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On the quantum no-signalling assisted zero-error classical simulation cost of non-commutative bipartite graphs

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Abstract—Using one channel to simulate another exactly with the aid of quantum no-signalling correlations has been studied recently. The one-shot no-signalling assisted classical zero-error simulation cost of non-commutative bipartite graphs has been formulated as semidefinite programing [Duan and Winter, arXiv:1409.3426]. It remains unknown whether the one-shot (or asymptotic) no-signalling assisted zero-error classical simulation cost for general non-commutative graphs is multiplicative (resp. additive) or not. In this paper we address these issues and give a general sufficient condition for the multiplicativity of the one-shot simulation cost and the additivity of the asymptotic simulation cost of non-commutative bipartite graphs, which include all known cases such as extremal graphs and classical-quantum graphs. Applying this condition, we exhibit a large class of so-called cheapest-full-rank graphs whose asymptotic zero-error simulation cost is given by the one-shot simulation cost. Finally, we disprove the multiplicativity of one-shot simulation cost by explicitly constructing a special class of qubit-qutrit non-commutative bipartite graphs.

I. INTRODUCTION

Channel simulation is a fundamental problem in information theory, which concerns how to use a channel $N$ from Alice (A) to Bob (B) to simulate another channel $M$ also from A to B [1]. Shannon’s celebrated noisy channel coding theorem determines the capability of any noisy channel $N$ to simulate an noiseless channel [2] and the dual theorem “reverse Shannon theorem” was proved recently [3]. According to different resources available between A and B, this simulation problem has many variants and the case when A and B share unlimited amount of entanglement has been completely solved [3]. To optimally simulate $M$ in the asymptotic setting, the rate is determined by the entanglement-assisted classical capacity of $N$ and $M$ [4], [5]. Furthermore, this rate cannot be improved even with no-signalling correlations or feedback [4].

In the zero-error setting [6], recently the quantum zero-error information theory has been studied and the problem becomes more complex since many unexpected phenomena were observed such as the super-activation of noisy channels [9], [10], [11], [12] as well as the assistance of shared entanglement in zero-error communication [7], [8].

Quantum no-signalling correlations (QNSC) are introduced as two-input and two-output quantum channels with the no-signalling constraints. And such correlations have been studied in the relativistic causality of quantum operations [13], [14], [15], [16]. Cubitt et al. [17] first introduced classical no-signalling correlations into the zero-error classical communication problem. They also observed a kind of reversibility between no-signalling assisted zero-error capacity and exact simulation [17]. Duan and Winter [18] further introduced quantum non-signalling correlations into the zero-error communication problem and formulated both capacity and simulation cost problems as semidefinite programing (SDPs) [21] which depend only on the non-commutative bipartite graph $K$. To be specific, QNSC is a bipartite partially positive and trace-preserving linear map $\Pi : L(A_i) \otimes L(B_i) \rightarrow L(A_o) \otimes L(B_o)$, where the subscripts $i$ and $o$ stand for input and output, respectively. Let the Choi-Jamiolkowski matrix of $\Pi$ be $\Omega_{A_i',A_o',B_i,B_o} = (1 \otimes \Phi_{B_i,B_i'}) \Pi (\Phi_{A_i,A_i'} \otimes 1_{B_i'}),\Phi_{A_o,A_o'} = |\Phi_{A_o,A_o'}\rangle \langle \Phi_{A_o,A_o'}|$, and $\Phi_{A_i,A_i'} = \sum_i |\Omega_i\rangle \langle \Omega_i|$. The unconstrained maximally-entangled state $\Omega_{A_i',A_o',B_i,B_o}$ is the un-normalized maximally-entangled state.

The following constraints are required for $\Pi$ to be QNSC [18]:

$$\Omega_{A_i',A_o',B_i,B_o} \geq 0, \quad \text{Tr}_{A_i,B_i} \Omega_{A_i',A_o',B_i,B_o} = 1_{A_i'B_i'};$$

$$\text{Tr}_{A_o,A_o'} \Omega_{A_i',A_o',B_i,B_o} X_{A_i'}^T = 0, \forall \text{Tr} X = 0,$$

$$\text{Tr}_{B_i,B_i'} \Omega_{A_i',A_o',B_i,B_o'} Y_{B_i'}^T = 0, \forall \text{Tr} Y = 0.$$

The new map $M_{A_i \rightarrow B_o} = \Pi_{A_i \otimes B_i \rightarrow A_o \otimes B_o} \circ \mathcal{L}_{A_i \rightarrow B_i}$ by composing $N$ and $\Pi$ can be constructed as illustrated in Figure 1. The simulation cost problem concerns how much zero-error communication is required to simulate a noisy channel exactly. Particularly, the one-shot zero-error classical simulation cost of $N$ assisted by $\Pi$ is the least noiseless symbols $m$ from $A_o$ to $B_i$ so that $M$ can simulate $N$. In [18], the one-shot simulation cost of a quantum channel $N$ is given by

$$\Sigma(N) = \min \text{Tr} T_{B_i}, \text{s.t. } J_{AB} \leq 1_A \otimes T_{B_i}.$$

(1)
Its SDP is
\[
\Sigma(\mathcal{N}) = \max \text{Tr}(J_{AB} U_{AB}), \quad \text{s.t. } U_{AB} \succeq 0, \quad \text{Tr}_A U_{AB} = \mathbb{1}_B,
\]
where \(J_{AB}\) is the Choi-Jamiołkowski matrix of \(\mathcal{N}\). By strong duality, the values of both the primal and the dual SDP coincide. The so-called “non-commutative graph theory” was first suggested in [25] as the non-commutative graph associated with the channel captures the zero-error communication properties, thus playing a similar role to confusability graph. Let \(\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger\) be a quantum channel from \(\mathcal{L}(A)\) to \(\mathcal{L}(B)\), where \(\sum_k E_k E_k^\dagger = \mathbb{1}_A\) and \(K = K(\mathcal{N}) = \text{span}\{E_k\}\) denotes the Choi-Kraus operator space of \(\mathcal{N}\). The zero-error classical capacity of a quantum channel in the presence of quantum feedback only depends on the Choi-Kraus operator space of the channel [19]. That is to say, the Choi-Kraus operator space plays a role that is quite similar to the bipartite graph. Such Choi-Kraus operator space \(K\) is alternatively called “non-commutative bipartite graph” since it is clear that any classical channel induces a bipartite graph and a confusability graph, while a quantum channel induces a non-commutative bipartite graph together with a non-commutative graph [18].

Back to the simulation cost problem, since there might be more than one channel with Choi-Kraus operator space included in \(K\), the exact simulation cost of the “cheapest” one among these channels was defined as the one-shot zero-error classical simulation cost of \(K\) [18]:
\[
\Sigma(K) = \min \{\Sigma(\mathcal{N}) : \mathcal{N}\text{' is quantum channel and } K(\mathcal{N}) < K\},
\]
where \(K(\mathcal{N}) < K\) means that \(K(\mathcal{N})\) is a subspace of \(K\). Then the one-shot zero-error classical simulation cost of a non-commutative bipartite graph \(K\) is given by [18]
\[
\Sigma(K) = \min \text{Tr} T_B \quad \text{s.t. } 0 \leq V_{AB} \leq \mathbb{1}_A \otimes T_B,
\]
\[
\text{Tr}_B V_{AB} = \mathbb{1}_A, \quad \text{Tr}(\mathbb{1} - P)_{AB} V_{AB} = 0.
\]
\[
(2)
\]

Its dual SDP is
\[
\Sigma(K) = \max \text{Tr} S_A \quad \text{s.t. } 0 \leq U_{AB}, \quad \text{Tr}_A U_{AB} = \mathbb{1}_B,
\]
\[
P_{AB}(S_A \otimes \mathbb{1}_B - U_{AB})P_{AB} \leq 0,
\]
where \(P_{AB}\) denotes the projection onto the support of the Choi-Jamiołkowski matrix of \(\mathcal{N}\). Then by strong duality, values of both the primal and the dual SDP coincide. It is evident that \(\Sigma(K)\) is sub-multiplicative, which means that for two non-commutative bipartite graphs \(K_1\) and \(K_2\), \(\Sigma(K_1 \otimes K_2) \leq \Sigma(K_1) \Sigma(K_2)\). Furthermore, the multiplicativity of \(\Sigma(K)\) for classical-quantum (cq) graphs as well as extremal graphs were known but the general case was left as an open problem [18]. By the regularization, the no-signalling assisted zero-error simulation cost is
\[
S_{0,NS}(K) = \inf_{n \geq 1} \frac{1}{n} \log \Sigma(K^\otimes n).
\]

As noted in previous work [18], [19],
\[
C_{0,NS}(K) \leq C_{\min E}(K) \leq S_{0,NS}(K),
\]
where \(C_{0,NS}(K)\) is the QSNC assisted classical zero-error capacity and \(C_{\min E}(K)\) is the minimum of the entanglement-assisted classical capacity [3], [20] of quantum channels \(\mathcal{N}\) such that \(K(\mathcal{N}) < K\).

Semidefinite programs [21] can be solved in polynomial time in the program description [22] and there exist several different algorithms employing interior point methods which can compute the optimum value of semidefinite programs efficiently [23], [24]. The CVX software package [28] for MATLAB allows one to solve semidefinite programs efficiently.

In this paper, we focus on the multiplicativity of \(\Sigma(K)\) for general non-commutative bipartite graph \(K\). We start from the simulation cost of two different graphs and give a sufficient condition which contains all the known multiplicative cases such as cq graphs and extremal graphs. Then we consider about the simulation cost \(\Sigma(K)\) when the “cheapest” subspace is full-rank and prove the multiplicativity of one-shot simulation cost in this case. We further explicitly construct a special class of non-commutative bipartite graphs \(K_\alpha\) whose one-shot simulation cost is non-multiplicative. We also find some more properties of \(K_\alpha\) as well as cheapest-low-rank graphs. Finally, we exhibit a lower bound in order to offer an estimation of the asymptotic simulation cost.

II. MAIN RESULTS

A. A sufficient condition of the multiplicativity of simulation cost

**Theorem 1** Let \(K_1\) and \(K_2\) be non-commutative bipartite graphs of two quantum channels \(\mathcal{N}_1 : \mathcal{L}(A_1) \to \mathcal{L}(B_1)\) and \(\mathcal{N}_2 : \mathcal{L}(A_2) \to \mathcal{L}(B_2)\) with support projections \(P_{A_1B_1}\) and \(P_{A_2B_2}\), respectively. Suppose the optimal solutions of SDP(3) for \(\Sigma(K_1)\) and \(\Sigma(K_2)\) are \(\{S_{A_1}, U_1\}\) and \(\{S_{A_2}, U_2\}\). If at least one of \(S_{A_1}\) and \(S_{A_2}\) satisfy
\[
P_{A_iB_i}(S_{A_i} \otimes \mathbb{1}_{B_i})P_{A_iB_i} \geq 0, \quad i = 1 \text{ or } 2,
\]
then
\[
\Sigma(K_1 \otimes K_2) = \Sigma(K_1) \Sigma(K_2).
\]

Furthermore,
\[
S_{0,NS}(K_1 \otimes K_2) = S_{0,NS}(K_1) + S_{0,NS}(K_2).
\]

**Proof** It is obvious that \(U_1 \otimes U_2 \succeq 0\) and \(\text{Tr}_{A_1A_2}(U_1 \otimes U_2) = \mathbb{1}_{B_1B_2}\). For convenience, let \(P_{A_1B_1} = P_1\) and...
\[ P_{A_2|B_2} = P_2. \] Without loss of generality, we assume that \( P_2(S_{A_1} \otimes 1_{B_2})P_2 \geq 0. \) From the last constraint of SDP(2), we have that \( P_1(P_1|A_1 \otimes 1_{B_1}|P_2 \leq P_2U_2P_2. \) Note that \( P_2(S_{A_1} \otimes 1_{B_1})P_2 \otimes (P_2A_2 \otimes 1_{B_2})P_2 \leq P_2U_1P_1 \otimes (P_2A_2 \otimes 1_{B_2})P_2. \) It is easy to see that

\[ P_1 \otimes P_2(S_{A_1} \otimes 1_{A_2} \otimes 1_{B_2|B_2})P_2 \otimes (P_2U_2P_2) \leq 0. \] (5)

Hence, \( \{S_{A_1} \otimes 1_{A_2} \otimes 1_{B_2|B_2}\} \) is a feasible solution of SDP(3) for \( \Sigma(K_1 \otimes K_2) \), which means that \( \Sigma(K_1 \otimes K_2) \geq \Sigma(K_1) \Sigma(K_2). \) Since \( \Sigma(K) \) is super-multiplicative, we can conclude that \( \Sigma(K_1 \otimes K_2) = \Sigma(K_1) \Sigma(K_2). \)

Furthermore, for \( K_2^{\otimes n} \), it is easy to see that \( \{S_{\otimes n}^{2n}, U_{2n}^{\otimes n}\} \) is a feasible solution of SDP(3) for \( \Sigma(K_2^{\otimes n}) \) and \( P_2^{\otimes n}(S_{A_2}^{\otimes n} \otimes 1_{B_2}^{\otimes n})P_2^{\otimes n} = 0. \) Therefore, \( \Sigma(K_2^{\otimes n}) = \Sigma(K_2)^n \) and

\[ \Sigma(K_1 \otimes K_2)^{\otimes n} = \Sigma(K_1^{\otimes n} \otimes K_2^{\otimes n}) = \Sigma(K_1)^n \Sigma(K_2)^n. \]

Hence,

\[ S_{0,NS}(K_1 \otimes K_2) = \inf_{n \geq 1} \frac{1}{n} \log \left( \frac{\Sigma(K_1^{\otimes n} \otimes K_2^{\otimes n})}{\Sigma(K_1 \otimes K_2)^{\otimes n}} \right) = \inf_{n \geq 1} \frac{1}{n} \log \left( \frac{\Sigma(K_1)^n \Sigma(K_2)^n}{\Sigma(K_1 \otimes K_2)^{\otimes n}} \right) = S_{0,NS}(K_1) + S_{0,NS}(K_2). \]

In [26], the activated zero-error no-signalling assisted capacity has been studied. Here, we consider about the corresponding simulation cost problem.

**Corollary 2** For any non-commutative bipartite graph \( K \), let \( \Delta_{\ell} = \sum_{i=1}^{\ell} |kk| |kk| \) be the non-commutative bipartite graph of a noisless channel with \( \ell \) symbols, then

\[ \Sigma(K \otimes \Delta_{\ell}) = \ell \Sigma(K), \]

which means that noisless channel cannot reduce the simulation cost of any other non-commutative bipartite graph.

**Proof** It is evident that \( \Delta_{\ell} \) satisfies the condition in Theorem 1. Then, \( \Sigma(K \otimes \Delta_{\ell}) = \ell \Sigma(K). \) □

**B. Simulation cost of the cheapest-full-rank non-commutative bipartite graph**

**Definition 3** Given a non-commutative bipartite graph \( K \) with support projection \( P_{AB} \). Assume the "cheapest channel" in this space is \( N_c \) with Choi-Jamiołkowski matrix \( J_{N_c} \). \( K \) is said to be **cheapest-full-rank** if there exists \( N_c \) such that \( \text{rank}(J_{N_c}) = \text{rank}(P_{AB}) \). Otherwise, \( K \) is said to be **cheapest-low-rank**.

**Lemma 4** For a quantum channel \( N \) with Choi-Jamiołkowski matrix \( J_{AB} \) and support projection \( P_{AB} \), if \( P_{AB}C_{AB} = P_{AB}D_{AB} \), then \( \text{Tr}(CJ_{AB}) = \text{Tr}(DJ_{AB}). \)

**Proof** It is easy to see that

\[ \text{Tr}(CJ_{AB}) = \text{Tr}(CP_{AB}J_{AB}P_{AB}) = \text{Tr}(P_{AB}CP_{AB}J_{AB}) = \text{Tr}(DP_{AB}J_{AB}) = \text{Tr}(DJ_{AB}). \]

**Proposition 5** For any non-commutative bipartite graph \( K \) with support projection \( P_{AB} \), suppose that the cheapest channel is \( N_c \) and the optimal solution of SDP (3) is \( \{S_A, U_{AB}\} \).

Assume that

\[ P_{AB}(S_A \otimes 1_B - U_{AB})P_{AB} = -W_{AB} \text{ and } W_{AB} \geq 0. \] (6)

Then, we have that

\[ \text{Tr}W_{AB}J_{AB} = 0, \]

and \( U_{AB} \) is also the optimal solution of \( \Sigma(N_c) \), where \( J_{AB} \) is the Choi-Jamiołkowski matrix of \( N_c \).

**Proof** On one hand, since \( N_c \) is the cheapest channel, \( \Sigma(K) \) will equal to \( \Sigma(N_c) \), also noting that \( \{S_A, U_{AB}\} \) is the optimal solution, we have

\[ \text{Tr}S_A = \Sigma(K) = \Sigma(N_c) \]

\[ = \max \text{Tr}J_{AB}V_{AB}, \text{ s.t. } V_{AB} \geq 0, \text{Tr}A V_{AB} = 1_B, \]

\[ \geq \text{Tr}J_{AB}U_{AB}. \] (8)

On the other hand, it is evident that \( W_{AB} = P_{AB}W_{P_{AB}} \), then \( P_{AB}U_{AB}P_{AB} = P_{AB}(W_{P_{AB}} + S_A \otimes 1_B)P_{AB} \). From Lemma 4, we can conclude that \( \text{Tr}U_{AB}J_{AB} = \text{Tr}(W_{AB} + S_A \otimes 1_B)J_{AB} = \text{Tr}W_{AB}J_{AB} + \text{Tr}(S_A \otimes 1_B)J_{AB} \).

For Choi-Jamiołkowski matrix \( J_{AB} \), we have that

\[ \text{Tr}(S_A \otimes 1_B)J_{AB} = \text{Tr}_A \text{Tr}[S_A \otimes 1_B]J_{AB} \]

\[ = \text{Tr}_A[S_A(\text{Tr}_B J_{AB})] = \text{Tr}_A S_A. \] (9)

then

\[ \text{Tr}U_{AB}J_{AB} = \text{Tr}W_{AB}J_{AB} + \text{Tr}S_A. \] (10)

Combining (8) and (10), and noting that \( W_{AB}, J_{AB} \geq 0 \), we can conclude that \( \text{Tr}W_{AB}J_{AB} = 0 \) and \( U_{AB} \) is also the optimal solution of \( \Sigma(N_c) \). □

**Theorem 6** For any cheapest-full-rank non-commutative bipartite graph \( K \), we have

\[ \Sigma(K) = \max \text{Tr}S_A \text{ s.t. } 0 \leq U_{AB}, \text{Tr}_A U_{AB} = 1_B, \text{P}_{AB}(S_A \otimes 1_B - U_{AB})P_{AB} = 0. \] (11)

Also, \( \Sigma(K \otimes K) = \Sigma(K) \Sigma(K) \). Consequently, \( S_{0,NS}(K) = \log \Sigma(K) \).

And for any other non-commutative bipartite graph \( K' \), \( S_{0,NS}(K \otimes K') = S_{0,NS}(K) + S_{0,NS}(K') \).

**Proof** We first assume that \( W \neq 0 \). Notice \( \text{rank}(J_{AB}) = \text{rank}(P_{AB}) \), it is easy to see that \( \text{Tr}W_{AB} > 0 \), which contradicts Eq. (7). Hence the assumption is false, and we can conclude that \( P_{AB}(S_A \otimes 1_B - U_{AB})P_{AB} = 0 \).

Then by Theorem 1, it is easy to see that \( \Sigma(K \otimes K) = \Sigma(K) \Sigma(K) \). Therefore,

\[ S_{0,NS}(K) = \inf_{n \geq 1} \frac{1}{n} \log \Sigma(K^{\otimes n}) = \log \Sigma(K). \]
Furthermore, for any other non-commutative bipartite graph $K'$, $S_{0,N}(K \otimes K') = S_{0,N}(K) + S_{0,N}(K')$. □

Noting that any rank-2 Choi-Kraus operator space is always cheapest-full-rank, we have the following immediate corollary.

**Corollary 7** For any rank-2 Choi-Kraus operator space $K$, $S_{0,N}(K) = \log \Sigma(K)$. And for any other non-commutative bipartite graph $K'$, $S_{0,N}(K \otimes K') = S_{0,N}(K) + S_{0,N}(K')$.

**C. The one-shot simulation cost is not multiplicative**

We will focus on the non-commutative bipartite graph $K_\alpha$ with support projection $P_{AB} = \sum_{j=0}^{2} |\psi_j\rangle\langle \psi_j|$, where $|\psi_0\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |12\rangle)$, $|\psi_1\rangle = \cos \alpha |02\rangle + \sin \alpha |11\rangle$, $|\psi_2\rangle = |10\rangle$.

To prove that $K_\alpha$ $(0 < \cos^2 \alpha < 1)$ is feasible to be a class of feasible non-commutative bipartite graphs, we only need to find a channel $\mathcal{N}$ with Choi-Jamiolkowski matrix $J_{AB}$ such that $P_{AB}J_{AB} = J_{AB}$ and $\text{rank}(P_{AB}) = \text{rank}(J_{AB})$. Assume that $J_{AB} = \sqrt{2} \sum_{j=0}^{2} a_j |\psi_j\rangle\langle \psi_j|$, then it is equivalent to prove that $\text{Tr}_B J_{AB} = 1_A$ and $J_{AB} \geq 0$ has a feasible solution. Therefore,

$$\frac{2}{3} a_0 + \cos^2 \alpha a_1 = 1, a_0 + a_1 + a_2 = 2, a_0, a_1, a_2 > 0.$$ 

Noting that when we choose $0 < a_1 < \frac{1}{2}$, $a_0 = \frac{3}{2}(1 - \cos^2 \alpha a_1)$ and $a_2 = \frac{1}{2} - \frac{1}{2} \cos^2 \alpha a_1$ will be positive, which means that there exists such $J_{AB}$. Hence, $K_\alpha$ is a feasible noncommutative bipartite graph.

**Theorem 8** There exists non-commutative bipartite graph $K$ such that $\Sigma(K \otimes K) < \Sigma(K)^2$.

**Proof** As we have shown above, it is reasonable to focus on $K_\alpha$. Then, by semidefinite programming assisted with useful tools CVX [28] and QETLAB [29], the gap between one-shot and two-shot average no-signalling assisted zero-error simulation cost of $K_\alpha (0.25 \leq \cos^2 \alpha \leq 0.35)$ is shown in Figure 2.

To be specific, when $\alpha = \pi/3$, it is clear that $\cos^2 \alpha = 1/4$ and $|\psi_1\rangle = \frac{1}{\sqrt{2}} |02\rangle + \frac{\sqrt{3}}{2} |11\rangle$. Assume that $S = 3.1102 |00\rangle\langle 00| - 0.5386 |11\rangle\langle 11|$ and $U = \frac{99}{50} |u_1\rangle\langle u_1| + \frac{11}{50} |u_2\rangle\langle u_2|$, where $|u_1\rangle = \frac{10}{3\sqrt{3}} |00\rangle + \frac{1}{3\sqrt{3}} |01\rangle + \frac{7}{\sqrt{3}4} |12\rangle$ and $|u_2\rangle = \frac{2}{\sqrt{5}4} |02\rangle - \frac{2}{\sqrt{17}4} |10\rangle + \frac{10}{3\sqrt{3}4} |11\rangle$, and it can be checked that $U \geq 0$, $\text{Tr}_A U = 1_B$ and $P_{AB} \Sigma(K_\alpha) = \Sigma(K_\alpha)$. Then $\{S, U\}$ is a feasible solution of SDP (3) for $\Sigma(K_\alpha)$, which means that $\Sigma(K_\alpha) \geq 2.5716$. Similarly, we can find a feasible solution of SDP (2) for $\Sigma(K_\alpha \otimes K_\alpha)$ through Matlab such that $\Sigma(K_\alpha \otimes K_\alpha)^{1/2} \leq 2.57$. (The code is available at [27].) Hence, there is a non-vanishing gap between $\Sigma(K_\alpha)$ and $\Sigma(K_\alpha \otimes K_\alpha)^{1/2}$. □

We have shown that one-shot simulation cost of cheapest-full-rank non-commutative bipartite graphs is multiplicative while there are counterexamples for cheapest-low-rank ones. However, not all cheapest-low-rank graphs have non-multiplicative simulation cost. Here is one trivial counterexample. Let $K = \text{span} \{|00\rangle, |11\rangle\}$, the cheapest channel is a constant channel $\mathcal{N}$ with $E_0 = |1\rangle\langle 0|$ and $E_1 = |1\rangle\langle 1|$. In this case, $\Sigma(K \otimes K) = \Sigma(K) \Sigma(K) = 1$. Actually, the simulation cost problem of cheapest-low-rank non-commutative bipartite graphs is complex since it is hard to determine the cheapest subspace under tensor powers. Therefore, it is difficult to calculate the asymptotic simulation cost of non-multiplicative cases.

In [19], $K$ is called non-trivial if there is no constant channel $N_0 : \rho \rightarrow |\beta\rangle\langle \beta|$ with $K(N_0) < K$, where $|\beta\rangle$ is a state vector. It was known that $K$ is non-trivial if and only if the no-signalling assisted zero-error capacity is positive, say $C_{0,N}(K) > 0$. Clearly we have the following result.

**Proposition 9** For any non-commutative bipartite graph $K$, $S_{0,N}(K) > 0$ if and only if $K$ is non-trivial.

**Proof** If $K$ is non-trivial, it is obvious that $S_{0,N}(K) \geq C_{0,N}(K) > 0$. Otherwise, $0 \leq S_{0,N}(K) \leq S_{0,N}(N_0) = 0$, which means that $S_{0,N}(K) = 0$. □

**D. A lower bound**

Let us introduce a revised SDP which has the same simplified form in cq-channel case:

$$\Sigma_-(K) = \max \text{Tr} S_A \quad \text{s.t.} \quad S_A \geq 0, U_{AB} \geq 0, \text{Tr}_A U_{AB} = 1_B, \quad P_{AB} (S_A \otimes 1_B - U_{AB}) P_{AB} \leq 0,$$

$$\Sigma_-(K_1 \otimes K_2) \geq \Sigma_-(K_1) \Sigma_-(K_2).$$

Fig. 2. The one-shot (red) and two-shot average (blue) no-signalling assisted zero-error simulation cost of $K_\alpha$ over the parameter $\alpha$. 
Consequently, $\Sigma_-(K_1)\Sigma_-(K_2) \leq \Sigma(K_1 \otimes K_2) \leq \Sigma(K_1)\Sigma(K_2)$.

**Proof** From SDP (12), noting that $P_{AB}(S_A \otimes I_B)P_{AB} \geq 0$, it is easy to prove $\Sigma_-(K_1 \otimes K_2) \geq \Sigma_-(K_1)\Sigma_-(K_2)$ by similar technique applied in Theorem 3. Therefore, $\Sigma_-(K_1)\Sigma_-(K_2) \leq \Sigma_-(K_1 \otimes K_2) \leq \Sigma(K_1 \otimes K_2) \leq \Sigma(K_1)\Sigma(K_2)$.

**Proposition 11** For a general non-commutative bipartite graph $K$,

$$\log \Sigma_-(K) \leq S_{0,N,S}(K) \leq \log \Sigma(K).$$

**Proof** By Lemma 10, it is easy to see that $\Sigma_-(K)^n \leq \Sigma(K^\otimes n) \leq \Sigma(K)^n$. Then, $\log \Sigma_-(K) \leq S_{0,N,S}(K) \leq \log \Sigma(K)$. Also, it is obvious that $S_{0,N,S}(K)$ will equal to $\log \Sigma(K)$ when $\Sigma_-(K) = \Sigma(K)$.

**III. Conclusions**

In sum, for two different non-commutative bipartite graphs, we give sufficient conditions for the multiplicativity of one-shot simulation cost as well as the additivity of the asymptotic simulation cost. The case of cheapest-full-rank non-commutative bipartite graphs has been completely solved while the cheapest-low-rank graphs have a more complex structure. We further show that the non-signalling assisted classical zero-error simulation cost of non-commutative bipartite graphs is not multiplicative. We provide a lower bound of $\Sigma(K)$ such that the asymptotic zero-error simulation cost can be estimated by $\log \Sigma_-(K) \leq S_{0,N,S}(K) \leq \log \Sigma(K)$.

It is of great interest to know whether the sufficient condition of multiplicativity in Theorem 1 is also necessary. Whether the asymptotic simulation cost of general non-commutative bipartite graphs is additive or equal to $\log \Sigma_-(K)$ also remains unknown.

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**References**

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