On the quantum no-signalling assisted zero-error classical simulation cost of non-commutative bipartite graphs

Xin Wang*, Runyao Duan*†
*Centre for Quantum Computation and Intelligent Systems
Faculty of Engineering and Information Technology
University of Technology Sydney (UTS), NSW 2007, Australia
†UTS-AMSS Joint Research Laboratory for Quantum Computation and Quantum Information Processing
Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100190, China
Email: xin.wang-8@student.uts.edu.au, runyao.duan@uts.edu.au

***Abstract*—Using one channel to simulate another exactly with the aid of quantum no-signalling correlations has been studied recently. The one-shot no-signalling assisted classical zero-error simulation cost of non-commutative bipartite graphs has been formulated as semidefinite programmings [Duan and Winter, arXiv:1409.3426]. It remains unknown whether the one-shot (or asymptotic) no-signalling assisted zero-error classical simulation cost for general non-commutative graphs is multiplicative (resp. additive) or not. In this paper we address these issues and give a general sufficient condition for the multiplicativity of the one-shot simulation cost and the additivity of the asymptotic simulation cost of non-commutative bipartite graphs, which include all known cases such as extremal graphs and classical-quantum graphs. Applying this condition, we exhibit a large class of so-called *cheapest-full-rank graphs* whose asymptotic zero-error simulation cost is given by the one-shot simulation cost. Finally, we disprove the multiplicativity of one-shot simulation cost by explicitly constructing a special class of qubit-qutrit non-commutative bipartite graphs.**

I. Introduction

Channel simulation is a fundamental problem in information theory, which concerns how to use a channel $\mathcal{N}$ from Alice (A) to Bob (B) to simulate another channel $\mathcal{M}$ also from A to B [1]. Shannon's celebrated noisy channel coding theorem determines the capability of any noisy channel $\mathcal{N}$ to simulate an noiseless channel [2] and the dual theorem "reverse Shannon theorem" was proved recently [3]. According to different resources available between A and B, this simulation problem has many variants and the case when A and B share unlimited amount of entanglement has been completely solved [3]. To optimally simulate $\mathcal{M}$ in the asymptotic setting, the rate is determined by the entanglement-assisted classical capacity of $\mathcal{N}$ and $\mathcal{M}$ [4], [5]. Furthermore, this rate cannot be improved even with no-signalling correlations or feedback [4].

In the zero-error setting [6] , recently the quantum zero-error information theory has been studied and the problem becomes more complex since many unexpected phenomena were observed such as the super-activation of noisy channels [9], [10], [11], [12] as well as the assistance of shared entanglement in zero-error communication [7], [8].

Quantum no-signalling correlations (QNSC) are introduced as two-input and two-output quantum channels with the no-signalling constraints. And such correlations have been studied in the relativistic causality of quantum operations [13], [14], [15], [16]. Cubitt et al. [17] first introduced classical no-signalling correlations into the zero-error classical communication problem. They also observed a kind of reversibility between no-signalling assisted zero-error capacity and exact simulation [17]. Duan and Winter [18] further introduced quantum non-signalling correlations into the zero-error communication problem and formulated both capacity and simulation cost problems as semidefinite programmings (SDPs) [21] which depend only on the non-commutative bipartite graph K. To be specific, QNSC is a bipartite completely positive and trace-preserving linear map $\Pi : \mathcal{L}(\mathcal{A}_i)\otimes\mathcal{L}(\mathcal{B}_i) \to \mathcal{L}(\mathcal{A}_o) \otimes \mathcal{L}(\mathcal{B}_o)$, where the subscripts i and o stand for input and output, respectively. Let the Choi-Jamiołkowski matrix of Π be $\Omega_{\mathcal{A}_i'\mathcal{A}_o\mathcal{B}_i'\mathcal{B}_o} = (\mathbb{1}_{\mathcal{A}_i'} \otimes \mathbb{1}_{\mathcal{B}_i'} \otimes \Pi)(\Phi_{\mathcal{A}_i\mathcal{A}_i'} \otimes \Phi_{\mathcal{B}_i\mathcal{B}_i'})$, where $\Phi_{\mathcal{A}_i\mathcal{A}_i'} = |\Phi_{\mathcal{A}_i\mathcal{A}_i'}\rangle\langle\Phi_{\mathcal{A}_i\mathcal{A}_i'}|$, and $|\Phi_{\mathcal{A}_i\mathcal{A}_i'}\rangle = \sum_k |k_{\mathcal{A}_i}\rangle|k_{\mathcal{A}_i'}\rangle$ is the un-normalized maximally-entangled state.The following constraints are required for Π to be QNSC [18]:

$$\Omega_{\mathcal{A}_i'\mathcal{A}_o\mathcal{B}_i'\mathcal{B}_o} \geq 0,\ \operatorname{Tr}_{\mathcal{A}_o\mathcal{B}_o} \Omega_{\mathcal{A}_i'\mathcal{A}_o\mathcal{B}_i'\mathcal{B}_o} = \mathbb{1}_{\mathcal{A}_i'\mathcal{B}_i'},$$
$$\operatorname{Tr}_{\mathcal{A}_o\mathcal{A}_i'} \Omega_{\mathcal{A}_i'\mathcal{A}_o\mathcal{B}_i'\mathcal{B}_o} X^T_{\mathcal{A}_i'} = 0, \forall \operatorname{Tr} X = 0,$$
$$\operatorname{Tr}_{\mathcal{B}_o\mathcal{B}_i'} \Omega_{\mathcal{A}_i'\mathcal{A}_o\mathcal{B}_i'\mathcal{B}_o} Y^T_{\mathcal{B}_i'} = 0, \forall \operatorname{Tr} Y = 0.$$

The new map $\mathcal{M}^{A_i\to B_o} = \Pi^{A_i\otimes B_i\to A_o\otimes B_o} \circ \mathcal{E}^{A_o\to B_i}$ by composing $\mathcal{N}$ and Π can be constructed as illustrated in Figure 1. The simulation cost problem concerns how much zero-error communication is required to simulate a noisy channel exactly. Particularly, the *one-shot* zero-error classical simulation cost of $\mathcal{N}$ assisted by Π is the least noiseless symbols m from A_o to B_i so that $\mathcal{M}$ can simulate $\mathcal{N}$. In [18], the one-shot simulation cost of a quantum channel $\mathcal{N}$ is given by

$$\Sigma(\mathcal{N}) = \min \operatorname{Tr} T_B,\ \text{s.t. } J_{AB} \leq \mathbb{1}_A \otimes T_B. \quad (1)$$

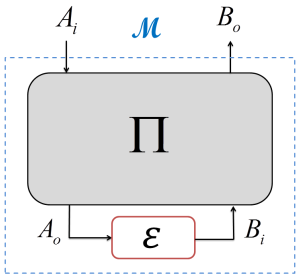

Fig. 1. Implementing a channel $\mathcal{M}$ using another channel $\mathcal{E}$ with QNSC Π between Alice and Bob.

Its dual SDP is

$$\Sigma(\mathcal{N}) = \max \operatorname{Tr}(J_{AB}U_{AB}), \text{ s.t. } U_{AB} \geq 0, \operatorname{Tr}_A U_{AB} = \mathbb{1}_B,$$

where J_{AB} is the Choi-Jamiołkowski matrix of $\mathcal{N}$. By strong duality, the values of both the primal and the dual SDP coincide. The so-called "non-commutative graph theory" was first suggested in [25] as the non-commutative graph associated with the channel captures the zero-error communication properties, thus playing a similar role to confusability graph. Let $\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger$ be a quantum channel from $\mathcal{L}(A)$ to $\mathcal{L}(B)$, where $\sum_k E_k^\dagger E_k = \mathbb{1}_A$ and $K = K(\mathcal{N}) = \operatorname{span}\{E_k\}$ denotes the Choi-Kraus operator space of $\mathcal{N}$. The zero-error classical capacity of a quantum channel in the presence of quantum feedback only depends on the Choi-Kraus operator space of the channel [19]. That is to say, the Choi-Kraus operator space plays a role that is quite similar to the bipartite graph. Such Choi-Kraus operator space K is alternatively called "non-commutative bipartite graph" since it is clear that any classical channel induces a bipartite graph and a confusability graph, while a quantum channel induces a non-commutative bipartite graph together with a non-commutative graph [18].

Back to the simulation cost problem, since there might be more than one channel with Choi-Kraus operator space included in K, the exact simulation cost of the "cheapest" one among these channels was defined as the one-shot zero-error classical simulation cost of K [18]: $\Sigma(K) = \min\{\Sigma(\mathcal{N}) : \mathcal{N}$ is quantum channel and $K(\mathcal{N}) < K\}$, where $K(\mathcal{N}) < K$ means that $K(\mathcal{N})$ is a subspace of K. Then the one-shot zero-error classical simulation cost of a non-commutative bipartite graph K is given by [18]

$$\begin{aligned} \Sigma(K) = \min \operatorname{Tr} T_B \quad \text{s.t.} \quad & 0 \leq V_{AB} \leq \mathbb{1}_A \otimes T_B, \\ & \operatorname{Tr}_B V_{AB} = \mathbb{1}_A, \\ & \operatorname{Tr}(\mathbb{1} - P)_{AB} V_{AB} = 0. \end{aligned} \tag{2}$$

Its dual SDP is

$$\begin{aligned} \Sigma(K) = \max \operatorname{Tr} S_A \quad \text{s.t.} \quad & 0 \leq U_{AB}, \ \operatorname{Tr}_A U_{AB} = \mathbb{1}_B, \\ & P_{AB}(S_A \otimes \mathbb{1}_B - U_{AB})P_{AB} \leq 0, \end{aligned} \tag{3}$$

where P_{AB} denotes the projection onto the support of the Choi-Jamiołkowski matrix of $\mathcal{N}$. Then by strong duality, values of both the primal and the dual SDP coincide. It is evident that $\Sigma(K)$ is sub-multiplicative, which means that for two non-commutative bipartite graphs K_1 and K_2, $\Sigma(K_1 \otimes K_2) \leq \Sigma(K_1)\Sigma(K_2)$. Furthermore, the multiplicativity of $\Sigma(K)$ for classical-quantum (cq) graphs as well as extremal graphs were known but the general case was left as an open problem [18]. By the regularization, the no-signalling assisted zero-error simulation cost is

$$S_{0,NS}(K) = \inf_{n \geq 1} \frac{1}{n} \log \Sigma\left(K^{\otimes n}\right).$$

As noted in previous work [18], [19],

$$C_{0,NS}(K) \leq C_{\text{minE}}(K) \leq S_{0,NS}(K),$$

where $C_{0,NS}(K)$ is the QSNC assisted classical zero-error capacity and $C_{\text{minE}}(K)$ is the minimum of the entanglement-assisted classical capacity [3], [20] of quantum channels $\mathcal{N}$ such that $K(\mathcal{N}) < K$.

Semidefinite programs [21] can be solved in polynomial time in the program description [22] and there exist several different algorithms employing interior point methods which can compute the optimum value of semidefinite programs efficiently [23], [24]. The CVX software package [28] for MATLAB allows one to solve semidefinite programs efficiently.

In this paper, we focus on the multiplicativity of $\Sigma(K)$ for general non-commutative bipartite graph K. We start from the simulation cost of two different graphs and give a sufficient condition which contains all the known multiplicative cases such as cq graphs and extremal graphs. Then we consider about the simulation cost $\Sigma(K)$ when the "cheapest" subspace is full-rank and prove the multiplicativity of one-shot simulation cost in this case. We further explicitly construct a special class of non-commutative bipartite graphs K_α whose one-shot simulation cost is non-multiplicative. We also exploit some more properties of K_α as well as cheapest-low-rank graphs. Finally, we exhibit a lower bound in order to offer an estimation of the asymptotic simulation cost.

II. Main Results

A. A sufficient condition of the multiplicativity of simulation cost

Theorem 1 *Let K_1 and K_2 be non-commutative bipartite graphs of two quantum channels $\mathcal{N}_1 : \mathcal{L}(A_1) \to \mathcal{L}(B_1)$ and $\mathcal{N}_2 : \mathcal{L}(A_2) \to \mathcal{L}(B_2)$ with support projections $P_{A_1B_1}$ and $P_{A_2B_2}$, respectively. Suppose the optimal solutions of SDP(3) for $\Sigma(K_1)$ and $\Sigma(K_2)$ are $\{S_{A_1}, U_1\}$ and $\{S_{A_2}, U_2\}$. If at least one of S_{A_1} and S_{A_2} satisfy*

$$P_{A_iB_i}(S_{A_i} \otimes \mathbb{1}_{B_i})P_{A_iB_i} \geq 0, i = 1 \text{ or } 2, \tag{4}$$

then

$$\Sigma(K_1 \otimes K_2) = \Sigma(K_1)\Sigma(K_2).$$

Furthermore,

$$S_{0,NS}(K_1 \otimes K_2) = S_{0,NS}(K_1) + S_{0,NS}(K_2).$$

Proof It is obvious that $U_1 \otimes U_2 \geq 0$ and $\operatorname{Tr}_{A_1A_2}(U_1 \otimes U_2) = \mathbb{1}_{B_1B_2}$. For convenience, let $P_{A_1B_1} = P_1$ and

$P_{A_2B_2} = P_2$. Without loss of generality, we assume that $P_2(S_{A_2} \otimes \mathbb{1}_{B_2})P_2 \geq 0$. From the last constraint of SDP(2), we have that $P_1(S_{A_1} \otimes \mathbb{1}_{B_1})P_1 \leq P_1U_1P_1$ and $P_2(S_{A_2} \otimes \mathbb{1}_{B_2})P_2 \leq P_2U_2P_2$. Note that $P_1(S_{A_1} \otimes \mathbb{1}_{B_1})P_1 \otimes P_2(S_{A_2} \otimes \mathbb{1}_{B_2})P_2 \leq P_1U_1P_1 \otimes P_2(S_{A_2} \otimes \mathbb{1}_{B_2})P_2$. It is easy to see that

$$\begin{aligned} & P_1 \otimes P_2 (S_{A_1} \otimes S_{A_2} \otimes \mathbb{1}_{B_1B_2} - U_1 \otimes U_2) P_1 \otimes P_2 \\ \leq & P_1U_1P_1 \otimes [P_2(S_{A_2} \otimes \mathbb{1}_{B_2})P_2 - P_2U_2P_2] \leq 0. \end{aligned} \tag{5}$$

Hence, $\{S_{A_1} \otimes S_{A_2}, U_1 \otimes U_2\}$ is a feasible solution of SDP(3) for $\Sigma(K_1 \otimes K_2)$, which means that $\Sigma(K_1 \otimes K_2) \geq \Sigma(K_1)\Sigma(K_2)$. Since $\Sigma(K)$ is sub-multiplicative, we can conclude that $\Sigma(K_1 \otimes K_2) = \Sigma(K_1)\Sigma(K_2)$.

Furthermore, for $K_2^{\otimes n}$, it is easy to see that $\{S_{A_2}^{\otimes n}, U_2^{\otimes n}\}$ is a feasible solution of SDP(3) for $\Sigma(K_2^{\otimes n})$ and $P_2^{\otimes n}(S_{A_2}^{\otimes n} \otimes \mathbb{1}_{B_2}{}^{\otimes n})P_2^{\otimes n} \geq 0$. Therefore, $\Sigma(K_2^{\otimes n}) = \Sigma(K_2)^n$ and

$$\Sigma[(K_1 \otimes K_2)^{\otimes n}] = \Sigma(K_1^{\otimes n} \otimes K_2^{\otimes n}) = \Sigma(K_1^{\otimes n})\Sigma(K_2^{\otimes n}).$$

Hence,

$$\begin{aligned} S_{0,NS}(K_1 \otimes K_2) &= \inf_{n \geq 1} \frac{1}{n} \log \Sigma\left(K_1^{\otimes n} \otimes K_2^{\otimes n}\right) \\ &= \inf_{n \geq 1} \frac{1}{n} \log \Sigma(K_1^{\otimes n})\Sigma(K_2^{\otimes n}) \\ &= S_{0,NS}(K_1) + S_{0,NS}(K_2). \end{aligned}$$

□

In [26], the activated zero-error no-signalling assisted capacity has been studied. Here, we consider about the corresponding simulation cost problem.

Corollary 2 *For any non-commutative bipartite graph K, let $\Delta_\ell = \sum_{k=1}^{\ell} |kk\rangle\langle kk|$ be the non-commutative bipartite graph of a noiseless channel with ℓ symbols, then*

$$\Sigma(K \otimes \Delta_\ell) = \ell\Sigma(K),$$

which means that noiseless channel cannot reduce the simulation cost of any other non-commutative bipartite graph.

Proof It is evident that Δ_ℓ satisfies the condition in Theorem 1. Then, $\Sigma(K \otimes \Delta_\ell) = \ell\Sigma(K)$. □

B. Simulation cost of the cheapest-full-rank non-commutative bipartite graph

Definition 3 *Given a non-commutative bipartite graph K with support projection P_{AB}. Assume the "cheapest channel" in this space is $\mathcal{N}_c$ with Choi-Jamiołkowski matrix $J_{\mathcal{N}_c}$. K is said to be* ***cheapest-full-rank*** *if there exists $\mathcal{N}_c$ such that $rank(J_{\mathcal{N}_c}) = rank(P_{AB})$. Otherwise, K is said to be* ***cheapest-low-rank****.*

Lemma 4 *For a quantum channel $\mathcal{N}$ with Choi-Jamiołkowski matrix J_{AB} and support projection P_{AB}, if $P_{AB}CP_{AB} = P_{AB}DP_{AB}$, then $\operatorname{Tr}(CJ_{AB}) = \operatorname{Tr}(DJ_{AB})$.*

Proof It is easy to see that

$$\begin{aligned} \operatorname{Tr}(CJ_{AB}) &= \operatorname{Tr}(CP_{AB}J_{AB}P_{AB}) = \operatorname{Tr}(P_{AB}CP_{AB}J_{AB}) \\ &= \operatorname{Tr}(P_{AB}DP_{AB}J_{AB}) = \operatorname{Tr}(DJ_{AB}). \end{aligned}$$

□

Proposition 5 *For any non-commutative bipartite graph K with support projection P_{AB}, suppose that the cheapest channel is $\mathcal{N}_c$ and the optimal solution of SDP (3) is $\{S_A, U_{AB}\}$. Assume that*

$$P_{AB}(S_A \otimes \mathbb{1}_B - U_{AB})P_{AB} = -W_{AB} \text{ and } W_{AB} \geq 0. \tag{6}$$

Then, we have that

$$\operatorname{Tr} W_{AB}J_{AB} = 0, \tag{7}$$

and U_{AB} is also the optimal solution of $\Sigma(\mathcal{N}_c)$, where J_{AB} is the Choi-Jamiołkowski matrix of $\mathcal{N}_c$.

Proof On one hand, since $\mathcal{N}_c$ is the cheapest channel, $\Sigma(K)$ will equal to $\Sigma(\mathcal{N}_c)$, also noting that $\{S_A, U_{AB}\}$ is the optimal solution, we have

$$\begin{aligned} \operatorname{Tr} S_A &= \Sigma(K) = \Sigma(\mathcal{N}_c) \\ &= \max \operatorname{Tr} J_{AB}V_{AB}, \text{ s.t. } V_{AB} \geq 0, \operatorname{Tr}_A V_{AB} = \mathbb{1}_B, \\ &\geq \operatorname{Tr} J_{AB}U_{AB}. \end{aligned} \tag{8}$$

On the other hand, it is evident that $W_{AB} = P_{AB}WP_{AB}$, then $P_{AB}U_{AB}P_{AB} = P_{AB}(W_{AB} + S_A \otimes \mathbb{1}_B)P_{AB}$. From Lemma 4, we can conclude that $\operatorname{Tr} U_{AB}J_{AB} = \operatorname{Tr}(W_{AB} + S_A \otimes \mathbb{1}_B)J_{AB} = \operatorname{Tr} W_{AB}J_{AB} + \operatorname{Tr}(S_A \otimes \mathbb{1}_B)J_{AB}$.

For Choi-Jamiołkowski matrix J_{AB}, we have that

$$\begin{aligned} \operatorname{Tr}(S_A \otimes \mathbb{1}_B)J_{AB} &= \operatorname{Tr}_A \operatorname{Tr}_B[(S_A \otimes \mathbb{1}_B)J_{AB}] \\ &= \operatorname{Tr}_A[S_A(\operatorname{Tr}_B J_{AB})] = \operatorname{Tr} S_A, \end{aligned} \tag{9}$$

then

$$\operatorname{Tr} U_{AB}J_{AB} = \operatorname{Tr} W_{AB}J_{AB} + \operatorname{Tr} S_A. \tag{10}$$

Combining (8) and (10), and noting that $W_{AB}, J_{AB} \geq 0$, we can conclude that $\operatorname{Tr} W_{AB}J_{AB} = 0$ and U_{AB} is also the optimal solution of $\Sigma(\mathcal{N}_c)$. □

Theorem 6 *For any cheapest-full-rank non-commutative bipartite graph K, we have*

$$\begin{aligned} \Sigma(K) = \max \operatorname{Tr} S_A \quad \text{s.t.} \quad & 0 \leq U_{AB},\ \operatorname{Tr}_A U_{AB} = \mathbb{1}_B, \\ & P_{AB}(S_A \otimes \mathbb{1}_B - U_{AB})P_{AB} = 0. \end{aligned} \tag{11}$$

Also, $\Sigma(K \otimes K) = \Sigma(K)\Sigma(K)$. Consequently, $S_{0,NS}(K) = \log \Sigma(K)$.

And for any other non-commutative bipartite graph K', $S_{0,NS}(K \otimes K') = S_{0,NS}(K) + S_{0,NS}(K')$.

Proof We first assume that $W \neq 0$. Notice $rank(J_{AB}) = rank(P_{AB})$, it is easy to see that $\operatorname{Tr} WJ_{AB} > 0$, which contradicts Eq. (7). Hence the assumption is false, and we can conclude that $P_{AB}(S_A \otimes \mathbb{1}_B - U_{AB})P_{AB} = 0$.

Then by Theorem 1, it is easy to see that $\Sigma(K \otimes K) = \Sigma(K)\Sigma(K)$. Therefore,

$$S_{0,\mathrm{NS}}(K) = \inf_{n \geq 1} \frac{1}{n} \log \Sigma(K^{\otimes n}) = \log \Sigma(K).$$

Furthermore, for any other non-commutative bipartite graph K', $S_{0,NS}(K\otimes K') = S_{0,\mathrm{NS}}(K) + S_{0,NS}(K')$. □

Noting that any rank-2 Choi-Kraus operator space is always cheapest-full-rank, we have the following immediate corollary.

Corollary 7 *For any rank-2 Choi-Kraus operator space K, $S_{0,NS}(K) = \log\Sigma(K)$. And for any other non-commutative bipartite graph K', $S_{0,NS}(K\otimes K') = S_{0,NS}(K) + S_{0,NS}(K')$.*

C. The one-shot simulation cost is not multiplicative

We will focus on the non-commutative bipartite graph K_α with support projection $P_{AB} = \sum_{j=0}^{2}|\psi_j\rangle\langle\psi_j|$, where $|\psi_0\rangle = \frac{1}{\sqrt{3}}(|00\rangle+|01\rangle+|12\rangle), |\psi_1\rangle = \cos\alpha|02\rangle+\sin\alpha|11\rangle, |\psi_2\rangle = |10\rangle$.

To prove that K_α $(0<\cos^2\alpha<1)$ is feasible to be a class of feasible non-commutative bipartite graphs, we only need to find a channel $\mathcal{N}$ with Choi-Jamiołkowski matrix J_{AB} such that $P_{AB}J_{AB} = J_{AB}$ and $\mathrm{rank}(P_{AB}) = \mathrm{rank}(J_{AB})$. Assume that $J_{AB} = \sum_{j=0}^{2} a_j|\psi_j\rangle\langle\psi_j|$, then it is equivalent to prove that $\mathrm{Tr}_B J_{AB} = \mathbb{1}_A$ and $J_{AB}\geq 0$ has a feasible solution. Therefore,

$$\frac{2}{3}a_0 + \cos^2\alpha a_1 = 1, a_0+a_1+a_2 = 2, a_0,a_1,a_2>0.$$

Noting that when we choose $0<a_1<\frac{1}{2}$, $a_0 = \frac{3}{2}(1-\cos^2\alpha a_1)$ and $a_2 = \frac{1-(2-3\cos^2\alpha)a_1}{2}$ will be positive, which means that there exists such J_{AB}. Hence, K_α is a feasible noncommutative bipartite graph.

Theorem 8 *There exists non-commutative bipartite graph K such that $\Sigma(K\otimes K)<\Sigma(K)^2$.*

Proof As we have shown above, it is reasonable to focus on K_α. Then, by semidefinite programming assisted with useful tools CVX [28] and QETLAB [29],the gap between one-shot and two-shot average no-signalling assisted zero-error simulation cost of $K_\alpha(0.25\leq\cos^2\alpha\leq 0.35)$ is shown in Figure 2.

To be specfic, when $\alpha=\pi/3$, it is clear that $\cos^2\alpha = 1/4$ and $|\psi_1\rangle = \frac{1}{2}|02\rangle + \frac{\sqrt{3}}{2}|11\rangle$. Assume that $S = 3.1102|0\rangle\langle 0| - 0.5386|1\rangle\langle 1|$ and $U = \frac{99}{50}|u_1\rangle\langle u_1| + \frac{51}{50}|u_2\rangle\langle u_2|$, where $|u_1\rangle = \frac{10}{3\sqrt{33}}|00\rangle + \frac{5}{3}\sqrt{\frac{2}{33}}|01\rangle + \frac{7}{3\sqrt{11}}|12\rangle$ and $|u_2\rangle = \frac{1}{\sqrt{51}}|02\rangle - \frac{5}{3}\sqrt{\frac{2}{17}}|10\rangle + \frac{10}{3\sqrt{17}}|11\rangle$, and it can be checked that $U\geq 0$, $\mathrm{Tr}_A U = \mathbb{1}_B$ and $P_{AB}(S_A\otimes\mathbb{1}_B - U_{AB})P_{AB}\leq 0$. Then $\{S,U\}$ is a feasible solution of SDP (3) for $\Sigma(K_{\pi/3})$, which means that $\Sigma(K_{\pi/3})\geq\mathrm{Tr}\,S = 2.5716$. Similarily, we can find a feasible solution of SDP (2) for $\Sigma(K_{\pi/3}\otimes K_{\pi/3})$ through Matlab such that $\Sigma(K_{\pi/3}\otimes K_{\pi/3})^{1/2}\leq 2.57$. (The code is available at [27].) Hence, there is a non-vanishing gap between $\Sigma(K_{\pi/3})$ and $\Sigma(K_{\pi/3}\otimes K_{\pi/3})^{1/2}$. □

We have shown that one-shot simulation cost of cheapest-full-rank non-commutative bipartite graphs is multiplicative while there are counterexamples for cheapest-low-rank

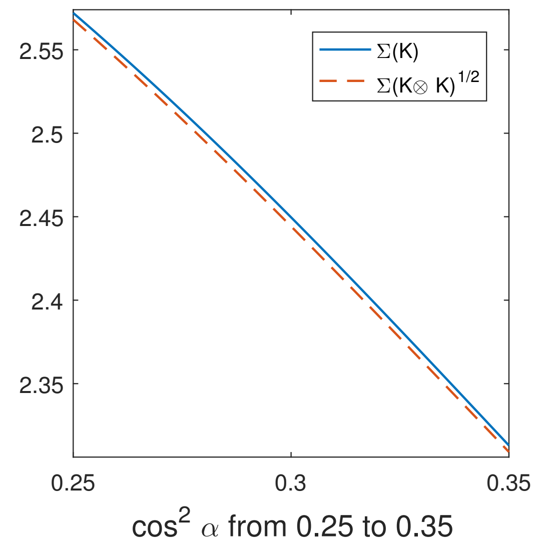

Fig. 2. The one-shot (red) and two-shot average (blue) no-signalling assisted zero-error simulation cost of K_α over the parameter α.

ones. However, not all cheapest-low-rank graphs have non-multiplicative simulation cost. Here is one trivial counterexample. Let $K = \mathrm{span}\{|0\rangle\langle 0|, |1\rangle\langle 0|, |1\rangle\langle 1|\}$, the cheapest channel is a constant channel $\mathcal{N}$ with $E_0 = |1\rangle\langle 0|$ and $E_1 = |1\rangle\langle 1|$. In this case, $\Sigma(K\otimes K) = \Sigma(K)\Sigma(K) = 1$. Actually, the simulation cost problem of cheapest-low-rank non-commutative bipartite graphs is complex since it is hard to determine the cheapest subspace under tensor powers. Therefore, it is difficult to calculate the asymptotic simulation cost of non-multiplicative cases.

In [19], K is called non-trivial if there is no constant channel $\mathcal{N}_0 : \rho\to|\beta\rangle\langle\beta|$ with $K(\mathcal{N}_0)<K$, where $|\beta\rangle$ is a state vector. It was known that K is non-trivial if and only if the no-signalling assisted zero-error capacity is positive, say $C_{0,NS}(K)>0$. Clearly we have the following result.

Proposition 9 *For any non-commutative bipartite graph K, $S_{0,NS}(K)>0$ if and only if K is non-trivial.*

Proof If K is non-trivial, it is obvious that $S_{0,NS}(K)\geq C_{0,NS}(K)>0$. Otherwise, $0\leq S_{0,NS}(K)\leq S_{0,NS}(\mathcal{N}_0) = 0$, which means that $S_{0,NS}(K) = 0$. □

D. A lower bound

Let us introduce a revised SDP which has the same simplified form in cq-channel case:

$$\Sigma^-(K) = \max\mathrm{Tr}\,S_A \quad \text{s.t.} \quad S_A\geq 0, U_{AB}\geq 0 \;\; \mathrm{Tr}_A U_{AB} = \mathbb{1}_B, \quad P_{AB}(S_A\otimes\mathbb{1}_B - U_{AB})P_{AB}\leq 0, \tag{12}$$

Lemma 10 *For any non-commutative bipartite graphs K_1 and K_2,*

$$\Sigma^-(K_1\otimes K_2)\geq\Sigma^-(K_1)\Sigma^-(K_2).$$

Consequently, $\Sigma^-(K_1)\Sigma^-(K_2) \le \Sigma(K_1 \otimes K_2) \le \Sigma(K_1)\Sigma(K_2)$.

Proof From SDP (12), noting that $P_{AB}(S_A \otimes \mathbb{1}_B)P_{AB} \ge 0$, it is easy to prove $\Sigma^-(K_1 \otimes K_2) \ge \Sigma^-(K_1)\Sigma^-(K_2)$ by similar technique applied in Theorem 3. Therefore, $\Sigma^-(K_1)\Sigma^-(K_2) \le \Sigma^-(K_1 \otimes K_2) \le \Sigma(K_1 \otimes K_2) \le \Sigma(K_1)\Sigma(K_2)$. □

Proposition 11 *For a general non-commutative bipartite graph K,*

$$\log \Sigma^-(K) \le S_{0,NS}(K) \le \log \Sigma(K).$$

Proof By Lemma 10, it is easy to see that $\Sigma^-(K)^n \le \Sigma(K^{\otimes n}) \le \Sigma(K)^n$. Then, $\log \Sigma^-(K) \le S_{0,NS}(K) \le \log \Sigma(K)$. Also, it is obvious that $S_{0,NS}(K)$ will equal to $\log \Sigma(K)$ when $\Sigma^-(K) = \Sigma(K)$. □

III. Conclusions

In sum, for two different non-commutative bipartite graphs, we give sufficient conditions for the multiplicativity of one-shot simulation cost as well as the additivity of the asymptotic simulation cost. The case of cheapest-full-rank non-commutative bipartite graphs has been completely solved while the cheapest-low-rank graphs have a more complex structure. We further show that the one-shot no-signalling assisted classical zero-error simulation cost of non-commutative bipartite graphs is not multiplicative. We provide a lower bound of $\Sigma(K)$ such that the asymptotic zero-error simulation cost can be estimated by $\log \Sigma^-(K) \le S_{0,NS}(K) \le \log \Sigma(K)$.

It is of great interest to know whether the sufficient condition of multiplicativity in Theorem 1 is also necessary. Whether the aysmptotic simulation cost of general non-commutative bipartite graphs is additive or equal to $\log \Sigma^-(K)$ also remains unknown.

Acknowledgments

We would like to thank Andreas Winter for his interest on this topic and for many insightful suggestions. XW would like to thank Ching-Yi Lai for helpful discussions on SDP. This work was partly supported by the Australian Research Council (Grant No. DP120103776 and No. FT120100449) and the National Natural Science Foundation of China (Grant No. 61179030).

References

[1] D. Kretschmann and R. F. Werner, “Tema con variazioni: quantum channel capacity”, *New J. Phys.* **6**:26, 2004.

[2] C. E. Shannon, “A mathematical theory of communication”, *Bell Syst. Tech. J.*, **27**(3):379-423, 1948.

[3] C. H. Bennett, P. W. Shor, J. A. Smolin and A. V. Thapliyal, “Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem”, *IEEE Trans. Inf. Theory* **48**(10):2637-2655, 2002.

[4] C. H. Bennett, I. Devetak, A. W. Harrow, P. W. Shor and A. Winter, “The Quantum Reverse Shannon Theorem and Resource Tradeoffs for Simulating Quantum Channels”, *IEEE Trans. Inf. Theory* **60**(3):2926-2959, 2014.

[5] M. Berta, M. Christandl and R. Renner, “The Quantum Reverse Shannon Theorem Based on One-Shot Information Theory”, *Commun. Math. Phys.* **306**(3):579-615, 2011.

[6] C. E. Shannon, “The zero-error capacity of a noisy channel”, *IRE Trans. Inf. Theory* **2**(3):8-19, 1956.

[7] T. S. Cubitt, D. Leung, W. Matthews, and A. Winter, “Improving zero-error classical communication with entanglement”, *Phys. Rev. Lett.* **104**:230503, 2010.

[8] D. Leung, L. Mančinska, W. Matthews, M. Ozols and A. Roy, “Entanglement can increase asymptotic rates of zero-error classical communication over classical channels”, *Commun. Math. Phys.* **311**:97-111, 2012.

[9] R. Duan, “Super-activation of zero-error capacity of noisy quantum channels”, *arXiv:0906.2526*, 2009.

[10] R. Duan and Y. Shi, “Entanglement between two uses of a noisy multipartite quantum channel enables perfect transmission of classical information”, *Phys. Rev. Lett.* **101**:020501, 2008.

[11] T. S. Cubitt, J. Chen and A. W. Harrow, “Superactivation of the asymptotic zero-error classical capacity of a quantum channel”, *IEEE Trans. Inf. Theory* **57**(12):8114-8126, 2011.

[12] T. S. Cubitt and G. Smith, “An extreme form of super-activation for quantum zero-error capacities”, *IEEE Trans. Inf. Theory* **58**(3):1953-1961, 2012.

[13] D. Beckman, D. Gottesman, M. A. Nielsen and J. Preskill, “Causal and localizable quantum operations”, *Phys. Rev. A* **64**:052309, 2001.

[14] T. Eggeling, D. Schlingemann and R. F. Werner, “Semicausal operations are semilocalizable”, *Europhys. Lett.* **57**(6):782-788, 2002.

[15] M. Piani, M. Horodecki, P. Horodecki and R. Horodecki, “Properties of quantum nonsignaling boxes”, *Phys. Rev. A* **74**:012305, 2006.

[16] O. Oreshkov, F. Costa, and Č. Brukner, “Quantum correlations with no causal order”, *Nature Comm.* **3**(10):1092, 2012.

[17] T. S. Cubitt, D. Leung, W. Matthews, and A. Winter, “Zero-error channel capacity and simulation assisted by non-local correlations”, *IEEE Trans. Inf. Theory* **57**(8):5509-5523, 2011.

[18] R. Duan and A. Winter, “Non-Signalling Assisted Zero-Error Capacity of Quantum Channels and an Information Theoretic Interpretation of the Lovász Number”, 2014, available at arXiv:1409.3426.

[19] R. Duan, S. Severini and A. Winter, “On zero-error communication via quantum channels in the presence of noiseless feedback”, 2015, available at arXiv:1502.02987.

[20] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V Thapliyal, “Entanglement-assisted classical capacity of noisy quantum channels,” *Phys. Rev. Lett.*, **83**(15):3081 (1999).

[21] L. Vandenberghe, S. Boyd, “Semidefinite Programming”, *SIAM Rev.* **38**(1):49-95 (1996).

[22] L. G. Khachiyan, Polynomial algorithms in linear programming, USSR Comput. Math. Math. Phys., **20**(1):5372, 1980.

[23] F. Alizadeh, “Interior point methods in semidefinite programming with applications to combinatorial optimization”, *SIAM J. Optim.*, **5**(1):1351, (1995).

[24] E. De Klerk, “Aspects of semidefinite programming: interior point algorithms and selected applications”, *Springer Science & Business Media*, 2002.

[25] R. Duan, S. Severini, and A. Winter, “Zero-error communication via quantum channels, non-commutative graphs and a quantum Lovász number”, *IEEE Trans. Inf. Theory* **59**(2):1164-1174 (2013).

[26] R. Duan and X. Wang, Activated zero-error classical capacity of quantum channels in the presence of quantum no-signalling correlations, 2015, available at arXiv:1505.00907.

[27] Xin Wang, Supplementary software for implementing the one-shot QSNC assisted zero-error simulation cost is not multiplicative, https://github.com/xinwang1/QSNC-simulation-cost

[28] Michael Grant and Stephen Boyd. CVX: Matlab software for disciplined convex programming, version 2.1. http://cvxr.com/cvx, 2014.

[29] Nathaniel Johnston, Alessandro Cosentino, and Vincent Russo. QETLAB: A MATLAB toolbox for quantum entanglement, http://qetlab.com, 2015.