Characterizing locally indistinguishable orthogonal product states

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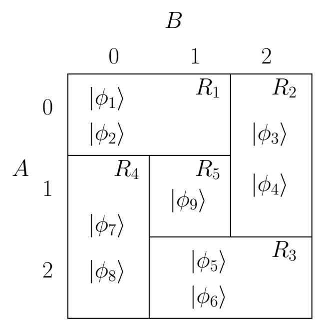
***Abstract*—Bennett et al. [C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, "Quantum nonlocality without entanglement", Physical Review A, vol. 59, no. 2, p. 1070, 1999] identified a set of orthogonal *product* states in the Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3$ such that reliably distinguishing those states requires non-local quantum operations. While more examples have been found for this counter-intuitive "nonlocality without entanglement" phenomenon, a complete and computationally verifiable characterization for all such sets of states remains unknown. In this paper, we give such a characterization for both $\mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.**

***Index Terms*—Nonlocality without entanglement, locally distinguishability, rectangular representation, locally unitarily equivalent.**



$$|i \pm j\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|i\rangle \pm |j\rangle), \quad 0 \le i < j \le 2$$
$$|\phi_{1,2}\rangle = |0\rangle|0 \pm 1\rangle$$
$$|\phi_{3,4}\rangle = |0 \pm 1\rangle|2\rangle$$
$$|\phi_{5,6}\rangle = |2\rangle|1 \pm 2\rangle$$
$$|\phi_{7,8}\rangle = |1 \pm 2\rangle|0\rangle$$
$$|\phi_9\rangle = |1\rangle|1\rangle$$

Fig. 1. The basis $\mathcal{B}_9$ for $\mathbb{C}^3 \otimes \mathbb{C}^3$ and its rectangular representation $(\mathcal{R}_9, \{|0\rangle, |1\rangle, |2\rangle\}, \{|0\rangle, |1\rangle, |2\rangle\}, U, V)$, where $\mathcal{R}_9 = \{R_i : 1 \le i \le 5\}$, V_{R_1}, U_{R_2}, V_{R_3}, and U_{R_4} are Hadamard and the other unitaries are Identities.

I. Introduction

A pure quantum state $|\phi\rangle_{AB}$ of a bipartite system AB is said to be entangled if it is not a product state, i.e., it cannot be represented as $|\alpha\rangle_A \otimes |\beta\rangle_B$, for some state $|\alpha\rangle_A$ and $|\beta\rangle_B$ of the system A and B, respectively. An entangled quantum state may generate measurement statistics that are inherently different from those generated by a classical process [1], [2]. This feature of entanglement is referred to as the nonlocality of quantum states. Dual to the notion of state nonlocality is the nonlocality of quantum operations. A natural definition of a local quantum operation on a multipartite quantum system is that of *Local Operations and Classical Communication (LOCC)* protocols, in which each party may apply to his system arbitrary quantum operations, while the inter-partite communication must be classical. It follows from the definition that if a quantum operation can be implemented by LOCC, it cannot create quantum entanglement. However, the reverse is false. That is, there exist quantum operations which cannot create entanglement and cannot be implemented by LOCC. This surprising fact was discovered by Bennett et al. [3] and was formulated as a problem of reliably distinguishing quantum states.

A set of state $\mathcal{E} = \{|\phi_i\rangle_{AB}\}_i$ is said to be *reliably distinguishable* by a quantum operation T if on each $|\phi_i\rangle_{AB}$, T outputs i with probability 1. The authors of [3] identified an orthonormal basis $\mathcal{B}_9$ for $\mathbb{C}^3 \times \mathbb{C}^3$, illustrated in Fig. 1, that cannot be reliably distinguished by LOCC. The important feature of the basis is that each base vector is a product state, thus the distinguishing operator cannot create entanglement.

The above property of nonlocal operations not necessarily creating entanglement is referred to as "nonlocality without entanglement", and has been studied by many authors subsequently [3]–[21]. A related discovery made by Horodecki et al. [22] is "more nonlocality with less entanglement" in the sense that sometimes reducing entanglement from the states to be distinguished can increase their indistinguishability. Formally, an *orthogonal product set (OPS)* is a set of multipartite product states that are pairwise orthogonal. An OPS that forms a basis is also called an *orthogonal product basis (OPB)*. Much effort has been devoted to searching for additional LOCC-indistinguishable OPSs. Besides $\mathcal{B}_9$, Ref. [3] also showed that $\mathcal{B}_9 - \{|1\rangle|1\rangle\}$ is not LOCC-distinguishable, either. All other known LOCC-indistinguishable OPSs belong to the following two classes.

Definition 1 ([4]): An *unextendable product basis (UPB)* is an OPS that is neither a complete basis nor a proper subset of any other OPS.

If $\mathcal{E}$ is an OPS in a multipartite Hilbert space $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$, then for each $1 \le i \le n$ denote by $\mathcal{E}_{A_i} = \{|\alpha_i\rangle \in \mathcal{H}_{A_i} : \exists |\alpha_1\rangle \in \mathcal{H}_{A_1}, \cdots, |\alpha_{i-1}\rangle \in \mathcal{H}_{A_{i-1}}, |\alpha_{i+1}\rangle \in \mathcal{H}_{A_{i+1}}, \cdots, |\alpha_n\rangle \in \mathcal{H}_{A_n},$ such that $|\alpha_1\rangle \cdots |\alpha_n\rangle \in \mathcal{E}\}$.

Definition 2 ([11]): An OPS $\mathcal{E}$ in $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_n}$ is *irreducible* if none of the set $\mathcal{E}_{A_i}$, $1 \le i \le n$, can be partitioned into two nonempty orthogonal subsets.

Theorem 3 ([4], [5], [11]): The following OPSs are LOCC-indistinguishable:

(1) An irreducible OPB ([11]).

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(2) A UPB ([4], [5]).

Ref. [11] indeed characterizes all LOCC-indistinguishable OPBs.

Theorem 4 ([11]): An OPB cannot be reliably distinguished by LOCC if and only if it contains an irreducible subset that spans a product space.

A direct corollary of Theorem 4 is that an OPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ or $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is LOCC-indistinguishable if and only if it is irreducible.

One of the main objectives of this line of research is to identify additional LOCC-indistinguishable OPSs, or more ambitiously, to give a complete and computationally verifiable characterization of all such OPSs. Clearly, any OPS in a $1 \otimes n$ system, $n \geq 1$, is LOCC-distinguishable. It was also known [3] that the same is true for any $2 \otimes n$ system, $n \geq 1$. Thus $\mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ are two of the smallest dimensional spaces where such a characterization was not known. In this paper, we obtain such characterizations for both spaces. Specifically, we show in Sec. II that when restricted to the $\mathbb{C}^3 \otimes \mathbb{C}^3$ space, the generalizations of $\mathcal{B}_9 - \{|1\rangle|1\rangle\}$, together with irreducible OPBs and UPBs, are the only possible LOCC-indistinguishable OPSs. A key step in the proof is to show that all irreducible OPBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$ must have a rectangular representation similar to that of $\mathcal{B}_9$. In Sec. III, we give a similar characterization of LOCC-indistinguishable OPSs in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ space by proving that all irreducible OPBs in this space must be locally unitarily equivalent to a particular basis.

We introduce some notions for the rest of the paper. For two vectors $|\alpha\rangle$ and $|\alpha'\rangle$, we write $|\alpha\rangle = |\alpha'\rangle$ if there exists a non-zero $c \in \mathbb{C}$ such that $|\alpha\rangle = c|\alpha'\rangle$. Two product states $|\alpha_1\rangle \cdots |\alpha_n\rangle$ and $|\alpha'_1\rangle \cdots |\alpha'_n\rangle$ in $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_n}$ are said to *align on A_i's component* if $|\alpha_i\rangle = |\alpha'_i\rangle$.

II. Characterization of locally indistinguishable OPS in $\mathbb{C}^3 \otimes \mathbb{C}^3$

To characterize all locally indistinguishable OPSs in $\mathbb{C}^3 \otimes \mathbb{C}^3$, we first generalize the set $\mathcal{B}_9 - \{|1\rangle|1\rangle\}$ to a broader class of OPSs having a similar structure.

Let $m, n \geq 1$ be integers. If $\mathcal{E}$ is an OPS in the $m \otimes n$ dimensional space and $|\mathcal{E}| = mn - 1$, then $\mathcal{E}$ can be extended to an OPB [5]. Denote by $\mathcal{E}^\perp$ the unique product state that extends $\mathcal{E}$ to a basis.

Lemma 5: Let $m, n \geq 1$ be integers. An OPS described below is LOCC-indistinguishable:

(3) An irreducible OPS $\mathcal{E}$ in $\mathbb{C}^m \otimes \mathbb{C}^n$ with $|\mathcal{E}| = mn - 1$ such that $\mathcal{E}^\perp$ does not align on either component with any element in $\mathcal{E}$.

Proof: Denote by $\mathcal{H}_A$ and $\mathcal{H}_B$ the state space of Alice and and Bob, respectively. Suppose $\mathcal{E} = \{|\alpha_i\rangle|\beta_i\rangle : 1 \leq i \leq mn-1\}$ and $\mathcal{E}^\perp = |\alpha_0\rangle|\beta_0\rangle$. Suppose that $\mathcal{E}$ can be reliably distinguished by an LOCC protocol. Fix such a protocol $\mathcal{P}$ that takes the smallest number of rounds of communication. Without loss of generality, assume that Alice sends the first message, which is the measurement outcome k of a Positive-Operator-Valued Measure (POVM)

$$\mathcal{M} \stackrel{\text{def}}{=} \{M_k : \mathcal{H}_A \to \mathcal{H}'_A\}_k,$$

where $\mathcal{H}'_A$ is Alice's state space after applying $\mathcal{M}$ and the operators M_k satisfy

$$\sum_k M_k^\dagger M_k = I_{\mathcal{H}_A}.$$

If for each k, there exists $\mu_k > 0$ such that $M_k^\dagger M_k = \mu_k I_{\mathcal{H}_A}$, then $\sum_k \mu_k = 1$ and each M_k is an isometric embedding. Thus $\mathcal{M}$ can be implemented by having Bob send the message instead: he generates a random number k with probability μ_k, sends it to Alice, who applies M_k to $\mathcal{H}_A$. This contradicts the assumption that $\mathcal{P}$ takes the smallest number of rounds. Therefore, there exists a k such that $M_k^\dagger M_k$ has $k_0 \geq 2$ number of distinct eigenvalues. Fix such a k for the rest of the proof.

Since the post-measurement states must remain orthogonal so that they can be reliably distinguished by the remaining steps of $\mathcal{P}$, we have

$$\langle\alpha_i|\langle\beta_i|(M_k^\dagger M_k \otimes I_{\mathcal{H}_B})|\alpha_j\rangle|\beta_j\rangle = 0$$

for all $1 \leq i < j \leq mn-1$. Note that $\mathcal{E}' \stackrel{\text{def}}{=} \mathcal{E} \cup \{\mathcal{E}^\perp\}$ is an OPB, thus for each i, $1 \leq i \leq mn-1$, there exist $\lambda_i, \lambda_i^0 \in \mathbb{C}$, such that

$$(M_k^\dagger M_k \otimes I_{\mathcal{H}_B})|\alpha_i\rangle|\beta_i\rangle = \lambda_i|\alpha_i\rangle|\beta_i\rangle + \lambda_i^0|\alpha_0\rangle|\beta_0\rangle.$$

Applying $\langle\alpha_0| \otimes I_{\mathcal{H}_B}$ on both components, we have

$$\langle\alpha_0|M_k^\dagger M_k|\alpha_i\rangle|\beta_i\rangle = \lambda_i\langle\alpha_0|\alpha_i\rangle|\beta_i\rangle + \lambda_i^0|\beta_0\rangle.$$

It follows that $\lambda_i^0 = 0$, since $|\beta_i\rangle \neq |\beta_0\rangle$. Therefore, $\mathcal{E}_A$ is a set of eigenstates of $M_k^\dagger M_k$.

If $\mathcal{E}_A$ does not span $\mathcal{H}_A$, let $|\alpha\rangle \in \mathcal{H}_A$ be a state orthogonal to $\mathrm{span}(\mathcal{E}_A)$. Let $|\beta\rangle \in \mathcal{H}_B$ be orthogonal to $|\beta_0\rangle$. Such $|\beta\rangle$ must exist since otherwise $\dim(\mathcal{H}_B) = 1$, and $\mathcal{E}$ would be reducible. Then $|\alpha\rangle|\beta\rangle$ is orthogonal to $\mathcal{E}'$, a contradiction to $\mathcal{E}'$ being a basis for $\mathcal{H}_A \otimes \mathcal{H}_B$. Therefore, $\mathcal{E}_A$ spans $\mathcal{H}_A$, and is a complete spectrum of $M_k^\dagger M_k$. It follows that $\mathcal{E}_A$ can be partitioned into k_0 number of pair-wise orthogonal subsets, each of which corresponds to a distinct eigenvalue of $M_k^\dagger M_k$. Since $k_0 \geq 2$, this contradicts the assumption that $\mathcal{E}$ is irreducible. Therefore, $\mathcal{E}$ is LOCC-indistinguishable. ■

As mentioned above, the $\mathbb{C}^3 \otimes \mathbb{C}^3$ space is one of the smallest spaces having LOCC-indistinguishable OPSs. We also know the following useful facts.

Proposition 6 ([5]): An OPS $\mathcal{E}$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$ is LOCC-distinguishable if $|\mathcal{E}| \leq 4$.

Theorem 7 ([4], [5]): Any UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ must have exactly 5 elements.

In what follows, we completely characterize all LOCC-indistinguishable OPSs in $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Theorem 8 (Main Theorem of Sec. II): An OPS in $\mathbb{C}^3 \otimes \mathbb{C}^3$ is LOCC-indistinguishable if and only if it belongs to one of the three classes (1), (2), and (3).

Combining the above three results, an LOCC-indistinguishable OPS in $\mathbb{C}^3 \otimes \mathbb{C}^3$ must have precisely

5, 8, or 9 elements, each of which corresponds to belong to the classes (2), (3) and (1), respectively. Whether or not an OPS is irreducible can be checked from the pairwise inner products of the state components. The same information can be used to determine if an OPS is an UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ [4], [5]. Therefore, whether or not an OPS belongs to (1), (2), or (3) can be determined computationally.

To prove Main Theorem 8, we first generalize the rectangular representation for $\mathcal{B}_9$ and derive some useful properties of the generalization. Let I and J be two sets. A subset $R \subseteq I \times J$ is a *rectangle* if $R = A \times B$ for some $A \subseteq I$ and $B \subseteq J$. If $R = A \times B$, denote by $I(R) \stackrel{\text{def}}{=} A$ and $J(R) \stackrel{\text{def}}{=} B$. A *rectangular decomposition* of $I \times J$ is a partition of $I \times J$ into rectangles. Fig. 1 illustrates a rectangular decomposition for $\{0, 1, 2\} \times \{0, 1, 2\}$. We refer to this decomposition as $\mathcal{R}_9$ and use the labeling scheme in the Figure for its elements.

Definition 9: Let $m, n \geq 1$ be integers, $I \stackrel{\text{def}}{=} \{0, 1, \cdots, n-1\}$, and $J \stackrel{\text{def}}{=} \{0, 1, \cdots, m-1\}$. Let $\mathcal{E}$ be an OPB of a product space $\mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim(\mathcal{H}_A) = n$ and $\dim(\mathcal{H}_B) = m$. A *rectangular representation* of $\mathcal{E}$ is a quintuple $(\mathcal{R}, \alpha, \beta, U, V)$ such that:

(a) $\mathcal{R}$ is a rectangular decomposition of $I \times J$.
(b) $\alpha = \{|\alpha_0\rangle, |\alpha_1\rangle, \cdots, |\alpha_{n-1}\rangle\}$ is an orthonormal basis for $\mathcal{H}_A$, and similarly, $\beta = \{|\beta_0\rangle, |\beta_1\rangle, \cdots, |\beta_{m-1}\rangle\}$ is an orthonormal basis for $\mathcal{H}_B$.
(c) U assigns each $R \in \mathcal{R}$ a unitary operator U_R on $\mathrm{span}\{|\alpha_i\rangle : i \in I(R)\}$, and similarly, V_R a unitary operator on $\mathrm{span}\{|\beta_j\rangle : j \in J(R)\}$.
(d) $\mathcal{E} = \{(U_R|\alpha_i\rangle) \otimes (V_R|\beta_j\rangle) : R \in \mathcal{R}, (i, j) \in R\}$.

It can be verified by direct inspection from Fig. 1 that $\mathcal{B}_9$ has a rectangular representation of which the rectangular decomposition is $\mathcal{R}_9$ and the unitary transformations are either Identity operators or Hadamard. Removing any state other than $|1\rangle|1\rangle$ from $\mathcal{B}_9$ results in an LOCC-distinguishable set. The same is true for any OPB having a rectangular representation using $\mathcal{R}_9$.

Proposition 10: Let $\mathcal{E}$ be an OPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ having a rectangular representation $(\mathcal{R}_9, \alpha, \beta, U, V)$. Suppose $|\alpha_1\rangle|\beta_1\rangle \in \mathcal{B}$ is the state corresponding to the 1×1 rectangle. Then any OPS obtained from $\mathcal{E}$ by removing some state other than $|\alpha_1\rangle|\beta_1\rangle$ is LOCC-distinguishable.

Proof: We denote the states in $\mathcal{E}$ by $\{|\phi_i\rangle : 1 \leq i \leq 9\}$ using the labeling scheme in Fig. 1. Without loss of generality, assume that $|\phi_1\rangle$ is the only state in $\mathcal{E}$ missing in $\mathcal{E}'$. By direct inspection, the following LOCC protocol identifies an unknown input state from $\mathcal{E}'$. Bob starts the protocol by measuring

$$\{|\beta_0\rangle\langle\beta_0|,\ I - |\beta_0\rangle\langle\beta_0|\}.$$

If the measurement outcome corresponds to the first operator, Alice measures

$$\left\{|\alpha_0\rangle\langle\alpha_0|,\ U_{R_4}|\alpha_1\rangle\langle\alpha_1|U_{R_4}^\dagger,\ U_{R_4}|\alpha_2\rangle\langle\alpha_2|U_{R_4}^\dagger\right\},$$

concluding that the input state is $|\phi_2\rangle$, $|\phi_7\rangle$, or $|\phi_8\rangle$ accordingly. In the other case, the protocol continues using a similar strategy. ∎

We now present our Main Lemma of this section, which characterizes irreducible OPBs (thus LOCC-indistinguishable OPBs) in terms of rectangular representations.

Lemma 11 (Main Lemma of Sec. II): Any irreducible OPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ has a rectangular representation using $\mathcal{R}_9$.

Proof: Let $\mathcal{E} = \{|\alpha_i\rangle|\beta_i\rangle : 1 \leq i \leq 9\}$ be an irreducible OPB in the $3 \otimes 3$ dimensional space $\mathcal{H}_A \otimes \mathcal{H}_B$. We will construct a rectangular representation

$$P = (\mathcal{R}_9, \{|0\rangle_A, |1\rangle_A, |2\rangle_A\}, \{|0\rangle_B, |1\rangle_B, |2\rangle_B\}, U, V)$$

for $\mathcal{E}$. For the sake of simplicity, in the following when $|\alpha_i\rangle = |\alpha_j\rangle$, we denote the state by $|\alpha_{i,j}\rangle$.

We first note that there exist two states $|\alpha_1\rangle|\beta_1\rangle$ and $|\alpha_2\rangle|\beta_2\rangle$ in $\mathcal{E}$ that are aligned on one component. (In fact, we can prove that in the $\mathbb{C}^3 \otimes \mathbb{C}^3$ space, there are at most 5 orthogonal product states such that no pair of them align on either component.) Assume that $|\alpha_1\rangle = |\alpha_2\rangle = |\alpha_{1,2}\rangle$; the other case would lead to the same conclusion. Then $|\beta_1\rangle \perp |\beta_2\rangle$. If there are 6 states whose component in $\mathcal{H}_A$ is orthogonal to $|\alpha_{1,2}\rangle$, then they must span $(\mathrm{span}\{|\alpha_{1,2}\rangle\})^\perp \otimes \mathcal{H}_B$, contradicting the assumption that $\mathcal{E}$ is irreducible. Thus there are $|\alpha_3\rangle, |\alpha_4\rangle \in \mathcal{E}_A$ with $\langle\alpha_{1,2}|\alpha_3\rangle \neq 0$ and $\langle\alpha_{1,2}|\alpha_4\rangle \neq 0$. This implies

$$\begin{aligned} |\beta_3\rangle &\perp \mathrm{span}\{|\beta_1\rangle, |\beta_2\rangle\}, \\ |\beta_4\rangle &\perp \mathrm{span}\{|\beta_1\rangle, |\beta_2\rangle\} \end{aligned}$$

and then $|\beta_3\rangle = |\beta_4\rangle = |\beta_{3,4}\rangle$.

Repeating the above argument, we find in $\mathcal{E}$ pairs of states $\{|\alpha_{5,6}\rangle|\beta_5\rangle, |\alpha_{5,6}\rangle|\beta_6\rangle\}$ and $\{|\alpha_7\rangle|\beta_{7,8}\rangle, |\alpha_8\rangle|\beta_{7,8}\rangle\}$ where $|\beta_5\rangle \perp |\beta_6\rangle$ and $|\alpha_7\rangle \perp |\alpha_8\rangle$. By direct inspection, $|\alpha_i\rangle|\beta_i\rangle$, $1 \leq i \leq 8$, must be distinct. Denote the remaining state in $\mathcal{E}$ by $|\alpha_9\rangle|\beta_9\rangle$.

Let

$$S_A \stackrel{\text{def}}{=} \{|\alpha_{1,2}\rangle, |\alpha_9\rangle, |\alpha_{5,6}\rangle\}.$$

We show that S_A is an orthonormal basis for $\mathcal{H}_A$. If $|\beta_9\rangle = |\beta_{3,4}\rangle$, then the set $\{|\alpha_3\rangle|\beta_{3,4}\rangle, |\alpha_4\rangle|\beta_{3,4}\rangle, |\alpha_9\rangle|\beta_9\rangle\}$ would span $\mathcal{H}_A \otimes \mathrm{span}\{|\beta_{3,4}\rangle\}$, contradicting $\mathcal{E}$ being irreducible. Thus $|\beta_9\rangle \neq |\beta_{3,4}\rangle$, implying that for some $i \in \{1, 2\}$, $\langle\beta_i|\beta_9\rangle \neq 0$. Thus $|\alpha_9\rangle \perp |\alpha_{1,2}\rangle$. Similarly, $|\alpha_9\rangle \perp |\alpha_{5,6}\rangle$. If $|\alpha_{1,2}\rangle \not\perp |\alpha_{5,6}\rangle$, then the states in $\{|\beta_i\rangle : i = 1, 2, 5, 6\}$ would be mutually orthogonal, contradicting $\dim(\mathcal{H}_B) = 3$. Thus $|\alpha_{1,2}\rangle \perp |\alpha_{5,6}\rangle$. Therefore, S_A is an orthonormal basis for $\mathcal{H}_A$. Similarly,

$$S_B \stackrel{\text{def}}{=} \{|\beta_{7,8}\rangle, |\beta_9\rangle, |\beta_{3,4}\rangle\}$$

is orthonormal in $\mathcal{H}_B$. Relabel S_A as $\{|i\rangle_A : 0 \leq i \leq 2\}$ and S_B as $\{|j\rangle_B : 0 \leq j \leq 2\}$ such that $|0\rangle_A = |\alpha_{1,2}\rangle$, $|0\rangle_B = |\beta_{7,8}\rangle$, etc.

Define the following unitaries as the Identity operator on the corresponding dimension 1 space: U_{R_1}, V_{R_2}, U_{R_3}, V_{R_4}, U_{R_5}, and V_{R_5}. Define

$$\begin{aligned} V_{R_1} &\stackrel{\text{def}}{=} |\beta_1\rangle\langle 0| + |\beta_2\rangle\langle 1|, & U_{R_2} &\stackrel{\text{def}}{=} |\alpha_3\rangle\langle 0| + |\alpha_4\rangle\langle 1|, \\ V_{R_3} &\stackrel{\text{def}}{=} |\beta_5\rangle\langle 1| + |\beta_6\rangle\langle 2|, & U_{R_4} &\stackrel{\text{def}}{=} |\alpha_7\rangle\langle 1| + |\alpha_8\rangle\langle 2|. \end{aligned}$$

This completes the construction of P. By direct inspection, P is a rectangular representation of $\mathcal{E}$. ∎

We are now ready to prove Main Theorem 8.

Proof of Theorem 8. Since the "if" direction is precisely the combination of Theorem 3 and Lemma 5, we need only to prove the "only if" direction. Suppose there exists an LOCC-indistinguishable OPS $\mathcal{E}$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$ not belonging to any of (1), (2), and (3). Then by Proposition 6 and the corollary of Theorems 4, we have $5 \leq |\mathcal{E}| \leq 8$. Furthermore, from Theorem 7, $\mathcal{E}$ is extensible to an OPB $\mathcal{E}'$. Since $\mathcal{E}'$ must be LOCC-indistinguishable (and thus irreducible), it has a rectangular representation using $\mathcal{R}_9$, by Lemma 11. Since $\mathcal{E}$ does not belong to Class (3), there exists a state $|\alpha\rangle|\beta\rangle$ in $\mathcal{E}' - \mathcal{E}$ not contained in the rectangle R_5. Thus $\mathcal{E}' - \{|\alpha\rangle|\beta\rangle\}$ is LOCC-distinguishable, by Proposition 10. So must be $\mathcal{E}$ since $\mathcal{E} \subseteq \mathcal{E}' - \{|\alpha\rangle|\beta\rangle\}$, which is a contradiction. Thus any LOCC-indistinguishable OPS must belong to (1), (2), or (3). □

Our method can also be used to give an alternative proof for the fact that there is no LOCC-indistinguishable OPSs in $2 \otimes n$ spaces observed in Ref. [3]. It remains an open problem to extend our result to the complete collection of LOCC-indistinguishable OPSs in spaces of a dimension higher than $3 \otimes 3$. To this end, it may be difficult to extend our technique as the rectangular representation lemma is not true for all dimensions. For example, for any θ, $0 < \theta < \pi/2$ and $\theta \neq \pi/4$, one can show that the following OPB in the $2 \otimes 4$ dimensional space does not have a rectangular representation:

$$\begin{aligned}
|\psi_1\rangle &= |0\rangle \otimes |0+1\rangle, \\
|\psi_2\rangle &= |0\rangle \otimes |0-1\rangle, \\
|\psi_3\rangle &= |1\rangle \otimes (\cos\theta|0\rangle + \sin\theta|1\rangle), \\
|\psi_4\rangle &= |1\rangle \otimes (\sin\theta|0\rangle - \cos\theta|1\rangle), \\
|\psi_5\rangle &= |0+1\rangle \otimes |2+3\rangle, \\
|\psi_6\rangle &= |0+1\rangle \otimes |2-3\rangle, \\
|\psi_7\rangle &= |0-1\rangle \otimes (\cos\theta|2\rangle + \sin\theta|3\rangle) \\
|\psi_8\rangle &= |0-1\rangle \otimes (\sin\theta|2\rangle - \cos\theta|3\rangle).
\end{aligned}$$

One may generalize the notion of rectangular representations through a recursive definition. Unfortunately, there also exist OPBs that do not admit such a generalized rectangular representation. We note that an even more general concept is that of *unwindability*, defined by DiVincenzo and Terhal [23]. Therefore, a deeper understanding of unwindable OPSs may lead to a better understanding of LOCC-indistinguishable OPSs in higher dimensions.

III. Characterization of locally indistinguishable OPS in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

This section is devoted to a complete characterization of LOCC-indistinguishable OPS in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ space. Known facts parallel to Proposition 6 and Theorem 7 in Sec. II are:

Proposition 12 ([5]): An OPS $\mathcal{E}$ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is LOCC-distinguishable if $|\mathcal{E}| \leq 3$.

Theorem 13 ([24]): Any UPB $\mathcal{E}$ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ has exactly 4 elements.

Let $\mathcal{B}_8 = \{|\psi_i\rangle : 1 \leq i \leq 8\}$ be an (irreducible) OPB in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ where

$$\begin{aligned}
|\psi_1\rangle &= |0\rangle \otimes |0\rangle \otimes |\gamma\rangle, \\
|\psi_2\rangle &= |0\rangle \otimes |0\rangle \otimes |\gamma^\perp\rangle, \\
|\psi_3\rangle &= |1\rangle \otimes |0\rangle \otimes |0\rangle, \\
|\psi_4\rangle &= |1\rangle \otimes |\beta\rangle \otimes |1\rangle, \\
|\psi_5\rangle &= |1\rangle \otimes |\beta^\perp\rangle \otimes |1\rangle, \\
|\psi_6\rangle &= |0\rangle \otimes |1\rangle \otimes |1\rangle, \\
|\psi_7\rangle &= |\alpha\rangle \otimes |1\rangle \otimes |0\rangle, \\
|\psi_8\rangle &= |\alpha^\perp\rangle \otimes |1\rangle \otimes |0\rangle,
\end{aligned}$$

and $|x\rangle \neq |0\rangle, |1\rangle$ for $x = \alpha, \beta, \gamma$. For any state $|x\rangle$ in a two dimensional Hilbert space, denote by $|x^\perp\rangle$ the unique state which is orthogonal to $|x\rangle$. Let

$$\mathcal{B}_6 = \{|\psi_i\rangle, i = 1, 2, 4, 5, 7, 8\}.$$

Proposition 14: A subset of $\mathcal{B}_8$ is LOCC-indistinguishable if and only if it contains $\mathcal{B}_6$.

Proof: For the "if" direction, we prove by contradiction that $\mathcal{B}_6$ is LOCC-indistinguishable. Suppose that $\mathcal{B}_6$ can be distinguished by an LOCC protocol. Without loss of generality, assume that the first step in the protocol is for Alice to apply a non-destructive measurement with the measurement elements M_0 and M_1, such that $M_0^\dagger M_0 \neq 0$ and $M_1^\dagger M_1 \neq 0$. Since $|\psi_7\rangle$ and $|\psi_8\rangle$ overlap on both Bob and Carol's components, $|\alpha\rangle$ and $|\alpha^\perp\rangle$ must remain orthogonal after the measurement. Therefore

$$\{M_0^\dagger M_0, M_1^\dagger M_1\} = \{|\alpha\rangle\langle\alpha|, |\alpha^\perp\rangle\langle\alpha^\perp|\}.$$

Since $|\alpha\rangle, |\beta\rangle, |\gamma\rangle \neq |0\rangle, |1\rangle$, $\langle 0|M_0^\dagger M_0|1\rangle \neq 0$, $\langle 0|\beta\rangle\langle\gamma|1\rangle \neq 0$. Thus $\langle\psi_1|M_0^\dagger M_0|\psi_4\rangle \neq 0$, contradicting to the assumption that $|\psi_1\rangle$ and $|\psi_4\rangle$ are be perfectly distinguished at the end of the protocol. Therefore $\mathcal{B}_6$ is LOCC-indistinguishable.

To prove the "only if" direction, we give an LOCC protocol to distinguish $\{|\psi_i\rangle : 2 \leq i \leq 8\}$ as follows; other cases are similar. Carol first performs a projective measurement according to the computational basis $\{|0\rangle, |1\rangle\}$ and broadcasts the measurement outcome to Alice and Bob. If the outcome corresponding to $|0\rangle$ is observed, then Alice and Bob know that the state they share is among the set

$$\{|0\rangle|0\rangle, |1\rangle|0\rangle, |\alpha\rangle|1\rangle, |\alpha^\perp\rangle|1\rangle\}$$

which can be further distinguished by LOCC between them. Similarly, if the outcome of $|1\rangle$ is observed, Alice and Bob can also determine the state by LOCC. ■

In what follows, we completely characterize all LOCC-indistinguishable OPSs in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Two sets of multipartite states $\mathcal{E}$ and $\mathcal{E}'$ in Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ are said to be locally unitarily equivalent if there exist unitary operators U_A, U_B, and U_C acting on $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$, respectively, such that

$$\mathcal{E}' = \{U_A|\alpha\rangle \otimes U_B|\beta\rangle \otimes U_C|\gamma\rangle : |\alpha\rangle|\beta\rangle|\gamma\rangle \in \mathcal{E}\}.$$

Note that locally unitarily equivalent sets have the same LOCC-distinguishability.

Theorem 15 (Main Theorem of Sec. III): An OPS in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is LOCC-indistinguishable if and only if it is a UPB or locally unitarily equivalent to a subset of $\mathcal{B}_8$ containing $\mathcal{B}_6$.

To prove this Main Theorem, we present first a lemma.

Lemma 16: Let $\mathcal{E}$ be a $2\otimes 2\otimes 2$ dimensional OPS in which any two states align on at most one component. Then $|\mathcal{E}| \leq 5$.

Proof: First fix a state, say, $|\psi_0\rangle = |\alpha_0\rangle|\beta_0\rangle|\gamma_0\rangle \in \mathcal{E}$. Let

$$\begin{aligned}\mathcal{E}_1 &= \{|\alpha\rangle|\beta\rangle|\gamma\rangle \in \mathcal{E} : |\alpha\rangle = |\alpha_0^\perp\rangle\},\\ \mathcal{E}_2 &= \{|\alpha\rangle|\beta\rangle|\gamma\rangle \in \mathcal{E} : |\beta\rangle = |\beta_0^\perp\rangle\},\\ \mathcal{E}_3 &= \{|\alpha\rangle|\beta\rangle|\gamma\rangle \in \mathcal{E} : |\gamma\rangle = |\gamma_0^\perp\rangle\}.\end{aligned}$$

Then $\mathcal{E} = \{|\psi_0\rangle\} \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$. Without loss of generality, assume $|\mathcal{E}_1| \geq |\mathcal{E}_2|, |\mathcal{E}_3|$. We claim that $|\mathcal{E}_1| \leq 2$. Otherwise $|\mathcal{E}_1'| \geq 3$ where

$$\mathcal{E}_1' = \{|\beta\rangle|\gamma\rangle : |\alpha_0^\perp\rangle|\beta\rangle|\gamma\rangle \in \mathcal{E}_1\}.$$

From orthogonality, we can easily show that there are two states in $\mathcal{E}_1'$ which align on Bob's or Carol's component, so the corresponding states in $\mathcal{E}$ align on at least two components, contradicting the assumption of $\mathcal{E}$. Furthermore, if $|\mathcal{E}_1| \leq 1$ then we are done since $|\mathcal{E}| \leq 4$. So we need only consider the case when $|\mathcal{E}_1| = 2$. Let us assume

$$\mathcal{E}_1 = \{|\alpha_0^\perp\rangle|\beta_1\rangle|\gamma_1\rangle, |\alpha_0^\perp\rangle|\beta_1^\perp\rangle|\gamma_2\rangle\}$$

where $|\gamma_1\rangle \neq |\gamma_2\rangle$. There are two cases to consider.

Case 1. $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$. In this case, $|\beta_0\rangle \neq |\beta_1\rangle, |\beta_1^\perp\rangle$. Then for any $|\alpha\rangle|\beta_0^\perp\rangle|\gamma\rangle \in \mathcal{E}_2$, we have $|\alpha\rangle = |\alpha_0\rangle$ since $|\gamma\rangle \not\perp |\gamma_i\rangle$ must hold for either $i = 1$ or 2. Thus $|\mathcal{E}_2| \leq 1$. If $\mathcal{E}_2 = \emptyset$ then we are done. Otherwise let $\mathcal{E}_2 = \{|\alpha_0\rangle|\beta_0^\perp\rangle|\gamma_3\rangle\}$ where $|\gamma_3\rangle \neq |\gamma_0\rangle$. Then for any $|\alpha\rangle|\beta\rangle|\gamma_0^\perp\rangle \in \mathcal{E}_3 - \mathcal{E}_1$, from

$$|\alpha\rangle|\beta\rangle|\gamma_0^\perp\rangle \perp |\alpha_0\rangle|\beta_0^\perp\rangle|\gamma_3\rangle$$

we have $|\beta\rangle = |\beta_0\rangle$ since $|\alpha\rangle \neq |\alpha_0^\perp\rangle$. Thus $|\mathcal{E}_3 - \mathcal{E}_1| \leq 1$, and then $|\mathcal{E}| \leq 5$.

Case 2. $\mathcal{E}_1 \cap \mathcal{E}_2 \neq \emptyset$. In this case $|\beta_0\rangle = |\beta_1\rangle$ or $|\beta_1^\perp\rangle$. Let us assume the former case. Then $|\gamma_0\rangle \neq |\gamma_1\rangle$ and $|\mathcal{E}_2 - \mathcal{E}_1| \leq 1$. Furthermore, for any $|\alpha\rangle|\beta\rangle|\gamma_0^\perp\rangle \in \mathcal{E}_3 - \mathcal{E}_2$, from

$$|\alpha\rangle|\beta\rangle|\gamma_0^\perp\rangle \perp |\alpha_0^\perp\rangle|\beta_0\rangle|\gamma_1\rangle$$

we have $|\alpha\rangle = |\alpha_0\rangle$ since $|\beta\rangle \neq |\beta_0^\perp\rangle$. Thus $|\mathcal{E}_3 - \mathcal{E}_2| \leq 1$, and then $|\mathcal{E}| \leq 5$. ■

The next lemma characterizes all irreducible OPBs (thus LOCC-indistinguishable OPBs) in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ space.

Lemma 17 (Main Lemma of Sec. III): Any irreducible OPB in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ space is locally unitarily equivalent to $\mathcal{B}_8$.

Proof: Let $\mathcal{E} = \{|\psi_i\rangle = |\alpha_i\rangle|\beta_i\rangle|\gamma_i\rangle : i = 1, \ldots, 8\}$ be an irreducible OPB in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. First, from Lemma 16, we assume $|\psi_1\rangle = |0\rangle|0\rangle|\gamma\rangle$ and $|\psi_2\rangle = |0\rangle|0\rangle|\gamma^\perp\rangle$ (We can always make such an assumption because of the local unitary equivalence). Let

$$\begin{aligned}\mathcal{E}_1 &= \{|\psi_i\rangle \in \mathcal{E} : |\alpha_i\rangle = |1\rangle\},\\ \mathcal{E}_2 &= \{|\psi_i\rangle \in \mathcal{E} : |\beta_i\rangle = |1\rangle\}.\end{aligned}$$

Then $\mathcal{E}_1 \cup \mathcal{E}_2 = \{|\psi_i\rangle : i = 3, \ldots, 8\}$. Assume $|\mathcal{E}_1| \geq |\mathcal{E}_2|$. We have $|\mathcal{E}_1| \leq 4$ from the constraint of dimension. Furthermore, if $|\mathcal{E}_1| = 4$ then for any $|\psi_i\rangle \in \mathcal{E}_2 - \mathcal{E}_1$, $|\alpha_i\rangle = |0\rangle$ by $|\psi_i\rangle \perp \mathcal{E}_1$, contradicting $\mathcal{E}$ being irreducible. Then we have $|\mathcal{E}_1| = 3$, and so $|\mathcal{E}_2| = 3$ and $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$.

Next we will show that there are exactly 1 state in $\mathcal{E}_2$ having $|0\rangle$ on Alice's component. Let

$$\mathcal{E}_2' = \{|\psi_i\rangle \in \mathcal{E}_2 : |\alpha_i\rangle = |0\rangle\}.$$

Then $|\mathcal{E}_2'| < 3$ from the assumption that $\mathcal{E}$ is irreducible. If $|\mathcal{E}_2'| = 0$, then for any $|\alpha\rangle|\beta\rangle|\gamma\rangle \in \mathcal{E}_2$, $|\alpha\rangle \neq |0\rangle$, so $|\beta\rangle|\gamma\rangle$ must be the unique state orthogonal to the set $\{|\beta_i\rangle|\gamma_i\rangle : |\psi_i\rangle \in \mathcal{E}_1\}$. So $|\mathcal{E}_2| \leq 2$, a contradiction. Furthermore, if $|\mathcal{E}_2'| = 2$ and let $\mathcal{E}_2' = \{|0\rangle|1\rangle|\gamma_i\rangle : i = 1, 2\}$ where $|\gamma_1\rangle \perp |\gamma_2\rangle$, then for the state $|\psi_k\rangle \in \mathcal{E}_2 - \mathcal{E}_2'$, $|\gamma_k\rangle$ should be simultaneously orthogonal to $|\gamma_1\rangle$ and $|\gamma_2\rangle$ (note that $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$), which is impossible. So we conclude that $|\mathcal{E}_2'| = 1$. Similarly, $|\mathcal{E}_1'| = 1$ where $\mathcal{E}_1' = \{|\psi_i\rangle \in \mathcal{E}_1 : |\beta_i\rangle = |0\rangle\}$.

Summarizing all the conditions derived above, we can assume that

$$\begin{aligned}|\psi_1\rangle &= |0\rangle \otimes |0\rangle \otimes |\gamma\rangle,\\ |\psi_2\rangle &= |0\rangle \otimes |0\rangle \otimes |\gamma^\perp\rangle,\\ |\psi_3\rangle &= |1\rangle \otimes |0\rangle \otimes |0\rangle,\\ |\psi_4\rangle &= |1\rangle \otimes |\beta_4\rangle \otimes |\gamma_4\rangle,\\ |\psi_5\rangle &= |1\rangle \otimes |\beta_5\rangle \otimes |\gamma_5\rangle,\\ |\psi_6\rangle &= |0\rangle \otimes |1\rangle \otimes |\gamma_6\rangle,\\ |\psi_7\rangle &= |\alpha_7\rangle \otimes |1\rangle \otimes |\gamma_7\rangle,\\ |\psi_8\rangle &= |\alpha_8\rangle \otimes |1\rangle \otimes |\gamma_8\rangle,\end{aligned}$$

where none of $|\alpha_7\rangle, |\alpha_8\rangle, |\beta_4\rangle, |\beta_5\rangle$ equals $|0\rangle$ or $|1\rangle$. Then we have $|\gamma_4\rangle = |\gamma_5\rangle = |1\rangle$ from the fact that

$$|\psi_3\rangle \perp \{|\psi_4\rangle, |\psi_5\rangle\},$$

and then $|\beta_4\rangle \perp |\beta_5\rangle$. Similarly, we can prove that $|\gamma_7\rangle = |\gamma_8\rangle = |\gamma_6^\perp\rangle$ and $|\alpha_7\rangle \perp |\alpha_8\rangle$. Furthermore, we find that $|\gamma_8\rangle = |0\rangle$ from the orthogonality of $|\psi_4\rangle$ and $|\psi_8\rangle$.

Let $|\alpha_7\rangle = |\alpha\rangle$ and $|\beta_4\rangle = |\beta\rangle$. Notice that the irreducibility of $\mathcal{E}$ implies $|x\rangle \neq |0\rangle, |1\rangle$ for $x = \alpha, \beta, \gamma$. So $\mathcal{E}$ is locally unitarily equivalent to $\mathcal{B}_8$. ■

We are now ready to prove Main Theorem 15.

Proof of Theorem 15. The "if" part is precisely the combination of Theorem 3 (2) and Proposition 14, and the simple fact that locally unitarily equivalent sets have the same LOCC-distinguishability, so we need only to prove the "only if" part.

Suppose there exists an LOCC-indistinguishable OPS $\mathcal{E}$ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ which is neither a UPB nor locally unitarily equivalent to a subset of $\mathcal{B}_8$ containing $\mathcal{B}_6$. Then by Proposition 12 and the corollary of Theorems 4, we have $4 \leq |\mathcal{E}| \leq 7$. Furthermore, from Theorem 13, $\mathcal{E}$ can be extended to an OPB $\mathcal{E}'$. Since $\mathcal{E}'$ must be LOCC-indistinguishable (and thus irreducible), it is locally unitarily equivalent to $\mathcal{B}_8$, by Lemma 17. Let $U_A \otimes U_B \otimes U_C$ be the local unitary transformation which relates $\mathcal{E}'$ to $\mathcal{B}_8$. Then there exists a state $|\alpha\rangle|\beta\rangle|\gamma\rangle$ in $\mathcal{E}' - \mathcal{E}$ such that $U_A|\alpha\rangle \otimes U_B|\beta\rangle \otimes U_C|\gamma\rangle$ is neither $|\psi_3\rangle$ nor $|\psi_6\rangle$. Thus $\mathcal{E}' - \{|\alpha\rangle|\beta\rangle|\gamma\rangle\}$ is LOCC-distinguishable, by Proposition 14. So must be $\mathcal{E}$ since $\mathcal{E} \subseteq \mathcal{E}' - \{|\alpha\rangle|\beta\rangle|\gamma\rangle\}$, which is a contradiction. □

Combining Theorems 13 and 15, an LOCC-indistinguishable OPS in the $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ space must have precisely 4, 6, 7, or 8 elements. Similar to the argument presented in Sec. II, whether or not an OPS in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is LOCC-indistinguishable can also be determined computationally.

IV. Conclusion

We present in this paper a complete and computationally verifiable characterization of all LOCC-indistinguishable OPSs in both $\mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ spaces. Our result can be interpreted as an indication that LOCC protocols are quite powerful. Along this line, Walgate *et al.* [15] proved that LOCC is sufficient to reliably distinguish *two* multipartite orthogonal pure states, even when they are entangled. When the two states are not orthogonal, LOCC protocols can reach the global optimality in either conclusive discrimination [16] or inconclusive but unambiguous discrimination [17]. Therefore, perhaps the whole class of LOCC-indistinguishable OPSs has much simpler structure than one may fear.

There are multipartite operators other than those distinguishing OPSs that do not create entanglement. Thus it remains an open problem to characterize all such operators that cannot be realized by LOCC, even in the $3 \otimes 3$ and $2 \otimes 2 \otimes 2$ dimension case.

We observe that if an OPB has a rectangular representation $(\mathcal{R}, \alpha, \beta, U, V)$, then there is a simple LOCC protocol to identify an unknown state given *two* copies of it: the first copy is projected to the bases α and β so that the rectangle R containing the state is identified, then the second copy is measured in the product basis $\{U_R|\alpha_i\rangle \otimes V_R|\beta_j\rangle : (i, j) \in R\}$. Given an OPS, determining the number of copies of an unknown state necessary to admit an LOCC distinguishing protocol is an interesting generalization of determining if it is LOCC-distinguishable.

Another interesting generalization is to determine the optimal probability of identifying an unknown state from a given OPS by LOCC. Finally, it remains possible that an operator cannot be realized by LOCC yet may be approximated to an arbitrary precision. Identifying such an operator or proving that none exists is a fascinating open problem.

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