1, \lambda_2 \in [0, 1]$, *if* $\lambda_1 \leqq \lambda_{2,}$ *then* $a_{\lambda_2} \subset a_{\lambda_1}$.

Definition 2.4. For every n-dimensional fuzzy numbers $\tilde{a}, \tilde{b}, \in F(R^n)$, we define

1. $\tilde{a} \succeqq \tilde{b}$ iff $a_\lambda^L \geqq b_\lambda^L$ and $a_\lambda^R \geqq b_\lambda^R, \lambda \in (0, 1]$;
2. $\tilde{a} \succeq \tilde{b}$ iff $a_\lambda^L \geq b_\lambda^L$ and $a_\lambda^R \geq b_\lambda^R, \lambda \in (0, 1]$;
3. $\tilde{a} \succ \tilde{b}$ iff $a_\lambda^L > b_\lambda^L$ and $a_\lambda^R > b_\lambda^R, \lambda \in (0, 1]$.

We call the binary relations $\overset{\succ}{=}$, $\succ$ and $\succ$ a fuzzy max order, a strict fuzzy max order and a strong fuzzy max order, respectively.

2.2 Fuzzy multi-objective linear bilevel programming model

Consider the following FMOLBP problem:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F : X \times Y \to F^*(R^s)$,

and $f : X \times Y \to F^*(R^t)$,

$$\min_{x \in X} F(x, y) = (\tilde{c}_{11}x + \tilde{d}_{11}y, \tilde{c}_{21}x + \tilde{d}_{21}y, \dots, \tilde{c}_{s1}x + \tilde{d}_{s1}y)^T \quad (2.1a)$$

$$\text{subject to } \tilde{A}_1 x + \tilde{B}_1 y \overset{\prec}{=} \tilde{b}_1 \quad (2.1b)$$

$$\min_{y \in Y} f(x, y) = (\tilde{c}_{12}x + \tilde{d}_{12}y, \tilde{c}_{22}x + \tilde{d}_{22}y, \dots, \tilde{c}_{t2}x + \tilde{d}_{t2}y)^T \quad (2.1c)$$

$$\text{subject to } \tilde{A}_2 x + \tilde{B}_2 y \overset{\prec}{=} \tilde{b}_2 \quad (2.1d)$$

where $\tilde{c}_{i1}, \tilde{c}_{j2} \in F^*(R^n)$, $\tilde{d}_{i1}, \tilde{d}_{j2} \in F^*(R^m)$, $i = 1, 2, \dots, s$, $j = 1, 2, \dots, t$, $\tilde{b}_1 \in F^*(R^p)$, $\tilde{b}_2 \in F^*(R^q)$, $\tilde{A}_1 = (\tilde{a}_{ij})_{p \times n}$, $\tilde{a}_{ij} \in F^*(R)$, $\tilde{B}_1 = (\tilde{b}_{ij})_{p \times m}$, $\tilde{b}_{ij} \in F^*(R)$, $\tilde{A}_2 = (\tilde{e}_{ij})_{q \times n}$, $\tilde{e}_{ij} \in F^*(R)$, $\tilde{B}_2 = (\tilde{s}_{ij})_{q \times m}$, $\tilde{s}_{ij} \in F^*(R)$.

For the sake of simplicity, we set $\tilde{X} \times \tilde{Y} = \{(x, y); \tilde{A}_1 x + \tilde{B}_1 \overset{\prec}{=} \tilde{b}_1, \tilde{A}_2 x + \tilde{B}_2 y \overset{\prec}{=} \tilde{b}_2\}$ and assume that $\tilde{X} \times \tilde{Y}$ is compact. In a FMOLBP problem, for each $(x, y) \in \tilde{X} \times \tilde{Y}$, the value of the objective functions $F(x, y) = (F_1(x, y), F_2(x, y), \dots, F_s(x, y))$ and $f(x, y) = (f_1(x, y), f_2(x, y), \dots, f_t(x, y))$ of the leader and the follower are s-dimensional and t-dimensional fuzzy numbers, respectively. Thus, we introduce the following concepts of optimal solutions to FMOLBP problems.

Definition 2.5. [41] A point $(x^*, y^*) \in \tilde{X} \times \tilde{Y}$ is said to be a complete optimal solution to the FMOLBP problem if it holds that $F(x^*, y^*) \overset{\prec}{=} F(x, y)$ and $f(x^*, y^*) \overset{\prec}{=} f(x, y)$ for all $(x, y) \in \tilde{X} \times \tilde{Y}$.

Definition 2.6. [41] A point $(x^*, y^*) \in \tilde{X} \times \tilde{Y}$ is said to be a Pareto optimal solution to the FMOLBP problem if there does not exist $(x, y) \in \tilde{X} \times \tilde{Y}$ such that $F(x^*, y^*) \succeq F(x, y)$ and $f(x^*, y^*) \succeq f(x, y)$ holds.

Definition 2.7. [41] A point $(x^*, y^*) \in \tilde{X} \times \tilde{Y}$ is said to be a weak Pareto optimal solution to the FMOLBP problem if there is no $(x, y) \in \tilde{X} \times \tilde{Y}$ such that $F(x^*, y^*) \succ F(x, y)$ and $f(x^*, y^*) \succ f(x, y)$ holds.

Associated with the FMOLBP problem, we now consider the following MOLBP problem:

$$
\begin{aligned}
&\text{For } x \in X \subset R^n, \quad y \in Y \subset R^m, \quad F : X \times Y \to F^*(R^s),\\
&\quad \text{and } f : X \times Y \to F^*(R^t),\\
&\quad \min_{x \in X} (F(x, y))_\lambda^{L(R)} = ((F_1(x, y))_\lambda^L, (F_1(x, y))_\lambda^R, \ldots, (F_s(x, y))_\lambda^L,\\
&\qquad\qquad (F_s(x, y))_\lambda^R)^T, \lambda \in [0,\ 1] \qquad (2.2a)\\
&\quad \text{subject to } A_{1\lambda}^L x + B_{1\lambda}^L y \leqq b_{1\lambda}^L, A_{1\lambda}^R x + B_{1\lambda}^R y \leqq b_{1\lambda}^R, \lambda \in [0,\ 1] \qquad (2.2b)\\
&\quad \min_{y \in Y} (f(x, y))_\lambda^{L(R)} = ((f_1(x, y))_\lambda^L, (f_1(x, y))_\lambda^R, \cdots (f_t(x, y))_\lambda^L,\\
&\qquad\qquad (f_t(x, y))_\lambda^R)^T, \lambda \in [0, 1] \qquad (2.2c)\\
&\quad \text{subject to } A_{2\lambda}^L x + B_{2\lambda}^L y \leqq b_{2\lambda}^L, A_{2\lambda}^R x + B_{2\lambda}^R y \leqq b_{2\lambda}^R, \lambda \in [0, 1] \qquad (2.2d)
\end{aligned}
$$

where $(F_i(x, y))_\lambda^L = c_{i1\lambda}^L x + d_{i1\lambda}^L y$, $(F_i(x, y))_\lambda^R = c_{i1\lambda}^R x + d_{i1\lambda}^R y$, $(f_j(x, y))_\lambda^L = c_{j2\lambda}^L x + d_{j12\lambda}^L y$ and$(f_j(x, y))_\lambda^R = c_{j2\lambda}^R x + d_{j12\lambda}^R y$, $\lambda \in [0, 1]$, $c_{i1\lambda}^L$, $c_{i1\lambda}^R$, $c_{j2\lambda}^L$, $c_{j2\lambda}^R \in R^n$, $d_{i1\lambda}^L$, $d_{i1\lambda}^R$, $d_{j2\lambda}^L$, $d_{j2\lambda}^R \in R^m$, $d_{i1\lambda}^L$, $d_{i1\lambda}^R$, $d_{j2\lambda}^L$, $d_{j2\lambda}^R \in R^m$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, t$, $b_{1\lambda}^L$, $b_{1\lambda}^R \in R^p$, $b_{2\lambda}^L$, $b_{2\lambda}^R \in R^q$, $A_{1\lambda}^L = (a_{ij_\lambda}^L)$, $A_{1\lambda}^R = (a_{ij_\lambda}^R) \in R^{p\times n}$, $A_{2\lambda}^L = (e_{ij_\lambda}^L)$, $A_{2\lambda}^R = (e_{ij_\lambda}^R) \in R^{q\times n}$, $B_{1\lambda}^L = (b_{ij_\lambda}^L)$, $B_{1\lambda}^R = (b_{ij_\lambda}^R) \in R^{p\times m}$, $B_{2\lambda}^L = (s_{ij_\lambda}^L)$, $B_{2\lambda}^R = (s_{ij_\lambda}^R) \in R^{q\times m}$.

For the sake of simplicity, we set $\underline{X} \times \underline{Y} = \{(x, y); A_{1\lambda}^L x + B_{1\lambda}^L \leqq b_{1\lambda}^L, A_{1\lambda}^R x + B_{1\lambda}^R \leqq b_{1\lambda}^R, A_{2\lambda}^L x + B_{2\lambda}^L \leqq b_{2\lambda}^L, A_{2\lambda}^R x + B_{2\lambda}^R \leqq b_{2\lambda}^R\}$ and assume that $\underline{X} \times \underline{Y}$ is compact. Obviously, $\tilde{X} \times \tilde{Y} = \underline{X} \times \underline{Y}$.

Definition 2.8. [41] A point $(x^*, y^*) \in \underline{X} \times \underline{Y}$ is said to be a complete optimal solution to the MOLBP problem if it holds that $((F_1(x^*, y^*))_\lambda^L$, $(F_1(x^*, y^*))_\lambda^R, \ldots, (F_s(x^*, y^*))_\lambda^L, (F_s(x^*, y^*))_\lambda^R)^T \leqq ((F_1(x, y))_\lambda^L, ((F_1(x, y))_\lambda^R, \ldots, (F_s(x, y))_\lambda^L, (F_s(x, y))_\lambda^R)^T$ and $((f_1(x^*, y^*))_\lambda^L, (f_1(x^*, y^*))_\lambda^R, \ldots,$ $(f_t(x^*, y^*))_\lambda^L$, $(f_t(x^*, y^*))_\lambda^R)^T \leqq ((f_1(x, y))_\lambda^L$, $((f_1(x, y))_\lambda^R, \ldots ((f_t(x, y))_\lambda^L, ((f_t(x, y))_\lambda^R)^T$ for $\lambda \in [0, 1]$ and $(x, y) \in \underline{X} \times \underline{Y}$.

Definition 2.9. [41] A point $(x^*, y^*) \in \underline{X} \times \underline{Y}$ is said to be a Pareto optimal solution to the MOLBP problem if there is no $(x, y) \in \underline{X} \times \underline{Y}$ such that $((F_1(x^*, y^*))_\lambda^L, (F_1(x^*, y^*))_\lambda^R, \ldots, (F_s(x^*, y^*))_\lambda^L$, $(F_s(x^*, y^*))_\lambda^R)^T \geq ((F_1(x, y))_\lambda^L, ((F_1(x, y))_\lambda^R, \ldots, (F_s(x, y))_\lambda^L, (F_s(x, y))_\lambda^R)^T$ or $((f_1(x^*, y^*))_\lambda^L$, $(f_1(x^*, y^*))_\lambda^R, \ldots, (f_t(x^*, y^*))_\lambda^L, (f_1(x^*, y^*))_\lambda^R)^T \geq ((f_1(x, y))_\lambda^L, (f_1(x, y))_\lambda^R, \ldots, (f_t(x, y))_\lambda^L), (f_t(x, y))_\lambda^R)^T$ hold.

Definition 2.10. [41] A point $(x^*, y^*) \in \underline{X} \times \underline{Y}$ is said to be a weak Pareto optimal solution to the MOLBP problem if there is no $(x, y) \in \underline{X} \times \underline{Y}$ such that $((F_1(x^*, y^*))_\lambda^L, (F_1(x^*, y^*))_\lambda^R, \ldots, (F_s(x^*, y^*))_\lambda^L, (F_s(x^*, y^*))_\lambda^R)^T > ((F_1(x, y))_\lambda^L, ((F_1(x, y))_\lambda^R, \ldots, (F_s(x, y))_\lambda^L, (F_s(x, y))_\lambda^R)^T$ or $((f_1(x^*, y^*))_\lambda^L, (f_1(x^*, y^*))_\lambda^R, \ldots, (f_t(x^*, y^*))_\lambda^L, (f_t(x^*, y^*))_\lambda^R)^T > ((f_1(x, y))_\lambda^L, (f_t(x, y))_\lambda^R, \ldots, (f_t(x, y))_\lambda^L, (f_t(x, y))_\lambda^R)^T$ hold.

Theorem 2.3. [41] *Let (x^*, y^*) be the optimal solution of the MOLBP problem defined by (2.2). Then it is also an optimal solution of the FMOLBP problem defined by (2.1).*

Theorem 2.4. [41] *For $x \in X \subset R^n$, $y \in Y \subset R^m$, if all the fuzzy parameters $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij}, \tilde{c}_{ij}, \tilde{b}_1, \tilde{b}_2$ and $\tilde{d}_{ij}$ have piecewise trapezoidal membership functions in the FMOLBP problem (2.1),*

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z_{\alpha_0}^L \\ \dfrac{\alpha_1 - \alpha_0}{z_{\alpha_1}^L - z_{\alpha_0}^L}(t - z_{\alpha_0}^L) + \alpha_0 & z_{\alpha_0}^L \leqq t < z_{\alpha_1}^L \\ \dfrac{\alpha_1 - \alpha_0}{z_{\alpha_2}^L - z_{\alpha_1}^L}(t - z_{\alpha_1}^L) + \alpha_1 & z_{\alpha_1}^L \leqq t < z_{\alpha_2}^L \\ \cdots & \cdots \\ \alpha & z_{\alpha_n}^L \leqq t < z_{\alpha_n}^R \\ \dfrac{\alpha_n - \alpha_{n-1}}{z_{\alpha_{n-1}}^R - z_{\alpha_n}^R}(-t + z_{\alpha_{n-1}}^R) + \alpha_{n-1} & z_{\alpha_n}^R \leqq t < z_{\alpha_{n-1}}^R \\ \cdots & \cdots \\ \dfrac{\alpha_0 - \alpha_1}{z_{\alpha_1}^R - z_{\alpha_0}^R}(-t + z_{\alpha_0}^R) + \alpha_0 & z_{\alpha_1}^R \leqq t \leqq z_{\alpha_0}^R \\ 0 & z_{\alpha_0}^R < t \end{cases} \tag{2.3}$$

where $\tilde{z}$ denotes $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij}, \tilde{c}_{ij}, \tilde{b}_1, \tilde{b}_2$ and $\tilde{d}_{ij}$ respectively, then, (x^*, y^*) is a complete optimal solution to the problem (2.1) if and only if (x^*, y^*) is a complete optimal solution to the MOLBP problem:

$$\min_{x \in X}(F_i(x, y))_{\alpha_j}^L = c_{i1\alpha_j}^L x + d_{i1\alpha_j}^L y, \quad i = 1, 2, \ldots, s, \; j = 0, 1, \ldots, n \tag{2.4a}$$

$$\min_{x \in X}(F_i(x, y))_{\alpha_j}^R = c_{i1\alpha_j}^R x + d_{i1\alpha_j}^R y, \quad i = 1, 2, \ldots, s, \; j = 0, 1, \ldots, n$$

subject to $A_{1\alpha_j}^L x + B_{1\alpha_j}^L y \leqq b_{1\alpha_j}^L, \quad j = 0, 1, \ldots, n$

$$A_{1\alpha_j}^R x + B_{1\alpha_j}^R y \leqq b_{1\alpha_j}^R, \quad j = 0, 1, \ldots, n \tag{2.4b}$$

$$\min_{y \in Y}(f_i(x, y))_{\alpha_j}^L = c_{i2\alpha_j}^L x + d_{i2\alpha_j}^L y, \quad i = 1, 2, \ldots, s, \; j = 0, 1, \ldots, n$$

$$\min_{y \in Y}(f_i(x, y))_{\alpha_j}^R = c_{i2\alpha_j}^R x + d_{i2\alpha_j}^R y, \quad i = 1, 2, \ldots, s, \; j = 0, 1, \ldots, n \tag{2.4c}$$

$$\text{subject to } A^L_{2\alpha_j}x + B^L_{2\alpha_j}y \leqq b^L_{2\alpha_j}, \quad j = 0, 1, \dots, n$$

$$A^R_{2\alpha_j}x + B^R_{2\alpha_j}y \leqq b^R_{2\alpha_j}, \quad j = 0, 1, \dots, n. \tag{2.4d}$$

We note

$$\bar{A}_1 x + \bar{B}_1 y \leqq \bar{b}_1 \tag{2.4b'}$$

$$\bar{A}_2 x + \bar{B}_2 y \leqq \bar{b}_2 \tag{2.4d'}$$

where

$$\bar{A}_1 = \begin{pmatrix} A^L_{1\alpha_0} \\ \vdots \\ A^L_{1\alpha_n} \\ A^R_{1\alpha_0} \\ \vdots \\ A^R_{1\alpha_n}, \end{pmatrix}, \bar{A}_2 = \begin{pmatrix} A^L_{2\alpha_0} \\ \vdots \\ A^L_{2\alpha_n} \\ A^R_{2\alpha_0} \\ \vdots \\ A^R_{2\alpha_n}, \end{pmatrix}, \bar{B}_1 = \begin{pmatrix} B^L_{1\alpha_0} \\ \vdots \\ B^L_{1\alpha_n} \\ B^R_{1\alpha_0} \\ \vdots \\ B^R_{1\alpha_n} \end{pmatrix}, \bar{B}_2 = \begin{pmatrix} B^L_{2\alpha_0} \\ \vdots \\ B^L_{2\alpha_n} \\ B^R_{2\alpha_0} \\ \vdots \\ B^R_{2\alpha_n} \end{pmatrix},$$

$$\bar{b}_1 = \begin{pmatrix} b^L_{1\alpha_0} \\ \vdots \\ b^L_{1\alpha_n} \\ b^R_{1\alpha_0} \\ \vdots \\ b^R_{1\alpha_n} \end{pmatrix}, \bar{b}_2 = \begin{pmatrix} b^L_{2\alpha_0} \\ \vdots \\ b^L_{2\alpha_n} \\ b^R_{2\alpha_0} \\ \vdots \\ b^R_{2\alpha_n} \end{pmatrix}.$$

Then we can re-write (2.4) by using

$$\min_{x\in X}(F_i(x,y))^L_{\alpha_j} = c^L_{i1\alpha_j}x + d^L_{i1\alpha_j}y, \quad i = 1, 2, \dots, s,\ j = 0, 1, \dots, n \tag{2.4$'a$}$$

$$\min_{x\in X}(F_i(x,y))^R_{\alpha_j} = c^R_{i1\alpha_j}x + d^R_{i1\alpha_j}y, \quad i = 1, 2, \dots, s,\ j = 0, 1, \dots, n$$

$$\text{subject to } \bar{A}_1 x + \bar{B}_1 y \leqq \bar{b}_1, \tag{2.4$'b$}$$

$$\min_{y\in Y}(f_i(x,y))^L_{\alpha_j} = c^L_{i2\alpha_j}x + d^L_{i2\alpha_j}y, \quad i = 1, 2, \dots, s,\ j = 0, 1, \dots, n$$

$$\min_{y\in Y}(f_i(x,y))^R_{\alpha_j} = c^R_{i2\alpha_j}x + d^R_{i2\alpha_j}y, \quad i = 1, 2, \dots, s,\ j = 0, 1, \dots, n \tag{2.4$'c$}$$

$$\text{subject to } \bar{A}_2 x + \bar{B}_2 y \leqq \bar{b}_2. \tag{2.4$'d$}$$

Theorem 2.5. [41] *For $x \in X \subset R^n$, $y \in Y \subset R^m$, if all the fuzzy parameters $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij}, \tilde{c}_{ij}, \tilde{b}_1, \tilde{b}_2$ and $\tilde{d}_{ij}$ have piecewise trapezoidal membership* *functions (2.3) in the FMOLBP problem (2.1), then (x^*, y^*) is a Pareto optimal solution to the problem (2.1) if and only if (x^*, y^*) is a Pareto optimal solution to the MOLBP problem (2.4′).*

Theorem 2.6. [41] *For $x \in X \subset R^n$, $y \in Y \subset R^m$, if all the fuzzy parameters $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij}, \tilde{c}_{ij}, \tilde{b}_1, \tilde{b}_2$ and $\tilde{d}_{ij}$ have piecewise trapezoidal membership functions (2.3) in the FMOLBP problem (2.1), then (x^*, y^*) is a weak Pareto optimal solution to the problem (2.1) if and only if (x^*, y^*) is a weak Pareto optimal solution to the MOLBP problem (2.4′).*

These definitions and theorems will be used in following sections to develop an approach for solving the FMOLBP problems.

3 AN APPROXIMATION *K*th-BEST APPROACH

To solve the FMOLBP problem (2.1), we need to solve its transformed form (2.4′). For solving (2.4′), we can use the method of weighting [21] to this problem, such that it becomes the following problem:

$$\min_{x\in X}(F(x,y)) = \sum_{j=1}^{s} w_{j1}\left(\sum_{i=0}^{n}(c^L_{1_{\alpha_i}}x + d^L_{1_{\alpha_i}}y) + \sum_{i=0}^{n}(c^R_{1_{\alpha_i}}x + d^R_{1_{\alpha_i}}y)\right) \tag{3.1a}$$

$$\text{subject to } \bar{A}_1 x + \bar{B}_1 y \leqq \bar{b}_1, \tag{3.1b}$$

$$\min_{y\in Y}(f(x,y)) = \sum_{j=1}^{t} w_{j2}\left(\sum_{i=0}^{n}(c^L_{2_{\alpha_i}}x + d^L_{2_{\alpha_i}}y) + \sum_{i=0}^{n}(c^R_{2_{\alpha_i}}x + d^R_{2_{\alpha_i}}y)\right) \tag{3.1c}$$

$$\text{subject to } \bar{A}_2 x + \bar{B}_2 y \leqq \bar{b}_2. \tag{3.1d}$$

In order to get a solution for above (3.1), we give a definition of optimal solution and related theorems as follows.

Definition 3.1. (a) Constraint region of the linear BP problem:

$$S = \{(x, y) : x \in X, y \in Y, \bar{A}_1 x + \bar{B}_1 y \leqq \bar{b}_1, \bar{A}_2 x + \bar{B}_2 y \leqq \bar{b}_2\}$$

(b) Feasible set for the follower for each fixed $x \in X$:

$$S(x) = \{y \in Y : \bar{B}_2 y \leqq \bar{b}_2 - \bar{A}_2 x\}$$

(c) Projection of S onto the leader's decision space:

$$S(X) = \{x \in X : \exists y \in Y, \bar{A}_1 x + \bar{B}_1 y \leqq \bar{b}_1, \bar{A}_2 x + \bar{B}_2 y \leqq \bar{b}_2\}$$

Follower's rational reaction set for $x \in S(X)$:

$$P(x) = \{y \in Y : y \in \text{argmin}\ [(f(x, \hat{y})) : \hat{y} \in S(x)]\}$$

where argmin $[f(x, \hat{y}) : \hat{y} \in S(x)] = \{y \in S(x) : (f(x, y)) \leqq (f(x, \hat{y})), \hat{y} \in S(x)\}$

(e) Inducible region:

$$IR = \{(x, y) : (x, y) \in S, y \in P(x)\}$$

The rational reaction set $P(x)$ defines the response while the inducible region IR represents the set over which the leader may optimize his objective. Thus in terms of the above notations, the linear BP problem can be written as

$$\min\{F(x, y) : (x, y) \in IR\}. \tag{3.2}$$

Theorem 3.1. *The inducible region can be written equivalently as piecewise linear equality constraint comprised of supporting hyperplanes of constraint region S.*

Proof. Let us begin by writing the inducible region of Definition 3.1(e) explicitly as follower:

$$\begin{aligned} IR = \{(x, y) &: (x, y) \in S, \\ &\bar{d}_2 y = \min[\bar{d}_2 \tilde{y} : \bar{B}_i \tilde{y} \leqq \bar{b}_i - \bar{A}_i x, i = 1, 2, \tilde{y} \geq 0]\}, \end{aligned} \tag{3.3}$$

where $\bar{c}_i = c_i + c_{i_0}^L + c_{i_0}^R, \bar{d}_i = d_i + d_{i_0}^L + d_{i_0}^R, i = 1, 2$. Now we define

$$Q(x) = \min\{\bar{d}_2 y : \bar{B}_i y \leq \bar{b}_i - \bar{A}_i x, i = 1, 2, y \geq 0\}. \tag{3.4}$$

Let us define

$$\bar{B} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix}.$$

We rewrite (3.4) as follows

$$Q(x) = \min\{\bar{d}_2 y : \bar{B} y \leqq \bar{b} - \bar{A} x, y \geq 0\}. \tag{3.5}$$

For each value of $x \in S(X)$, the resulting feasible region to problem (2.3) is nonempty and compact. Thus $Q(x)$, which is a linear program parameterized in x, always has a solution. From duality theory, we get

$$\max\{u(\bar{A}x - b) : u\bar{B} \geqq -\bar{d}_2, u \geq 0\}, \tag{3.6}$$

which has the same optimal value as (3.1) at the solution u^*. Let $u^1, \ldots, u^s$ be a listing of all the vertices of the constraint region of (3.6) given by

$U = \{u : u\bar{B} \geqq -\bar{d}_2, u \geq 0\}$. Because we know that a solution to (3.6) occurs at a vertex of U, we get the equivalent problem

$$\max\{u^j(\bar{A}x - \bar{b}) : u^j \in \{u^1, \ldots, u^s\}\}, \tag{3.7}$$

which demonstrates that $Q(x)$ is a piecewise linear function. Rewriting IR as

$$IR = \{(x, y) \in S : Q(x) - \bar{d}_2 y = 0\}, \tag{3.8}$$

yields the desired result. □

By this theorem, we give the following corollaries:

Corollary 3.1. *The linear BP problem (3.1) is equivalent to minimizing F over a feasible region comprised of a piecewise linear equality constraint.*

Proof. From (3.2) and Theorem 2.6, we have the desired result. □

Corollary 3.2. *A solution for the linear BP problem occurs at a vertex of IR.*

Proof. A linear BP programming can be written (3.2). Since F is linear, if a solution exists, one must occur at a vertex of IR. The proof is completed. □

Now, we give a very important theorem which is the core for proposing an approximation Kth-best approach.

Theorem 3.2. *The solution (x^*, y^*) of the linear BP problem occurs at a vertex of S.*

Proof. Let $(x^1, y^1), \ldots, (x^r, y^r)$ be the distinct vertices of S. Since any point in S can be written a convex combination of these vertices, let $(x^*, y^*) = \sum_{i=1}^{r} \alpha_i (x^i, y^i)$, where $\sum_{i=1}^{r} \alpha_i = 1, \alpha_i \geq 0, i = 1, \ldots, \bar{r}$ and $\bar{r} \leq r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (2.3) at (x^*, y^*) in their piecewise linear form (2.4′).

$$\begin{aligned} 0 &= Q(x^*) - \bar{d}_2 y^* \\ &= Q\left(\sum_i \alpha_i x^i\right) - \bar{d}_2 \left(\sum_i \alpha_i y^i\right) \\ &\leq \sum_i \alpha_i Q(x^i) - \sum_i \alpha_i \bar{d}_2 y^i \end{aligned}$$

by convexity of $Q(x)$

$$= \sum_i \alpha_i (Q(x^i) - \bar{d}_2 y^i).$$

But by definition,

$$Q(x^i) = \min_{y \in S(x^i)} \bar{d}_2 y \leq \bar{d}_2 y^i.$$

Therefore, $Q(x^i) - \bar{d}_2 y^i \leq 0, i = 1, \ldots, \bar{r}$. Noting that $\alpha_i \geq 0, i = 1, \ldots, \bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q(x^i) - \bar{d}_2 y^i = 0$ for all i. This implies that $(x^i, y^i) \in IR, i = 1, \ldots, \bar{r}$ and (x^*, y^*) can be written as a convex combination of points in *IR*. Because (x^*, y^*) is a vertex of *IR*, a contradiction results unless $\bar{r} = 1$. □

We therefore give the following corollary.

Corollary 3.3. *If x is an extreme point of IR, it is an extreme point of S.*

Proof: Let (x^*, y^*) be an extreme point of *IR* and assume that it is not an extreme point of *S*. Let $(x^1, y^1), \ldots, (x^r, y^r)$ be the distinct vertices of *S*. Since any point in *S* can be written a convex combination of these vertices, let $(x^*, y^*) = \sum_{i=1}^{r} \alpha_i (x^i, y^i)$, where $\sum_{i=1}^{r} \alpha_i = 1, \alpha_i \geq 0, i = 1, \ldots, \bar{r}$ and $\bar{r} \leq r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (2.3) at (x^*, y^*) in their piecewise linear form (2.4′).

$$
\begin{aligned}
0 &= Q(x^*) - \bar{d}_2 y^* \\
&= Q\left(\sum_i \alpha_i x^i\right) - \bar{d}_2 \left(\sum_i \alpha_i y^i\right) \\
&\leq \sum_i \alpha_i Q(x^i) - \sum_i \alpha_i \bar{d}_2 y^i
\end{aligned}
$$

by convexity of $Q(x)$

$$
= \sum_i \alpha_i (Q(x^i) - \bar{d}_2 y^i).
$$

But by definition,

$$
Q(x^i) = \min_{y \in S(x^i)} \bar{d}_2 y \leq \bar{d}_2 y^i.
$$

Therefore, $Q(x^i) - \bar{d}_2 y^i \leq 0, i = 1, \ldots, \bar{r}$. Noting that $\alpha_i \geq 0, i = 1, \ldots, \bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q(x^i) - \bar{d}_2 y^i = 0$ for all i. This implies that $(x^i, y^i) \in IR, i = 1, \ldots, \bar{r}$ and (x^*, y^*) can be written as a convex combination of points in *IR*. Because (x^*, y^*) is an extreme point of *IR*, a contradiction results unless $\bar{r} = 1$. This means that (x^*, y^*) is an extreme point of *S*. The proof is completed. □

Theorem 2.6 and Corollary 3.3 have provided theoretical foundation for our new approach. It means that by searching extreme points on the constraint region *S*, we can efficiently find an optimal solution for a linear BP problem. The basic idea of our extended properties approach is that according to the objective function of the upper level, we descendent order all the extreme points on S, and select the first extreme point to check if it is on the inducible region IR. If yes, the current extreme point is the optimal solution. If not, select the next one and check.

More specifically, let $(x_{[1]}, y_{[1]}), \ldots, (x_{[N]}, y_{[N]})$ denote the N ordered extreme points to the linear programming problem

$$\min\{\bar{c}_1 x + \bar{d}_1 y : (x, y) \in S\}, \tag{3.9}$$

such that

$$\bar{c}_1 x_{[i]} + \bar{d}_1 y_{[i]} \leq \bar{c}_1 x_{[i+1]} + \bar{d}_1 y_{[i+1]}, \quad i = 1, \ldots, N-1.$$

Let $\tilde{y}$ denote the optimal solution to the following problem

$$\min(f(x[i], y) : y \in S(x_{[i]})). \tag{3.10}$$

We only need to find the smallest $i (i \in \{1, \ldots, N\})$ under which $y_{[i]} = \tilde{y}$.

Let write (3.10) as follows

$$\begin{aligned} &\min f(x, y) \\ &\text{subject to } y \in S(x) \\ &\qquad\qquad x = x_{[i]}. \end{aligned}$$

From Definition 3.1(a) and (c), we have

$$\min f(x, y) = \bar{c}_2 x + \bar{d}_2 y \tag{3.11a}$$

$$\text{subject to } \bar{A}_1 x + \bar{B}_1 y \leq \bar{b}_1 \tag{3.11b}$$

$$\bar{A}_2 x + \bar{B}_2 y \leq \bar{b}_2 \tag{3.11c}$$

$$x = x_{[i]} \tag{3.11d}$$

$$y \geq 0. \tag{3.11e}$$

To solve (3.11), the first is select one ordered extreme point $(x_{[i]}, y_{[i]})$, then solve (3.11) to obtain the optimal solution $\tilde{y}$. If $\tilde{y} = y_{[i]}$, $(x_{[i]}, y_{[i]})$ is the global optimum to (3.1). Otherwise, check the next extreme point.

Based on above definition of optimal solution and Theorem 3.2, we propose an approximation Kth-best approach for solving FMOLBP problem (2.1) as follows.

The approximation *K*th-best approach:

Step 1 Given weights $w_{j1} (j = 1, 2, \ldots, s)$ and $w_{j2} (j = 1, 2, \ldots, t)$ for the objectives of the leader and the follower respectively and let $\sum_{j=1}^{s} w_{j1} = 1$and $\sum_{j=1}^{t} w_{j2} = 1$.

Step 2 Transform the problem (2.1) to the problem (2.4′).

Step 3 Set $l = 1$ and a range of errors $\varepsilon > 0$.

Step 4 Let the interval [0, 1] be decomposed into 2^{l-1} equal sub-intervals with $(2^{l-1} + 1)$ nodes $\lambda_i (i = 0, \ldots, 2^{l-1})$ which are arranged in the order of $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{2^{l-1}} = 1$.

Step 5 Transform the problem (2.4′) to the problem (3.1) by the weighting method and solve $(\text{MOLBP})^{\,l}_{2,}$ i.e. (3.1) by using the extended Kth-best approach [29] for obtaining an optimal solution $(x, y)_{2^l}$.

Step 6 Put $i \leftarrow 1$. Solve (3.9) with the simplex method to obtain the optimal solution $(x_{[1]}, y_{[1]})$. Let $W = \{(x_{[1]}, y_{[1]})\}$ and $T = \phi$. Go to Step 7.

Step 7 Solve (3.11) with the bounded simplex method. Let $\tilde{y}$ denote the optimal solution to (3.11). If $\tilde{y} = y_{[i]}$ stop; $(x_{[i]}, y_{[i]})$ is the global optimum to (3.1) with $K^* = i$. Otherwise, go to Step 8.

Step 8 Let $W_{[i]}$ denote the set of adjacent extreme points of $(x_{[i]}, y_{[i]})$ such that $(x, y) \in W_{[i]}$ implies $\bar{c}_1 x + \bar{d}_1 y \geq \bar{c}_1 x_{[i]} + \bar{d}_1 y_{[i]}$. Let $T = T \cup \{(x_{[i]}, y_{[i]})\}$ and $W = (W \cup W_{[i]}) \backslash T$. Go to Step 9.

Step 9 Set $i \leftarrow i+1$ and choose $(x_{[i]}, y_{[i]})$ so that $f x_{[i]} + g y_{[i]} = \min\{\bar{c}_1 x + \bar{d}_1 y : (x, y) \in W\}$. Go to Step 7.

Step 10 $l = l + 1$, repeat Step 4 to Step 9 to solve $(\text{MOLBP})_{2^{l+1}}$.

Step 11 If $\|(x, y)_{2^{l+1}} - (x, y)_{2^l}\| < \varepsilon$, then the solution (x^*, y^*) of the FMOLBP problem is $(x, y)_{2^{l+1}}$, otherwise, update l to $2l$ and go back to Step 10.

Step 12 Show the result of problem (2.1), stops.

4 ILLUSTRATIVE EXAMPLES

We give examples here to illustrate how to use the proposed FMOLBP model and the approximation Kth-best approach solving a FMOLBP problem in practice. Example 1 mainly shows how to build a FMOLBP model for a real problem, and Example 2 gives all details to solve a FMOLBP problem by the proposed approximate Kth-best approach.

Example 1. In a company, the CEO is as the leader, and the heads of branches of the company are as the follower in making an annual budget for the company. Obviously, the leader (the CEO)'s decision will be affected by the reactions of the follower (heads of branches). Each of the CEO's possible decisions is influenced by the various reactions of the heads. In order to arrive an optimal solution (better strategies) for the CEO's decision on the annual budget, we establish a bilevel decision making model.

The CEO has two main objectives: 1) to maximize the net profits, represented by $F_1(x, y)$ and 2) to maximize the quality of products, by $F_2(x, y)$, but subject to some constraints including the requirements of material, marking cost, labor cost, working hours and so on. The heads of branches, as the follower, attempts to 1) maximize their net profit, $f_1(x, y)$, and 2) maximize work satisfactory $f_2(x, y)$. The CEO understands that for each policy he may make, these heads will have a new reaction to deal with by optimizing their objective $\max_{y\in Y}(f_1(x, y),\ f_2(x, y))$.

When modeling the bilevel decision problem, the main difficulty is to set up parameters for the objectives and constraints of both the leader and the follower. We can only estimate some values such as material cost, labor cost, according to our experience and previous data. For some items, the values can be only assigned by linguistic terms, such as ‘about \$1000’. This is a common case in any organizational decision practice. By using fuzzy numbers to describe these uncertain values and linguistic terms in parameters, a FMOLBP model can be established for this decision problem.

Let $x = (x_1, x_2)^T \in R^2$ be the CEO’s decision variables, and $y = (y_1, y_2, y_3)^T \in R^3$ be the branch heads’ decision variables, and $X = \{x \geq 0\}$, $Y = \{y \geq 0\}$, we can build the following model for the decision problem:

$$
\begin{aligned}
&\max_{x\in X} F_1(x, y) = (\tilde{1}, \tilde{9})(x_1, x_2)^T + (\widetilde{10}, \tilde{1}, \tilde{3})(y_1, y_2, y_3)^T\\
&\max_{x\in X} F_2(x, y) = (\tilde{9}, \tilde{2})(x_1, x_2)^T + (\tilde{2}, \tilde{7}, \tilde{4})(y_1, y_2, y_3)^T\\
&\text{subject to } (\tilde{3}, \tilde{9})(x_1, x_2)^T + (\tilde{9}, \tilde{5}, \tilde{3})(y_1, y_2, y_3)^T \leq \widetilde{1039}\\
&\qquad (-\tilde{4}, -\tilde{1})(x_1, x_2)^T + (\tilde{3}, -\tilde{3}, \tilde{2})(y_1, y_2, y_3)^T \leq \widetilde{94}\\
&\qquad \max_{y\in Y} f_1(x, y) = (\tilde{4}, \tilde{6})(x_1, x_2)^T + (\tilde{7}, \tilde{4}, \tilde{8})(y_1, y_2, y_3)^T\\
&\qquad \max_{y\in Y} f_2(x, y) = (\tilde{6}, \tilde{4})(x_1, x_2)^T + (\tilde{8}, \tilde{7}, \tilde{4})(y_1, y_2, y_3)^T\\
&\qquad \text{subject to } (\tilde{3}, -\tilde{9})(x_1, x_2)^T + (-\tilde{9}, -\tilde{4}, \tilde{0})(y_1, y_2, y_3)^T \leq \widetilde{61}\\
&\qquad\qquad (\tilde{5}, \tilde{9})(x_1, x_2)^T + (\widetilde{10}, -\tilde{1}, -\tilde{2})(y_1, y_2, y_3)^T \leq \widetilde{924}\\
&\qquad\qquad (\tilde{3}, -\tilde{3})(x_1, x_2)^T + (\tilde{0}, \tilde{1}, \tilde{5})(y_1, y_2, y_3)^T \leq \widetilde{420}
\end{aligned}
$$

In this model, the unified form for all membership functions of the parameters of the objective functions and constraints is as follows:

$$
\mu_{\tilde{\alpha}}(x) = \begin{cases} 0 & x < a \text{ or } c < x \\ (x^2 - a^2)/(b^2 - a^2) & a \leq x < b \\ 1 & b \\ (c^2 - x^2)/(c^2 - d^2) & b < x \leq c \end{cases}
$$

$\tilde{c}_{ij}$	1	2	3	4	5
1	(0, 1, 2)	(8, 9, 12)	(9, 10, 13)	(0.5, 1, 2.5)	(2, 3, 6)
2	(8, 9, 12)	(1, 2, 5)	(1, 2, 5)	(6, 7, 10)	(3, 4, 7)
3	(2, 4, 5)	(4, 6, 7)	(5, 7, 8)	(2, 4, 5)	(6, 8, 9)
	(4, 6, 7)	(2, 4, 5)	(6, 8, 9)	(5, 7, 8)	(2, 4, 5)

TABLE 4.1
Membership functions of fuzzy objective functions' parameters

$\tilde{a}_{ij}$	1	2	3	4	5
1	(2, 3, 5)	(8, 9, 11)	(8, 9, 11)	(4, 5, 7)	(2, 3, 5)
2	(−6,−4,−3)	(−2, −1, −0.5)	(2, 3, 5)	(−5, −3, −2)	(−4, −2, −1)
3	(2, 3, 5)	(−11, −9, −8)	(−11, −9, −8)	(−6, −4, −3)	(0, 0, 0)
4	(4, 5, 7)	(8, 9, 11)	(9, 10, 12)	(0.5, 1, 2)	(−4, −2, −1)
5	(2, 3, 5)	(−5, −3, −2)	(0, 0, 0)	(0.5, 1, 2)	(4, 5, 7)

TABLE 4.2
Membership functions of fuzzy constraints' parameters

$\tilde{b}_i$	1
1	(1038, 1039, 1041)
2	(93, 94, 96)
3	(60, 61, 63)
4	(923, 924, 926)
5	(419, 420, 422)

TABLE 4.3
Membership functions of fuzzy right-hand-side's parameters

For simplicity, we only represent the above form of membership function as (a, b, c). Then, for the example, all membership functions of fuzzy parameters of the objective functions and constraints are to be represented in the quadruple pair form and listed in Tables 4.1, 4.2, and 4.3, respectively.

Now, We first given the weights for the two fuzzy objectives of the leader are (0.5, 0.5) and of the follower (0.5, 0.5) and the interval [0, 1] be decomposed into 2^{l-1} mean sub-intervals with $(2^{l-1}+1)$ nodes $\lambda_i (i = 0, \ldots, 2^{l-1})$ which is arranged in the order of $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{2^{l-1}} = 1$ and a range of errors $\varepsilon = 10^{-6} > 0$. Then we can solve this problem by using the proposed approximation Kth-best approach. The solution of the problem is

$x_1 = 146.2955$, $x_2 = 28.9394$ and $y_1 = 0$, $y_2 = 67.9318$, $y_3 = 0$ such that

$$\max_{x\in X} F_1(x, y) = 164.2955 \times \tilde{1} + 28.9394 \times \tilde{9} + 67.9318 \times \tilde{1}$$

$$\max_{x\in X} F_2(x, y) = 164.2955 \times \tilde{9} + 28.9394 \times \tilde{2} + 67.9318 \times \tilde{7}$$

$$\min_{y\in Y} f_1(x, y) = 164.2955 \times \tilde{4} + 28.9394 \times \tilde{6} + 67.9318 \times \tilde{4}$$

$$\min_{y\in Y} f_2(x, y) = 164.2955 \times \tilde{6} + 28.9394 \times \tilde{4} + 67.9318 \times \tilde{7}.$$

Example 2. Consider the following FMOLBP problem with $x \in R^1$, $y \in R^1$, and $X = \{x \geq 0\}$, $Y = \{y \geq 0\}$,

$$\begin{aligned}
&\min_{x\in X} F_1(x, y) = -\tilde{1}x + \tilde{2}y \\
&\min_{x\in X} F_2(x, y) = \tilde{2}x - \tilde{4}y \\
&\text{subject to } -\tilde{1}x + \tilde{3}y \leq \tilde{4} \\
&\qquad \min_{y\in Y} f_1(x, y) = -\tilde{1}x + \tilde{2}y \\
&\qquad \min_{y\in Y} f_2(x, y) = \tilde{2}x - \tilde{1}y \\
&\qquad \text{subject to } \tilde{1}x - \tilde{1}y \leq \tilde{0} \\
&\qquad\qquad -\tilde{1}x - \tilde{1}y \leq \tilde{0}
\end{aligned}$$

where

$$\mu_{\tilde{1}}(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \leqq t < 1 \\ 2 - t & 1 \leqq t < 2 \\ 0 & 2 \leqq t \end{cases}, \quad \mu_{\tilde{2}}(t) = \begin{cases} 0 & t < 1 \\ t - 1 & 1 \leqq t < 2 \\ 3 - t & 2 \leqq t < 3 \\ 0 & 3 \leqq t \end{cases},$$

$$\mu_{\tilde{3}}(t) = \begin{cases} 0 & t < 2 \\ t - 2 & 2 \leqq t < 3 \\ 4 - t & 3 \leqq t < 4 \\ 0 & 4 \leqq t \end{cases}, \quad \mu_{\tilde{4}}(t) = \begin{cases} 0 & t < 3 \\ t - 3 & 3 \leqq t < 4 \\ 5 - t & 4 \leqq t < 5 \\ 0 & 5 \leqq t \end{cases},$$

$$\mu_{\tilde{0}}(t) = \begin{cases} 0 & t < -1 \\ t + 1 & -1 \leqq t < 0 \\ 1 - t^2 & 0 \leqq t < 1 \\ 0 & 1 \leqq t \end{cases}.$$

We now solve this problem by using the proposed approximation Kth-best approach.

Step 1 Given the weights for the two fuzzy objectives of the leader are (0.5, 0.5) and of the follower (0.5, 0.5).

Step 2 The FMOLBP problem is first transformed to the following associated MOLBP problem by using Theorem 2.3

$$
\begin{aligned}
&\min_{x\in X}(F_1(x,y))_\lambda^L = (-\tilde{1})_\lambda^L x + \tilde{2}_\lambda^L y, \quad \lambda \in [0,1]\\
&\min_{x\in X}(F_1(x,y))_\lambda^R = (-\tilde{1})_\lambda^R x + \tilde{2}_\lambda^R y, \quad \lambda \in [0,1]\\
&\min_{x\in X}(F_2(x,y))_\lambda^L = \tilde{2}_\lambda^L x + (-\tilde{4})_\lambda^L y, \quad \lambda \in [0,1]\\
&\min_{x\in X}(F_2(x,y))_\lambda^R = \tilde{2}_\lambda^R x + (-\tilde{4})_\lambda^R y, \quad \lambda \in [0,1]\\
&\text{subject to } (-\tilde{1})_\lambda^L x + \tilde{3}_\lambda^L y \leqq \tilde{4}_\lambda^L,\ (-\tilde{1})_\lambda^R x + \tilde{3}_\lambda^R y \leqq \tilde{4}_\lambda^R, \quad \lambda \in [0,1]\\
&\qquad \min_{y\in Y}(f_1(x,y))_\lambda^L = \tilde{2}_\lambda^L x + (-\tilde{1})_\lambda^L y, \quad \lambda \in [0,1]\\
&\qquad \min_{y\in Y}(f_1(x,y))_\lambda^R = \tilde{2}_\lambda^R x + (-\tilde{1})_\lambda^R y, \quad \lambda \in [0,1]\\
&\qquad \min_{y\in Y}(f_2(x,y))_\lambda^L = (-\tilde{1})_\lambda^L x + \tilde{2}_\lambda^L y, \quad \lambda \in [0,1]\\
&\qquad \min_{y\in Y}(f_2(x,y))_\lambda^R = (-\tilde{1})_\lambda^R x + \tilde{2}_\lambda^R y, \quad \lambda \in [0,1]\\
&\qquad \text{subject to } \tilde{1}_\lambda^L x + (-\tilde{1})_\lambda^L y \leqq \tilde{0}_\lambda^L,\ \tilde{1}_\lambda^R x + (-\tilde{1})_\lambda^R y \leqq \tilde{0}_\lambda^R, \quad \lambda \in [0,1]\\
&\qquad\qquad (-\tilde{1})_\lambda^L x + (-\tilde{1})_\lambda^L y \leqq \tilde{0}_\lambda^L,\ (-\tilde{1})_\lambda^R x + (-\tilde{1})_\lambda^R y\\
&\qquad\qquad\qquad \leqq \tilde{0}_\lambda^R, \quad \lambda \in [0,1]
\end{aligned}
$$

Step 3 Set $l = 1$ and $\varepsilon = 10^{-6} > 0$.

Step 4 Let the interval [0, 1] be decomposed into 2^{l-1} equal sub-intervals with $(2^{l-1}+1)$ nodes $\lambda_i (i = 0, \ldots, 2^{l-1})$ which is arranged in the order of $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{2^{l-1}} = 1$. We get the following MOLBP problem

$$
\begin{aligned}
&\min_{x\in X}(F_1(x,y))_1^{L(R)} = -1x + 2y\\
&\min_{x\in X}(F_1(x,y))_0^{L} = -2x + y\\
&\min_{x\in X}(F_1(x,y))_0^{R} = 0x + 3y\\
&\min_{x\in X}(F_2(x,y))_1^{L(R)} = 2x - 4y\\
&\min_{x\in X}(F_2(x,y))_0^{L} = 1x - 5y
\end{aligned}
$$

$$
\begin{aligned}
&\min_{x\in X}(F_2(x,y))_0^R = 3x - 3y\\
&\text{subject to } -1x + 3y \le 4\\
&\qquad -2x + 2y \le 3\\
&\qquad 0x + 4y \le 5\\
&\qquad \min_{y\in Y}(f_1(x,y))_1^{L(R)} = 2x - 1y\\
&\qquad \min_{y\in Y}(f_1(x,y))_0^{L} = 1x - 2y\\
&\qquad \min_{y\in Y}(f_1(x,y))_0^{R} = 3x - 0y\\
&\qquad \min_{y\in Y}(f_2(x,y))_1^{L(R)} = -1x + 2y\\
&\qquad \min_{y\in Y}(f_2(x,y))_0^{L} = -2x + 1y\\
&\qquad \min_{y\in Y}(f_2(x,y))_0^{L} = 0x + 3y\\
&\qquad \text{subject to } 1x - 1y \le 0\\
&\qquad\qquad 0x - 2y \le -1\\
&\qquad\qquad 2x - 0y \le 1\\
&\qquad\qquad -1x - 1y \le 0\\
&\qquad\qquad -2x - 2y \le -1.
\end{aligned}
$$

Step 5 We solve this MOLBP problem by using the extended Kth-best approach [29] and the method of weighting.

$$
\begin{aligned}
&\min_{x\in X} F(x,y) = 3x - 6y\\
&\text{subject to } -1x + 3y \le 4\\
&\qquad -2x + 2y \le 3\\
&\qquad 0x + 4y \le 5\\
&\qquad \min_{y\in Y} f(x,y) = 3x + 3y\\
&\qquad \text{subject to } 1x - 1y \le 0\\
&\qquad\qquad 0x - 2y \le -1\\
&\qquad\qquad 2x - 0y \le 1\\
&\qquad\qquad -1x - 1y \le 0\\
&\qquad\qquad -2x - 2y \le -1.
\end{aligned}
$$

According to the extended Kth-best approach, let us rewrite it as follows in (3.7)

$$\begin{aligned}
\min F(x, y) &= 3x - 6y \\
\text{subject to } & -1x + 3y \le 4 \\
& -2x + 2y \le 3 \\
& 0x + 4y \le 5 \\
& 1x - 1y \le 0 \\
& 0x - 2y \le -1 \\
& 2x - 0y \le 1 \\
& -1x - 1y \le 0 \\
& -2x - 2y \le -1 \\
& x \ge 0,\ y \ge 0.
\end{aligned}$$

Step 6 Let $i = 1$, and solve the above problem with the simplex method to obtain the optimal solution $(x_{[1]}, y_{[1]}) = (0, 1.25)$. Let $W = \{(0, 1.25)\}$ and $T = \phi$. Go to Step 7.

Loop 1:
Step 7 By (3.9), we have

$$\begin{aligned}
\min f(x, y) &= 3x + 3y \\
\text{subject to } & -1x + 3y \le 4 \\
& -2x + 2y \le 3 \\
& 0x + 4y \le 5 \\
& 1x - 1y \le 0 \\
& 0x - 2y \le -1 \\
& 2x - 0y \le 1 \\
& -1x - 1y \le 0 \\
& -2x - 2y \le -1 \\
& x = 0 \\
& y \ge 0.
\end{aligned}$$

Using the bounded simplex method, we have $\tilde{y} = 0.5$. Because of $\tilde{y} \ne y_{[i]}$, we go to Step 8.

Step 8 We have $W_{[i]} = \{(0.5, 1.25), (0, 0.5), (0, 1.25)\}$, $T = \{(0, 1.25)\}$ and $W = \{(0, 0.5), (0.5, 1.25)\}$, then go to Step 9.

Step 9 Update $i = 2$, and choose $(x_{[i]}, y_{[i]}) = (0.5, 1.25)$, then go to Step 7.

Loop 2:
Step 7 By (3.9)

$$\begin{aligned}
\min f(x, y) &= 3x + 3y \\
\text{subject to } &- 1x + 3y \le 4 \\
&- 2x + 2y \le 3 \\
&0x + 4y \le 5 \\
&1x - 1y \le 0 \\
&0x - 2y \le -1 \\
&2x - 0y \le 1 \\
&- 1x - 1y \le 0 \\
&- 2x - 2y \le -1 \\
&x = 0.5 \\
&y \ge 0.
\end{aligned}$$

Using the bounded simplex method, we have $\tilde{y} = 0.5$. Because of $\tilde{y} \ne y_{[i]}$, we go to Step 5.

Step 8 We have $W_{[i]} = \{(0.5, 1.25), (0.5, 0.5), (0, 1.25)\}$, $T = \{(0, 1.25), (0.5, 1.25)\}$ and $W = \{(0, 0.5), (0.5, 0.5)\}$, then go to Step 9.

Step 9 Update $i = 3$, and choose $(x_{[i]}, y_{[i]}) = (0, 0.5)$, then go to Step 7.

Loop 3:
Step 7 By (3.9), we have

$$\begin{aligned}
&\min f(x, y) = 3x + 3y \\
&\text{subject to } -1x + 3y \le 4 \\
&\qquad -2x + 2y \le 3 \\
&\qquad 0x + 4y \le 5 \\
&\qquad 1x - 1y \le 0 \\
&\qquad 0x - 2y \le -1 \\
&\qquad 2x - 0y \le 1 \\
&\qquad -1x - 1y \le 0 \\
&\qquad -2x - 2y \le -1 \\
&\qquad x = 0 \\
&\qquad y \ge 0.
\end{aligned}$$

Using the bounded simplex method, we have $\tilde{y} = 0.5$. Because of $\tilde{y} = y_{[i]}$, we stop here. $(x_{[i]}, y_{[i]}) = (0, 0.5)$ is the global solution to this Example.

By examining above procedure, we found that the optimal solution occurs at the point $(x^*, y^*) = (0, 0.5)$ with

$$\min_{x \in X} F_1(x, y) = 1x - 2y = -1$$
$$\min_{x \in X} F_2(x, y) = 0x - 3y = -1.3$$
$$\min_{x \in X} F_3(x, y) = 2x - 1y = -0.5$$
$$\min_{y \in Y} f_1(x, y) = 0.5$$
$$\min_{y \in Y} f_2(x, y) = 1$$

Step 10 Set $l = 2$ and we solve the following MOLBP problem

$$\begin{aligned}
&\min_{x \in X}(F_1(x, y))_1^{L(R)} = -1x + 2y \\
&\min_{x \in X}(F_1(x, y))_{\frac{1}{2}}^{L} = -\frac{3}{2}x + \frac{3}{2}y \\
&\min_{x \in X}(F_1(x, y))_0^{L} = -2x + 1y \\
&\min_{x \in X}(F_1(x, y))_{\frac{1}{2}}^{R} = -\frac{\sqrt{2}}{2}x + \frac{5}{2}y \\
&\min_{x \in X}(F_1(x, y))_0^{R} = 0x + 3y \\
&\min_{x \in X}(F_2(x, y))_1^{L(R)} = 2x - 4y \\
&\min_{x \in X}(F_2(x, y))_{\frac{1}{2}}^{L} = \frac{3}{2}x - \frac{9}{2}y \\
&\min_{x \in X}(F_2(x, y))_0^{L} = 1x - 5y \\
&\min_{x \in X}(F_2(x, y))_{\frac{1}{2}}^{L} = \frac{5}{2}x - \frac{7}{2}y \\
&\min_{x \in X}(F_2(x, y))_0^{R} = 3x - 3y \\
&\text{subject to } -1x + 3y \leq 4 \\
&\qquad -\frac{3}{2}x + \frac{5}{2}y \leq \frac{7}{2} \\
&\qquad -2x + 2y \leq 3 \\
&\qquad -\frac{\sqrt{2}}{2}x + \frac{7}{2}y \leq \frac{9}{2}
\end{aligned}$$

$$0x + 4y \leq 5$$

$$\min_{y \in Y}(f_1(x, y))_1^{L(R)} = 2x - 1y$$

$$\min_{y \in Y}(f_1(x, y))_{\frac{1}{2}}^{L} = \frac{3}{2}x - \frac{3}{2}y$$

$$\min_{y \in Y}(f_1(x, y))_0^{L} = 1x - 2y$$

$$\min_{y \in Y}(f_1(x, y))_{\frac{1}{2}}^{R} = \frac{5}{2}x - \frac{\sqrt{2}}{2}y$$

$$\min_{y \in Y}(f_1(x, y))_0^{R} = 3x - 0y$$

$$\min_{y \in Y}(f_2(x, y))_1^{L(R)} = -1x + 2y$$

$$\min_{y \in Y}(f_2(x, y))_{\frac{1}{2}}^{L} = -\frac{3}{2}x + \frac{3}{2}y$$

$$\min_{y \in Y}(f_2(x, y))_0^{L} = -2x + 1y$$

$$\min_{y \in Y}(f_2(x, y))_{\frac{1}{2}}^{R} = -\frac{\sqrt{2}}{2}x + \frac{5}{2}y$$

$$\min_{y \in Y}(f_2(x, y))_0^{R} = 0x + 3y$$

subject to $1x - 1y \leq 0$

$$\frac{\sqrt{2}}{2}x - \frac{3}{2}y \leq -\frac{1}{2}$$

$$0x - 2y \leq -1$$

$$\frac{3}{2}x - \frac{\sqrt{2}}{2}y \leq \frac{\sqrt{2}}{2}$$

$$2x - 0y \leq 1$$

$$-\frac{3}{2}x - \frac{3}{2}y \leq -\frac{1}{2}$$

$$-1x - 1y \leq 0$$

$$-\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \leq \frac{\sqrt{2}}{2}$$

$$-2x - 2y \leq -1.$$

We solve this MOLBP problem by using the extended Kth-best approach and the method of weighting.

$$\min_{x \in X} F(x, y) = \left(3 + \frac{5-\sqrt{2}}{2}\right) x - 10y$$

$$\begin{aligned}
\text{subject to } & -1x + 3y \le 4 \\
& -\frac{3}{2}x + \frac{5}{2}y \le \frac{7}{2} \\
& -2x + 2y \le 3 \\
& -\frac{\sqrt{2}}{2}x + \frac{7}{2}y \le \frac{9}{2} \\
& 0x + 4y \le 5 \\
& \min_{y \in Y} f(x, y) = \left(\frac{5-\sqrt{2}}{2} + 3\right) x + \left(\frac{5-\sqrt{2}}{2} + 3\right) y \\
& \text{subject to } 1x - 1y \le 0 \\
& \qquad \frac{\sqrt{2}}{2}x - \frac{3}{2}y \le -\frac{1}{2} \\
& \qquad 0x - 2y \le -1 \\
& \qquad \frac{3}{2}x - \frac{\sqrt{2}}{2}y \le \frac{\sqrt{2}}{2} \\
& \qquad 2x - 0y \le 1 \\
& \qquad -\frac{3}{2}x - \frac{3}{2}y \le -\frac{1}{2} \\
& \qquad -1x - 1y \le 0 \\
& \qquad -\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \le \frac{\sqrt{2}}{2} \\
& \qquad -2x - 2y \le -1
\end{aligned}$$

The optimal solution occurs at the point $(x^*, y^*) = (0, 0.5)$ with

$$\min_{x \in X} (F_1(x, y))_1^{L(R)} = 1$$

$$\min_{x \in X} (F_1(x, y))_{\frac{1}{2}}^{L} = 0.75$$

$$\min_{x \in X} (F_1(x, y))_0^{L} = 0.5$$

$$\min_{x \in X} (F_1(x, y))_{\frac{1}{2}}^{R} = 1.25$$

$$\min_{x\in X}(F_1(x,y))_0^R = 1.5$$

$$\min_{x\in X}(F_2(x,y))_1^{L(R)} = -2$$

$$\min_{x\in X}(F_2(x,y))_{\frac{1}{2}}^L = -2.25$$

$$\min_{x\in X}(F_2(x,y))_0^L = -2.5$$

$$\min_{x\in X}(F_2(x,y))_{\frac{1}{2}}^L = -1.75$$

$$\min_{x\in X}(F_2(x,y))_0^R = -1.5$$

$$\min_{y\in Y}(f_1(x,y))_1^{L(R)} = -0.5$$

$$\min_{y\in Y}(f_1(x,y))_{\frac{1}{2}}^L = -0.75$$

$$\min_{y\in Y}(f_1(x,y))_0^L = -1$$

$$\min_{y\in Y}(f_1(x,y))_{\frac{1}{2}}^R = -\frac{\sqrt{2}}{4}$$

$$\min_{y\in Y}(f_1(x,y))_0^R = 0$$

$$\min_{y\in Y}(f_2(x,y))_1^{L(R)} = 1$$

$$\min_{y\in Y}(f_2(x,y))_{\frac{1}{2}}^L = 0.75$$

$$\min_{y\in Y}(f_2(x,y))_0^L = 0.5$$

$$\min_{y\in Y}(f_2(x,y))_{\frac{1}{2}}^R = 1.25$$

$$\min_{y\in Y}(f_2(x,y))_0^R = 1.5.$$

Step 10 When $x = 0$, $y = 0.5$, we have $\|(x, y)_{2^2} - (x, y)_{2^1}\| = 0 < \varepsilon$.

Step 11 The solution of the problem is $x = 0$, $y = 0.5$ such that

$$\min_{x\in X} F_1(x,y) = 0.5 \times \tilde{2}$$

$$\min_{x\in X} F_2(x,y) = -0.5 \times \tilde{4}$$

$$\min_{y\in Y} f_1(x,y) = 0.5 \times \tilde{2}$$

$$\min_{y\in Y} f_2(x,y) = -0.5 \times \tilde{1}.$$

5 CONCLUSION AND FURTHER STUDY

Following our previous research [29,31,41], this paper proposes a fuzzy number based approximate Kth-best approach to solve proposed FMOLBP problem. Two examples are given to illustrate how to establish a FMOLBP model and how to use the proposed approach. Further study will include the development of fuzzy multi-objective multi-follower bilevel programming problems.

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