# $l^\infty$-Bounded Robustness for Nonlinear Systems : Analysis and Synthesis

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**Abstract**

The purpose of this paper is to describe systematic analysis and design tools for robust control problems with $l^\infty$ criteria. We first generalize the Hill-Moylan-Willems framework for dissipative systems to accommodate $l^\infty$ criteria, and then derive state feedback and measurement feedback synthesis procedures for $l^\infty$ robust control problems. The information state framework is used for the measurement feedback robust control problem. Necessary and sufficient conditions are proved, and new synthesis procedures using dynamic programming are presented.
**Keywords:** Nonlinear robust control, $l^\infty$ criteria, dissipative systems, information states, dynamic programming, controller synthesis.

## 1 Introduction

Techniques for the design of robust control systems and indeed for optimal control in general have primarily made use of integral-type performance criteria. These criteria are sometimes referred to as *soft criteria*, since a bound on the performance integral need not guarantee that a given output quantity meets absolute bound specifications or constraints. In some applications it is important for outputs to meet hard constraints in the time domain, such as applications where an absolute regulation error is required to be always less than a specified amount. Further, persistent input signals may be present that do not have finite energy. These situations can be formulated in terms of $L^\infty$-type (or $l^\infty$-type) criteria, which might be called *hard criteria*.

Methods for analysis and design using hard criteria have been considered for some time, mostly for linear systems. We mention here a small selection of results in the literature. The state feedback control problem to force the state of the closed-loop system

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(with uncertainties) to stay in a specified region was considered in [3, 5]. The recursive state estimation for linear system with bounded input disturbance and measurement noise was studied in [4]. The $l^1$ optimal control problem was introduced by [29], and solutions were obtained using linear programming in [10], [11], [8], [9]. These solutions [27] can be infinite dimensional and dynamic, although finite dimensional approximation methods were developed. In [26], [27], near optimal memoryless nonlinear state feedback solutions were obtained, using controlled invariance kernels and viability theory. These results were very interesting, given that the plants were described by linear dynamics. More recently, in [12] dynamic programming equations were derived for the state feedback problem for linear systems. In [28] set-valued observers were considered, and a separation structure controller was derived for linear systems. Here, the controller was a static function of the set-valued observer state. Also, we mention the papers [13] and [24], which considered the problems of $L^\infty$ worst-case analysis and rejection of persistent bounded disturbance for nonlinear systems.

Our objective in this paper is to describe systematic analysis and design tools for robust control problems with hard criteria. We begin by generalizing the Hill-Moylan-Willems framework for dissipative systems, originally developed for integral performance criteria, to accommodate $l^\infty$ criteria. This framework is powerful and widely used for a range of stability and robustness problems. The generalization of the dissipation property to the $l^\infty$ case is completely characterized in terms of a dynamic programming equation (or inequality) related to equations [13, (14) and (15)]. This is done in a way which makes use of a formal analogy between integrals and max-plus integrals (involving the (essential) supremum of a function on an interval), with links to the optimal control problems studied in [2], [7].

This analysis framework is then developed to derive state feedback and measurement feedback synthesis procedures by exploiting connections with optimal control and game theory. In the state feedback case, related results are available for linear systems, e.g. [12, Section V]. For the measurement feedback $l^\infty$ robust control problem, we employ the information state framework [21], [22], [15], and obtain dynamic controllers that feed back the information state. The information state is a generalization of observer or filter, with a state computable from measurement data. For the special case of what we refer to as the uniform $l^\infty$ bounded dissipation problem (essentially specified in [28, Definition 4.1]), it is shown that the controllers can be chosen to feed back only a set-valued state estimate, to which the information state reduces, consistent with the separation structure of [28, Theorem 4.1].

The results in this paper are expressed in terms of dynamic programming equations (or inequalities). We prove necessary and sufficient conditions in terms of them. Thus a particular dynamic programming equation (or inequality) has a solution when the corresponding control property holds or problem is solvable, as is the case with the bounded real lemma and $H^\infty$ control, [14]. Conversely, if a given dynamic programming equation (or inequality) has a solution (satisfying mild technical conditions), then the corresponding control property holds or problem is solvable. We do not address the issue of finding solutions to the dynamic programming equations (or inequalities); as is well known in dynamic programming, explicit solutions are not generally available and approximate or numerical methods are required, see, e.g. [6], [13]. We do, however, illustrate the synthesis procedure by applying it to simple linear and bilinear examples. Interestingly, the certainty equivalence principle as used in linear $H^\infty$ control [21], [1], [15] does not in general usefully apply in this $l^\infty$ context. Further applications and examples are reported in the publications [18], [19].

This paper considers discrete time nonlinear systems for technical simplicity. We do, however, present some of the analogous continuous time equations and inequalities for comparison. The continuous time case is more technical and is considered in a separate paper.

# 2 Analysis

## 2.1 $l^\infty$ Bounded Dissipation

Consider the nonlinear discrete-time system

$$\begin{aligned} \xi_{k+1} &= f(\xi_k, w_k) \\ z_k &= g(\xi_k, w_k) \end{aligned} \tag{2.1}$$

Here, $\xi_k \in \mathbf{R}^n, w_k \in \mathbf{W} \subset \mathbf{R}^s$, and $z_k \in \mathbf{R}$ are the state, disturbance input and performance output quantity, respectively.

We employ the following notation:

$$\begin{aligned} w_{0,k-1} &= \{w_0, \cdots, w_{k-1}\}, \forall k \geq 0, \\ \mathcal{W}_{0,k-1} &= \{w_{0,k-1} : w_i \in \mathbf{W}, 0 \leq i \leq k-1\}, k \geq 0, \\ \mathcal{W}_{0,\infty} &= \{w_{0,\infty} : w_i \in \mathbf{W}\}. \end{aligned} \tag{2.2}$$

We adopt the convention that sets of signal sequences corresponding to the index $k = 0$ are empty, so that $\mathcal{W}_{0,-1} = \emptyset$. We also take the supremum over an empty set to equal $-\infty$.

The following definition is motivated by the disturbance rejection problem specified by [28, Definition 4.1], the worst case analysis of [13, Section IIB], the $l^1$ performance specification formulated in [29], [10], [11], [8], [9] and the cost functions in [2]. It is one possible definition of dissipation-like properties with $l^\infty$ criteria. The dissipative systems framework was developed by Willems, Hill and Moylan [31], [16], [17].

**Definition 2.1** *Given $B_0 \subset \mathbf{R}^n$, the system (2.1) is* $l^\infty$-bounded (LIB) dissipative with respect to $B_0$ *if there exists a $\beta : B_0 \to \mathbf{R}$ such that*

$$z_k \leq \beta(x_0), \quad \forall \xi_0 = x_0 \in B_0, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0. \tag{2.3}$$

## 2.2 Storage Functions

Denote

$$\bar{\mathbf{R}} \stackrel{\triangle}{=} \mathbf{R} \cup \{+\infty\}\,. \tag{2.4}$$

For a function $V : \mathbf{R}^n \to \bar{\mathbf{R}}$, denote

$$\mathrm{dom}V \triangleq \{x \in \mathbf{R}^n : V(x) < +\infty\}. \tag{2.5}$$

We denote $T(A) = \{f(x, w) : x \in A, w \in \mathbf{W}\}$ where $A \subset \mathbf{R}^n$. A subset $S \subset \mathbf{R}^n$ is called a $T$-invariant set if $T(S) \subset S$.

**Definition 2.2** *Let $V : \mathbf{R}^n \to \bar{\mathbf{R}}$, $S \subset \mathrm{dom}V \subset \mathbf{R}^n$ be a $T$-invariant set. For system (2.1), $V$ is called a* storage function on $S$ *if*

$$V(x) \geq \max\{\max_{0 \leq i \leq k-1} g(\xi_i, w_i), V(\xi_k)\}, \forall x \in S, \forall w_{0,k-1} \in \mathcal{W}_{0,k-1}, \forall k \geq 0, \tag{2.6}$$

*where $\xi_\cdot$ denotes the state trajectory of (2.1) with disturbance $w$ and initial condition $\xi_0 = x$. Inequality (2.6) is called the* LIB dissipation inequality.

In the general theory of dissipative systems, two particular storage functions are of special interest, viz. the available storage and the required supply. In our present context, the *available storage* $V_a : \mathbf{R}^n \to \bar{\mathbf{R}}$ is defined by

$$V_a(x) = \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \max_{0 \leq i \leq k-1} g(\xi_i, w_i), \tag{2.7}$$

where $\xi_\cdot$ denotes the state trajectory of (2.1) with disturbance $w$ and initial condition $\xi_0 = x$ (this is a generalization of the usual definition of available storage [31]).

**Lemma 2.3** *The available storage $V_a(x)$ is a storage function on* $\mathrm{dom}V_a$ *for the system (2.1). Moreover, for any storage function $V$ on $S$, we have*

$$S \subset \mathrm{dom}V_a, \quad V_a(x) \leq V(x), \forall x \in S. \tag{2.8}$$

Proof. The proof of the fact that $V_a$ satisfies (2.6) is similar to the proofs of [31, Theorems 1 and 2] and is omitted here. If $x \in \mathrm{dom}V_a$, i.e. $V_a(x) < +\infty$, then from (2.6), $\forall w \in \mathbf{W}$, $V_a(f(x, w)) \leq V_a(x) < +\infty$, i.e. $f(x, w) \in \mathrm{dom}V_a$, so $\mathrm{dom}V_a$ is a $T$-invariant set. The minimal property follows from the following observation. If $V$ is a storage function on $S$ as in Definition 2.2, then by (2.6) $\forall x \in S$,

$$V(x) \geq \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \max_{0 \leq i \leq k-1} g(\xi_i, w_i) = V_a(x).$$

Hence $S \subset \mathrm{dom}V_a$. □

The following theorem shows how storage functions characterize the LIB dissipation property.

**Theorem 2.4** *The system (2.1) is LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$ if and only if there exists a function $V : \mathbf{R}^n \to \bar{\mathbf{R}}$ and a $T$-invariant set $S$ satisfying $B_0 \subset S \subset \mathrm{dom}V$ such that $V$ is a storage function on $S$.*

Proof. If system (2.1) is LIB dissipative with respect to $B_0$, then there exists a $\beta : B_0 \to \mathbf{R}$ such that

$$g(\xi_k, w_k) = z_k \leq \beta(x_0), \quad \forall x_0 \in B_0, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0,$$

hence from (2.7) we have $V_a(x_0) \leq \beta(x_0)$ for all $x_0 \in B_0$. Thus $B_0 \subset \mathrm{dom}V_a$, and from Lemma 2.3, $V_a$ is a storage function on $\mathrm{dom}V_a$ with $B_0 \subset \mathrm{dom}V_a$, as required.

Conversely, if $V$ is a storage function on $S$, then

$$z_k = g(\xi_k, w_k) \leq \max_{0 \leq i \leq k} g(\xi_i, w_i) \leq V(x_0) < +\infty, \quad \forall x_0 \in S, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0,$$

Since $B_0 \subset S$, $\beta(x_0) \stackrel{\triangle}{=} V(x_0)$ satisfies (2.3). □

**Corollary 2.5** *System (2.1) is LIB dissipative with respect to $B_0$ if and only if $B_0 \subset \mathrm{dom}V_a$.*

## 2.3 Dynamic Programming Inequality

We now give an "infinitesimal", or, precisely, a one-step, dynamic programming inequality that characterizes storage functions, and hence by Theorem 2.4 the LIB dissipation property.

For $V : \mathbf{R}^n \to \bar{\mathbf{R}}$ and $S \subset \mathrm{dom}V \subset \mathbf{R}^n$, the *LIB dynamic programming equation* is

$$V(x) = \sup_{w \in \mathbf{W}} \max\{g(x,w), V(f(x,w))\}, \forall x \in S. \tag{2.9}$$

The analogous *LIB dynamic programming inequality* is

$$V(x) \geq \sup_{w \in \mathbf{W}} \max\{g(x,w), V(f(x,w))\}, \forall x \in S. \tag{2.10}$$

**Remark 2.6** The analogous LIB dynamic programming equation for the continuous time system $\dot{\xi} = f(\xi, w)$, $z = g(\xi, w)$, is the partial differential equation [20] (see also [2], [7], [13]):

$$\sup_{w \in \mathbf{W}} \max\{g(x,w) - V(x), \nabla V(x) f(x,w)\} = 0, \quad \forall x \in S \subset \mathrm{dom}V. \tag{2.11}$$

□

**Theorem 2.7** *Given a function $V : \mathbf{R}^n \to \bar{\mathbf{R}}$ and a $T$-invariant set $S \subset \mathrm{dom}V$, $V$ is a storage function on $S$ if and only if $(V, S)$ is a solution of (2.10).*

Proof. The necessity is obvious. Now consider the sufficiency, assume $(V, S)$ satisfying (2.10). Let $k \geq 0$ and select $w_{0,k-1} \in \mathcal{W}_{0,k-1}$ and $x_0 \in S$. This determines a trajectory $\xi_i$, $0 \leq i \leq k$. Since $S$ is a $T$-invariant set, iterating (2.10) we find that

$$V(x_0) \geq V(\xi_i), \ \forall\ 0 \leq i \leq k; \quad V(\xi_i) \geq g(\xi_i, w_i), \ \forall\ 0 \leq i \leq k-1.$$

Therefore $V(x_0) \geq V(\xi_k)$ and $V(x_0) \geq \max_{0 \leq i \leq k-1} g(\xi_i, w_i)$. This implies (2.6). □

**Theorem 2.8** *(Necessity) If system (2.1) is LIB dissipative with respect to $B_0$, then the available storage $V_a$ defined by (2.7) satisfies*

*(i)* $B_0 \subset \mathrm{dom} V_a$;

*(ii)* $V_a(x) \geq \sup_{w \in \mathbf{W}} g(x, w), \quad \forall x \in \mathbf{R}^n$;

*(iii)* $\mathrm{dom} V_a$ *is a $T$-invariant set and the dynamic programming relation holds:*

$$V_a(x) = \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \max\{\max_{0 \leq i \leq j-1} g(\xi_i, w_i), V_a(\xi_j)\}, \quad \forall x \in \mathrm{dom} V_a, \forall j \geq 0. \quad (2.12)$$

*i.e. $V_a$ solve the dynamic programming equation (2.9) with $S = \mathrm{dom} V_a$.*

*(iv) If $(V, S)$ satisfies the LIB dynamic programming inequality (2.10), then*

$$S \subset \mathrm{dom} V_a, \quad V_a(x) \leq V(x), \forall x \in S.$$

*(Sufficiency) If there exists $V : \mathbf{R}^n \to \bar{\mathbf{R}}$ and $T$-invariant set $S$ satisfying the LIB dynamic programming inequality (2.10), then for system (2.1),*

$$z_k \leq V(x_0), \quad \forall \xi_0 = x_0 \in S, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0.$$

*Moreover, if $B_0 \subset S$, then system (2.1) is LIB dissipative with respect to $B_0$.*

Proof. (i) and (ii) are obvious by the definition of $V_a$, and (iv) follows from Lemma 2.3 and Theorem 2.7. Part (iii) follows from Lemma 2.3 and standard dynamic programming techniques. The proof of the sufficiency assertion comes directly from Theorem 2.4 and Theorem 2.7. □

## 2.4 Performance and Stability

The notion of LIB dissipation abstracts the approach to worst case analysis in [13]; the definition (2.7) of available storage corresponds to the function defined by equations (2) and (3) in [13]. Storage functions for LIB dissipative systems can be used to analyze $L^\infty$ gain functions and induced $L^\infty$ gains over bounded signals [13].

In many applications asymptotic stability to an equilibrium, or stability about an equilibrium, is an issue. We say that the system (2.1) has an *equilibrium* at $x = 0$ if $0 \in \mathbf{W}$, $f(0, 0) = 0$, $g(0, 0) = 0$. The next theorem is an example of a stability theorem for LIB dissipative systems.

**Theorem 2.9** *Let $V$ be a storage function on a $T$-invariant set $S \subset \mathrm{dom} V$ for an LIB dissipative system (2.1) with equilibrium $x = 0$, where $g(x, w) = |x|$.*

*(i) If $x_0 \in S$, then the state is bounded as follows:*

$$|\xi_k| \leq V(\xi_k) \leq V(x_0), \ \forall k \geq 0 \quad (2.13)$$

*whenever $w_{0,\infty} \in \mathcal{W}_{0,\infty}$.*

*(ii) Assume in addition that $V$ is continuous and $V(0) = 0$. Then for any $w_{0,\infty} \in \mathcal{W}_{0,\infty}$ the system (2.1) is stable, and $V$ is a positive definite Lyapunov function.*

*(iii) Assume in addition that $V(f(x,0)) = V(x) \Rightarrow x = 0$, and that $f(x,0)$ is continuous. Then the system (2.1) is asymptotically stable when $w = 0$.*

Proof. Since $V$ is a storage function on $S$, from Definition 2.2, if $\xi_0 = x_0 \in S$, then $\forall k \geq 0, \xi_k \in S$ and

$$|\xi_k| = g(\xi_k, w_k) \leq V(\xi_k); \quad V(\xi_k) \leq V(x_0).$$

This immediately implies (2.13).

If also $V$ is continuous and $V(0) = 0$, then given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_0| < \delta$ implies $V(x_0) < \varepsilon$. Then (2.13) implies $|\xi_k| < \varepsilon$ for all $k \geq 0$.

Since $V(x_0) \geq |x_0|$ and $V(\xi_{k+1}) \leq V(\xi_k), \forall k \geq 0$, $V$ is a positive definite Lyapunov function.

When $V(f(x,0)) = V(x) \Rightarrow x = 0$, $f(x,0)$ is continuous and $w = 0$, since $\xi_k$ is bounded, from the Invariance Principle [23, Chapter 1, Theorem 6.3], $\xi_k \to 0$ as $k \to \infty$. So the system is asymptotically stable when $w = 0$. □

# 3 State Feedback Synthesis

With the tools developed in the previous section in hand, we turn to the problem of finding state feedback controllers achieving LIB dissipation for the closed loop.

## 3.1 Problem

Consider the nonlinear discrete-time system

$$\begin{aligned} \xi_{k+1} &= f(\xi_k, u_k, w_k) \\ z_k &= g(\xi_k, u_k, w_k) \end{aligned} \tag{3.1}$$

Here, $\xi_k \in \mathbf{R}^n, u_k \in \mathbf{U} \subset \mathbf{R}^m, w_k \in \mathbf{W} \subset \mathbf{R}^s$ and $z_k \in \mathbf{R}$ are the state, control input, disturbance and performance output quantity, respectively. In addition to the notation of the previous section, we define $x_{0,k}$, $\mathcal{X}_{0,\infty}$, $u_{0,k}$, $\mathcal{U}_{0,k}$, $\mathcal{U}_{0,\infty}$, analogously to (2.2).

**Assumption 3.1** *Assume that*

$$\breve{g}(x) \triangleq \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} g(x,u,w) > -\infty, \quad \forall x \in \mathbf{R}^n. \tag{3.2}$$

An *admissible* state feedback controller is a causal map $K : \mathcal{X}_{0,\infty} \to \mathcal{U}_{0,\infty}$, meaning that for each time $k \geq 0$ if $x^1, x^2 \in \mathcal{X}_{0,\infty}$ and $x^1_l = x^2_l$ for all $0 \leq l \leq k$ then $K(x^1)_k = K(x^2)_k$, i.e., the control at time $k$ is independent of future states. Denote by $\mathcal{K}_{sf}$ the class of such admissible state feedback controllers. We sometimes abuse notation by writing $u_k = K(x_{0,k})$ or $u = K(x)$.

**Problem:** Given $B_0 \subset \mathbf{R}^n$, find a state feedback controller $K \in \mathcal{K}_{sf}$ such that the closed-loop system is LIB dissipative with respect to $B_0$.

For the closed-loop system, we define

$$\beta_a^K(x) \stackrel{\triangle}{=} \sup_{k\geq 0} \sup_{w_{0,k}\in\mathcal{W}_{0,k}} \{g(\xi_k, u_k, w_k) : u = K(\xi)\},\ \forall x \in \mathbf{R}^n \tag{3.3}$$

where $\xi_\cdot$ denotes the corresponding state trajectory of (3.1) with disturbance $w$ and initial condition $\xi_0 = x$. In fact, $\beta_a^K(x)$ is the available storage of the closed-loop system (with controller $K$).

Further define the *state feedback value function*

$$V_a(x) = \inf_{K\in\mathcal{K}_{sf}} \beta_a^K(x),\ \forall x \in \mathbf{R}^n. \tag{3.4}$$

By Assumption 3.1, $V_a(x) \geq \inf_{u\in\mathbf{U}} \sup_{w\in\mathbf{W}} g(x, u, w) = \breve{g}(x) > -\infty, \forall x \in \mathbf{R}^n$. i.e. $V_a : \mathbf{R}^n \to \bar{\mathbf{R}}$.

## 3.2 Dynamic Programming Solution

In Section 2.3 we saw the importance of the LIB dynamic programming inequality (2.10) and equation (2.9) in characterizing LIB dissipation. A similar inequality and equation arises when dynamic programming techniques are applied to the minimax game specified by (3.4). As we shall see, it will be useful to consider the dynamic programming equation or inequality as holding on a subset of the domain of the solution function $V$.

Let $V : \mathbf{R}^n \to \bar{\mathbf{R}}$ and $S \subset \mathrm{dom}V \subset \mathbf{R}^n$. The *state feedback dynamic programming equation* is

$$V(x) = \inf_{u\in\mathbf{U}} \sup_{w\in\mathbf{W}} \max\{g(x, u, w), V(f(x, u, w))\}, \forall x \in S. \tag{3.5}$$

The analogous *state feedback dynamic programming inequality* is

$$V(x) \geq \inf_{u\in\mathbf{U}} \sup_{w\in\mathbf{W}} \max\{g(x, u, w), V(f(x, u, w))\}, \forall x \in S. \tag{3.6}$$

**Remark 3.2** The analogous partial differential equation for the continuous time system $\dot{\xi} = f(\xi, u, w)$, $z = g(\xi, u, w)$, is

$$\inf_{u\in\mathbf{U}} \sup_{w\in\mathbf{W}} \max\{g(x, u, w) - V(x),\ \nabla V(x)f(x, u, w)\} = 0, \quad \forall x \in S \subset \mathrm{dom}V. \tag{3.7}$$

□

The main results of state feedback case are listed below. The proofs are similar to the proofs of corresponding results in [22] and are omitted here due to the limited space, see also [15] and the proofs of the results of measurement feedback case, Section 4.

**Theorem 3.3** *(Necessity) If there exists a state feedback controller $K_0 \in \mathcal{K}_{sf}$ such that the closed-loop system (with $K = K_0$) is LIB dissipative with respect to $B_0$, then the function $V_a : \mathbf{R}^n \to \bar{\mathbf{R}}$ defined by (3.4) satisfies:*

*(i)* $B_0 \subset \mathrm{dom}V_a$;

*(ii)* $V_a(x) \geq \breve{g}(x), \ \forall x \in \mathbf{R}^n$, *where $\breve{g}(x)$ is defined in (3.2);*

*(iii) The dynamic programming relation holds*

$$V_a(x) = \inf_{K \in \mathcal{K}_{sf}} \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \{\max\{\max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j)\} : \xi_0 = x, u_i = K(\xi_{0,i})\}, \quad \forall x \in \mathrm{dom}V_a, \forall j \geq 0, \tag{3.8}$$

*i.e. $V_a$ solves the dynamic programming equation (3.5), with $S = \mathrm{dom}V_a$.*

**Definition 3.4** *Given a function $V : \mathbf{R}^n \to \bar{\mathbf{R}}$ and a nonempty set $S \subset \mathrm{dom}V \subset \mathbf{R}^n$, the pair $(V, S)$ is called a* good solution *of the dynamic programming inequality (3.6) if it satisfies*

*(i) $(V, S)$ is a solution of the dynamic programming inequality (3.6) and there exists $\mathbf{u}^* : S \to \mathbf{U}$ such that*

$$\begin{aligned} &\sup_{w \in \mathbf{W}} \max\{g(x, \mathbf{u}^*(x), w), V(f(x, \mathbf{u}^*(x), w))\} \\ = &\inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} \max\{g(x, u, w), V(f(x, u, w))\}, \forall x \in S. \end{aligned} \tag{3.9}$$

*(ii) $S$ is an invariant set under the closed-loop dynamics when the controller is $\mathbf{u}^*(x)$, i.e. $\forall x \in S, \forall w \in \mathbf{W}, \ f(x, \mathbf{u}^*(x), w) \in S$.*

A controller $K^* \in \mathcal{K}_{sf}$ can be defined by $K^*(x_.)_k = \mathbf{u}^*(x_k)$, for $x_. \in \mathcal{X}_{0,\infty}$ (static state feedback). In this definition, if $S \neq \mathbf{R}^n$ we specify $\mathbf{u}^*(x)$ arbitrarily for $x \notin S$.

**Theorem 3.5** *(Sufficiency) If $(V, S)$ is a good solution of the dynamic programming inequality (3.6), then the closed-loop system (with $K = K^*$) satisfies*

$$z_k \leq V(x_0), \ \forall x_0 \in S, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0.$$

*Moreover, if $B_0 \subset S$, then the closed-loop system is LIB dissipative with respect to $B_0$.*

**Corollary 3.6** *If $(V, S)$ is a good solution of the dynamic programming inequality (3.6), then we have*

$$S \subset \mathrm{dom}V_a; \quad V_a(x) \leq V(x), \forall x \in S. \tag{3.10}$$

*where $V_a$ is the state feedback value function defined in (3.4).*

# 4 Measurement Feedback Synthesis

We turn now to the measurement feedback synthesis problem for the LIB dissipation property. The solution to the problem requires suitable state estimation, in addition to some *concepts* from the state feedback solution.

## 4.1 Problem Statement

Consider the nonlinear discrete-time system:

$$\begin{aligned} \xi_{k+1} &= f(\xi_k, u_k, w_k) \\ z_k &= g(\xi_k, u_k, w_k) \\ y_k &= h(\xi_k, u_k, w_k) \end{aligned} \tag{4.1}$$

Here, $\xi_k \in \mathbf{R}^n, u_k \in \mathbf{U} \subset \mathbf{R}^m, w_k \in \mathbf{W} \subset \mathbf{R}^s, y_k \in \mathbf{R}^p$ and $z_k \in \mathbf{R}$ are the state, control input, disturbance input, measurement output and performance measure, respectively.

We continue to make Assumptions 3.1. We also denote:

$$\mathbf{Y} = \text{range}\{h\}. \tag{4.2}$$

In addition to the notation of the previous sections, we define $y_{0,k}$, $\mathcal{Y}_{0,k}$, $\mathcal{Y}_{0,\infty}$, analogously to (2.2).

An *admissible* measurement feedback controller is a causal map $K : \mathcal{Y}_{0,\infty} \to \mathcal{U}_{0,\infty}$, meaning that for each time $k > 0$ if $y^1, y^2 \in \mathcal{Y}_{0,\infty}$ and $y^1_l = y^2_l$ for all $0 \leq l \leq k-1$ then $K(y^1)_k = K(y^2)_k$, i.e., the control at time $k$ is independent of current and future measurements. Denote by $\mathcal{K}_{mf}$ the class of such admissible controllers. We sometimes abuse notation by writing $u_k = K(y_{0,k-1})$ or $u = K(y)$.

**Problem:** Given $B_0 \subset \mathbf{R}^n$, find a measurement feedback controller $K \in \mathcal{K}_{mf}$ such that the closed-loop system is LIB dissipative with respect to $B_0$.

For a given controller $K \in \mathcal{K}_{mf}$, the available storage of the closed-loop system is

$$\beta_a^K(x_0) = \sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} g(\xi_k, u_k, w_k), \forall x_0 \in \mathbf{R}^n \tag{4.3}$$

where $\xi_\cdot$ denotes the state trajectory of (4.1) with input $u = K(y)$, disturbance $w$ and initial condition $\xi_0 = x_0$.

**Problem Restatement:** Given $B_0 \subset \mathbf{R}^n$, choose $K \in \mathcal{K}_{mf}$ such that $B_0 \subset \text{dom}\beta_a^K$.

## 4.2 Problem Restatement in Terms of a Cost Function

We will solve the optimal $K$ problem using a minimax cost function and information state methods in subsequent sections. The aim of this section is to define a suitable cost function and relate it to LIB dissipation.

Denote

$$\tilde{\mathbf{R}} \stackrel{\triangle}{=} \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}. \tag{4.4}$$

and

$$\tilde{\chi} \stackrel{\triangle}{=} \{p : \mathbf{R}^n \to \tilde{\mathbf{R}}\}. \tag{4.5}$$

For $p \in \tilde{\chi}$, denote

$$\langle p \rangle \stackrel{\triangle}{=} \sup_{x \in \mathbf{R}^n} p(x) \tag{4.6}$$

Notice that $-\infty \leq \langle p \rangle \leq +\infty$. For $p \in \tilde{\chi}$, denote

$$\text{support}p \stackrel{\triangle}{=} \{x \in \mathbf{R}^n \ : \ p(x) > -\infty\} \tag{4.7}$$

Also, for nonempty set $M \subset \mathbf{R}^n$, denote

$$\delta_M(x) \stackrel{\triangle}{=} \begin{cases} 0, & \text{if} \quad x \in M, \\ -\infty, & \text{if} \quad x \notin M. \end{cases} \tag{4.8}$$

Then $\delta_M \in \tilde{\chi}$, $\langle \delta_M \rangle = 0$ and support$\delta_M = M$.

Define, for $p \in \tilde{\chi}$ and $u \in \mathbf{U}$, $y \in \mathbf{Y}$,

$$G(p, u, y) \stackrel{\triangle}{=} \sup_{x \in \mathbf{R}^n} \sup_{w \in \mathbf{W}} \{p(x) + g(x, u, w) : h(x, u, w) = y\}. \tag{4.9}$$

Similar to $\breve{g}(x)$ in the state feedback case, for $p \in \tilde{\chi}$, we define

$$\breve{G}(p) \stackrel{\triangle}{=} \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} G(p, u, y). \tag{4.10}$$

We can prove from Assumption 3.1 that $\breve{G}(p) = -\infty \Leftrightarrow p \equiv -\infty$.

For $p \in \tilde{\chi}$, controller $K \in \mathcal{K}_{mf}$, define the *cost function*

$$J_p(K) \stackrel{\triangle}{=} \sup_{k \geq 0} \sup_{x_0 \in \mathbf{R}^n} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} \{p(x_0) + g(\xi_k, u_k, w_k)\} \tag{4.11}$$

where $\xi_\cdot$ denotes the state trajectory of (4.1) with input $u = K(y)$, disturbance $w$ and initial condition $\xi_0 = x_0$.

The cost function $J_p(K)$ enjoys the following simple properties, and in particular encodes the LIB dissipation property. (see also Lemmas 3.1.2 and 3.1.3 in [15])

**Lemma 4.1** *For $p \in \tilde{\chi}$, the cost function satisfies*

$$J_p(K) = \sup_{x_0 \in \text{support}p} \left\{ p(x_0) + \beta_a^K(x_0) \right\}. \tag{4.12}$$

Proof. By the definition of $J_p(K)$, if $x_0 \notin$ support$p$, then $p(x_0) = -\infty$ and

$$\sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} \{p(x_0) + g(\xi_k, u_k, w_k)\} = -\infty.$$

If $x_0 \in$ support$p$, then

$$\sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} \{p(x_0) + g(\xi_k, u_k, w_k)\} = p(x_0) + \beta_a^K(x_0).$$

Hence the equality (4.12) holds. □

**Lemma 4.2** *The function $p \to J_p(K)$ satisfies (i) Domination. $J_p(K) \geq \breve{G}(p)$, where $\breve{G}(p)$ is defined in (4.10); $J_p(K) = -\infty \Leftrightarrow p \equiv -\infty$. (ii) Monotonicity. $p_1 \geq p_2 \Longrightarrow J_{p_1}(K) \geq J_{p_2}(K)$. (iii) Additive homogeneity. $\forall c \in \mathbf{R}, J_{p+c}(K) = J_p(K) + c$.*

PROOF. These structural properties come directly from Lemma 4.1 and the definition of $\breve{G}(p)$ in (4.10). □

**Lemma 4.3** *The closed-loop system (with controller $K \in \mathcal{K}_{mf}$) is LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$, if and only if $\exists p \in \tilde{\chi}$ with* $\text{support} p = B_0$ *such that*

$$J_p(K) \leq 0. \tag{4.13}$$

PROOF. From Lemma 4.1,

$$\begin{aligned} J_p(K) \leq 0 &\Leftrightarrow \sup_{x_0 \in \text{support} p} \left\{ p(x_0) + \beta_a^K(x_0) \right\} \leq 0 \\ &\Leftrightarrow \beta_a^K(x_0) \leq -p(x_0), \forall x_0 \in \text{support} p \\ &\Leftrightarrow \beta_a^K(x_0) \leq -p(x_0) < +\infty, \forall x_0 \in B_0 \end{aligned}$$

□

**Problem Restatement I:** Given $B_0 \subset \mathbf{R}^n$, choose controller $K \in \mathcal{K}_{mf}$ such that

$$J_p(K) \leq 0 \tag{4.14}$$

for some $p \in \tilde{\chi}$ with $\text{support} p = B_0$.

**Optimal** $K$ **Problem I:** Choose controller $K \in \mathcal{K}_{mf}$ such that $J_p(K)$ is the smallest (over $\mathcal{K}_{mf}$) for some $p \in \tilde{\chi}$ with $\text{support} p = B_0$.

## 4.3 Equivalent Formulation Using Information States

To solve the LIB problem, we introduce a state estimator quantity from which a suitable controller can be determined. This state quantity must be computable from the measurements $(u_., y_.)$ available to the controller, and it must characterize the LIB property. The information state framework of [21], [22], [15] is employed, and we recast the measurement feedback LIB dissipation problem in terms of an equivalent state feedback problem, where the *new* state is an information state.

**Definition 4.4** *For $p_0 \in \tilde{\chi}$, $j \geq 0$, $u_{0,j-1} \in \mathcal{U}_{0,j-1}$, $y_{0,j-1} \in \mathcal{Y}_{0,j-1}$, we define the* information state $p_j : \mathbf{R}^n \to \tilde{\mathbf{R}}$ *by*

$$p_j(x) = \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \sup_{x_0 \in \mathbf{R}^n} \{ p_0(x_0) : \xi_0 = x_0, \xi_j = x, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq j-1 \} \tag{4.15}$$

*where $\xi_i$ satisfies*

$$\xi_{i+1} = f(\xi_i, u_i, w_i), \quad 0 \leq i \leq j-1. \tag{4.16}$$

The next lemma shows that the information state characterizes the LIB dissipation property.

**Lemma 4.5** *The closed-loop system (with controller $K \in \mathcal{K}_{mf}$) is LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$, if and only if there exists $p_0 \in \tilde{\chi}$ with* support$p_0 = B_0$ *such that*

$$G(p_k, u_k, y_k) \leq 0, \quad \forall k \geq 0, \tag{4.17}$$

*where $p_k$ is defined by (4.15) with initial state $p_0$ and $u = K(y)$, $y$ is the measurement output of (4.1) for any initialization $x_0 \in B_0$ and disturbance $w$, and $G(p, u, y)$ is defined in (4.9).*

Proof. Assume (4.17) holds as stated. $\forall x_0 \in B_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}$, denote by $\xi_k$, $u_k$ and $y_k$ the corresponding state, control and measurement output trajectories of the closed-loop system. By Definition 4.4 we have

$$p_0(x_0) + g(\xi_k, u_k, w_k) \leq p_k(\xi_k) + g(\xi_k, u_k, w_k) \leq G(p_k, u_k, y_k) \leq 0.$$

i.e.

$$z_k = g(\xi_k, u_k, w_k) \leq -p_0(x_0).$$

This implies that the closed-loop system is LIB dissipative with respect to $B_0$ where $\beta(x_0) = -p_0(x_0)$.

Conversely, suppose that the closed-loop system is LIB dissipative with respect to $B_0$ and $\beta(x_0)$. Choose $p_0(x_0) = -\beta(x_0)$, then support$p_0 = B_0$ and $\forall k \geq 0$,

$$\begin{aligned}
& G(p_k, u_k, y_k) \\
= & \sup_{x \in \mathbf{R}^n} \sup_{w \in \mathbf{W}} \{p_k(x) + g(x, u_k, w) : h(x, u_k, w) = y_k\} \\
= & \sup_{x \in \mathbf{R}^n} \sup_{w_k \in \mathbf{W}} \{ \sup_{w_{0,k-1}} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) : \xi_0 = x_0, \xi_k = x, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq k-1\} \\
& + g(x, u_k, w_k) : h(x, u_k, w_k) = y_k\} \\
= & \sup_{w_{0,k}} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) + g(\xi_k, u_k, w_k) : \xi_0 = x_0, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq k\} \\
= & \sup_{w_{0,k}} \sup_{x_0 \in B_0} \{g(\xi_k, u_k, w_k) - \beta(x_0) : \xi_0 = x_0, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq k\} \\
\leq & \ 0.
\end{aligned}$$

□

It can be readily checked that the information state satisfies a recursion of the form

$$p_{j+1}(x) = F(p_j, u_j, y_j)(x), \quad j \geq 0 \tag{4.18}$$

where

$$F(p, u, y)(x) = \sup_{w \in \mathbf{W}} \sup_{\xi \in \mathbf{R}^n} \{p(\xi) : f(\xi, u, w) = x, h(\xi, u, w) = y\} \tag{4.19}$$

(see, e.g., [22, Lemma 4.4]).

**Remark 4.6** The analogous partial differential equation for the continuous time system $\dot{\xi} = f(\xi, u, w)$, $z = g(\xi, u, w)$, $y = h(\xi, u, w)$ is

$$\frac{\partial}{\partial t} p_t(x) = \sup\{-\nabla p_t(x) \cdot f(x, u, w) \ : \ w \in \mathbf{W} \text{ and } y = h(x, u, w)\} \tag{4.20}$$

Thus

$$\dot{p} = F(p, u, y)$$

where $F$ is defined by the RHS of (4.20). □

In the definition of information state, $u$ and $y$ are independent. Now consider the case when the controller $K$ is known, and $u = K(y)$.

For $p \in \tilde{\chi}$, define

$$\bar{J}_p(K) \stackrel{\triangle}{=} \sup_{k \geq 0} \sup_{y_{0,k} \in \mathcal{Y}_{0,k}} \{G(p_k, u_k, y_k) : p_0 = p, u = K(y)\} \tag{4.21}$$

where $p_k(x)$ are information states obtained by (4.15), and $G(p, u, y)$ is defined in (4.9).

The relation between the cost function and the information state is given in the following theorem. The proof is similar to the proof of [15, Theorem 3.1.8] and is omitted here.

**Theorem 4.7** *We have, for all $K \in \mathcal{K}_{mf}$,*

$$\bar{J}_p(K) = J_p(K). \tag{4.22}$$

**Problem Restatement II:** Given $B_0 \subset \mathbf{R}^n$, choose a controller $K \in \mathcal{K}_{mf}$ such that

$$\bar{J}_p(K) \leq 0.$$

for some $p \in \tilde{\chi}$ with support$p = B_0$.

**Optimal $K$ Problem II:** Choose a controller $K \in \mathcal{K}_{mf}$ such that $\bar{J}_p(K)$ is the smallest for some $p \in \tilde{\chi}$ with support$p = B_0$.

## 4.4 Dynamic Programming Solution

In this section we show how to synthesize LIB dissipative controllers by finding optimal minimax controllers solving the Optimal $K$ Problem II. We will make use of a dynamic programming equation and inequality analogous to (3.5) and (3.6).

Let $W : \tilde{\chi} \to \tilde{\mathbf{R}}$, and define

$$\mathrm{dom}W \stackrel{\triangle}{=} \{p \in \tilde{\chi} : -\infty < W(p) < +\infty\} \tag{4.23}$$

where $\tilde{\chi}$ is the function space defined by (4.5). Let $\tilde{S} \subset \mathrm{dom}W \subset \tilde{\chi}$. The *measurement feedback dynamic programming equation* is

$$W(p) = \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{G(p, u, y), W(F(p, u, y))\}, \quad \forall p \in \tilde{S}, \tag{4.24}$$

where $G(p,u,y)$ and $F(p,u,y)$ are defined in (4.9) and (4.19). The *measurement feedback dynamic programming inequality* is

$$W(p) \geq \inf_{u\in\mathbf{U}} \sup_{y\in\mathbf{Y}} \max\{G(p,u,y), W(F(p,u,y))\}, \quad \forall p \in \tilde{S}. \tag{4.25}$$

**Remark 4.8** Notice that (3.5), (3.6) and (4.24), (4.25) have the same *form*, respectively.

**Remark 4.9** The analogous partial differential equation for the continuous time system $\dot{\xi} = f(\xi,u,w)$, $z = g(\xi,u,w)$, $y = h(\xi,u,w)$ is

$$\inf_{u\in\mathbf{U}} \sup_{y\in\mathbf{Y}} \max\{G(p,u,y) - W(p), \nabla W(p)F(p,u,y)\} = 0, \quad \forall p \in \tilde{S} \subset \text{dom}W. \tag{4.26}$$

□

### 4.4.1 Necessity

For $p \in \tilde{\chi}$, define the *measurement feedback value function*

$$W_a(p) \stackrel{\triangle}{=} \inf_{K\in\mathcal{K}_{mf}} \bar{J}_p(K) = \inf_{K\in\mathcal{K}_{mf}} \sup_{k\geq 0} \sup_{y_{0,k}\in\mathcal{Y}_{0,k}} \{G(p_k,u_k,y_k) : p_0 = p, u = K(y)\} \tag{4.27}$$

where the minimization ranges over the class of all the admissible measurement feedback controllers $\mathcal{K}_{mf}$. Notice that $W_a : \tilde{\chi} \to \tilde{\mathbf{R}}$, i.e., $-\infty \leq W_a(p) \leq +\infty$.

For a given controller $K \in \mathcal{K}_{mf}$ and $B \subset \mathbf{R}^n$, denote

$$\mathcal{Y}_{0,k}(K,B) \stackrel{\triangle}{=} \{y_{0,k} : \exists x_0 \in B, \exists w_{0,k} \in \mathcal{W}_{0,k}, \text{ s.t. } y_i = h(\xi_i,u_i,w_i), 0 \leq i \leq k\}. \tag{4.28}$$

i.e. $\mathcal{Y}_{0,k}(K,B)$ denotes all the possible measurement output $y_{0,k}$ of the closed-loop system (with controller $K \in \mathcal{K}_{mf}$) where the initial state $x_0$ contains in the set $B$.

**Theorem 4.10** *Assume that there exists an admissible measurement feedback controller $K_0$ such that the closed-loop system is LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$, $\beta_{K_0}(x_0)$. Then:*

*(i) The set* $\text{dom}W_a$ *is nonempty,* $p_0 = -\beta_{K_0} \in \text{dom}W_a$.

*(ii) The following structural properties hold:*

*(ii-a)* $W_a$ *dominates* $\breve{G}$*:* $W_a(p) \geq \breve{G}(p), \forall p \in \tilde{\chi}$*, where* $\breve{G}(p)$ *is defined in (4.10);* $W_a(p) = -\infty \Leftrightarrow p(x) \equiv -\infty$.

*(ii-b)* $W_a$ *is monotone:* $\forall p_1, p_2 \in \tilde{\chi}$*, if* $p_1 \geq p_2$*, then* $W_a(p_1) \geq W_a(p_2)$*. Moreover, if* $p_1 \in \text{dom}W_a$ *and* $\langle p_2 \rangle > -\infty$*, then* $p_2 \in \text{dom}W_a$.

*(ii-c)* $W_a$ *is additive homogeneous:* $\forall c \in \mathbf{R}, \forall p \in \text{dom}W_a, W_a(p+c) = W_a(p) + c$.

*(iii) Fix $p \in \tilde{\chi}$ and assume $\bar{J}_p(K_0)$ is finite, then*

$$\begin{aligned} \bar{J}_p(K_0) \geq \max\{\max_{0\leq i\leq j-1} G(p_i, u_i, y_i), W_a(p_j)\} \text{ and } p_j \in \text{dom}W_a, \\ \forall j \geq 0, \forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_0, \text{support}p), u = K_0(y), p_0 = p. \end{aligned} \tag{4.29}$$

*where $\mathcal{Y}_{0,k}(K, B)$ is defined by (4.28).*

*(iv) Fix $p \in \text{dom}W_a$, if $K_\varepsilon$ is an $\varepsilon$-optimal controller (i.e., $\bar{J}_p(K_\varepsilon) \leq W_a(p) + \varepsilon$), then*

$$\begin{aligned} W_a(p) + \varepsilon \geq \max\{\max_{0\leq i\leq j-1} G(p_i, u_i, y_i), W_a(p_j)\} \text{ and } p_j \in \text{dom}W_a, \\ \forall j \geq 0, \forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_\varepsilon, \text{support}p), u = K_\varepsilon(y), p_0 = p. \end{aligned} \tag{4.30}$$

*(v) The dynamic programming relation holds:*

$$W_a(p) = \inf_{K\in\mathcal{K}_{mf}} \sup_{y_{0,j-1}\in\mathcal{Y}_{0,j-1}} \{\max\{\max_{0\leq i\leq j-1} G(p_i, u_i, y_i), W_a(p_j)\} : p_0 = p, u = K(y)\}, \quad \forall p \in \text{dom}W_a, \forall j \geq 0 \tag{4.31}$$

*i.e. $W_a$ solves the dynamic programming equation (4.24), with $\tilde{S} = \text{dom}W_a$.*

Proof. (i) From Lemma 4.2 and Theorem 4.7, $\forall p \in \tilde{\chi}, \forall K, \bar{J}_p(K) = J_p(K) \geq \breve{G}(p)$, hence

$$W_a(p) = \inf_{K\in\mathcal{K}_{mf}} \bar{J}_p(K) \geq \breve{G}(p).$$

For $p_0 = -\beta_{K_0}$,

$$-\infty < \breve{G}(-\beta_{K_0}) = \breve{G}(p_0) \leq W_a(p_0) = \inf_{K\in\mathcal{K}_{mf}} \bar{J}_{p_0}(K) \leq \bar{J}_{p_0}(K_0) \leq 0 \tag{4.32}$$

hence $p_0 \in \text{dom}W_a$, so $\text{dom}W_a$ is nonempty.

(ii-a) The domination is proved in (i). If $p(x) \equiv -\infty$, then $\bar{J}_p(K) = -\infty$ for any $K$, hence $W_a(p) = -\infty$. If $\langle p\rangle > -\infty$, then $W_a(p) \geq \breve{G}(p) > -\infty$.

For (ii-b), from Lemma 4.2, $p_1 \geq p_2 \Rightarrow \forall K, \bar{J}_{p_1}(K) \geq \bar{J}_{p_2}(K) \Rightarrow W_a(p_1) \geq W_a(p_2)$. Moreover, if $p_1 \in \text{dom}W_a$ and $\langle p_2\rangle > -\infty$, then $-\infty < \breve{G}(p_2) \leq W_a(p_2) \leq W_a(p_1) < +\infty$, hence $p_2 \in \text{dom}W_a$. (Notice that for $p \in \tilde{\chi}$, $\langle p\rangle > -\infty$ if and only if $\breve{G}(p) > -\infty$.)

(ii-c) is obvious since $\forall K \in \mathcal{K}_{mf}, \bar{J}_{p+c}(K) = \bar{J}_p(K) + c$. (Lemma 4.2)

(iii) If $j = 0$, $\bar{J}_p(K_0) \geq W_a(p_0) \geq \breve{G}(p_0) = \breve{G}(p) > -\infty$, hence $p_0 \in \text{dom}W_a$. (If $p \equiv -\infty$, then $\bar{J}_p(K_0) = -\infty$ is not finite, so $\langle p\rangle > -\infty$)

Now fix $j > 0$ and fix $y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_0, \text{support}p)$, then we can obtain $p_j$ by $p_0 = p$ and $u_i = K_0(y_{0,i-1}), 0 \leq i \leq j-1$. From $y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_0, \text{support}p)$, $\langle p_j\rangle > -\infty$.

For any $k \geq 0$ and any $\tilde{y}_{0,k}$, denote

$$\hat{y}_i = \begin{cases} y_i, & 0 \leq i \leq j-1 \\ \tilde{y}_{i-j}, & i \geq j \end{cases} \tag{4.33}$$

and

$$K_0^j(\tilde{y}_{0,k-1}) = K_0(\hat{y}_{0,k+j-1}), \quad k \geq 0. \tag{4.34}$$

Then we can prove that

$$\begin{aligned}
\bar{J}_p(K_0) &= \sup_{k\geq 0}\sup_{\bar{y}_{0,k}}\{G(p_k,u_k,\bar{y}_k): p_0=p, u=K_0(\bar{y})\}\\
&\geq \sup_{k\geq 0}\sup_{\hat{y}_{0,k}}\{G(p_k,u_k,\hat{y}_k): p_0=p, u=K_0(\hat{y})\}\\
&= \max\{\sup_{0\leq k\leq j-1}\{G(p_k,u_k,y_k): p_0=p, u_i=K_0(y_{0,i-1}), 0\leq i\leq j-1\},\\
&\quad \sup_{k\geq j}\sup_{\hat{y}_{0,k}}\{G(p_k,u_k,\hat{y}_k): p_0=p, u=K_0(\hat{y})\}\}\\
&= \max\{\{\max_{0\leq k\leq j-1} G(p_k,u_k,y_k): p_0=p, u_i=K_0(y_{0,i-1}), 0\leq i\leq j-1\},\\
&\quad \sup_{k\geq 0}\sup_{\tilde{y}_{0,k}}\left\{G(p_k,u_k,\tilde{y}_k): p_0=p_j, u=K_0^j(\tilde{y})\right\}\}\\
&= \max\{\{\max_{0\leq k\leq j-1} G(p_k,u_k,y_k): p_0=p, u_i=K_0(y_{0,i-1}), 0\leq i\leq j-1\}, \bar{J}_{p_j}(K_0^j)\}\\
&\geq \max\{\{\max_{0\leq k\leq j-1} G(p_k,u_k,y_k): p_0=p, u_i=K_0(y_{0,i-1}), 0\leq i\leq j-1\}, W_a(p_j)\}.
\end{aligned}$$

Hence

$$+\infty > \bar{J}_p(K_0) \geq W_a(p_j) \geq \breve{G}(p_j) > -\infty$$

and $p_j \in \mathrm{dom}W_a$.

(iv) Apply part (iii) with $K_0 = K_\varepsilon$.

(v) $\forall p \in \mathrm{dom}W_a, W_a(p)$ is finite, $\forall\varepsilon > 0$, $\exists K_\varepsilon$ such that $\bar{J}_p(K_\varepsilon) \leq W_a(p)+\varepsilon$ ($K_\varepsilon$ is an $\varepsilon$-optimal controller), from (iv), $\forall j \geq 0$,

$$\begin{gathered}
W_a(p)+\varepsilon \geq \max\{\max_{0\leq i\leq j-1} G(p_i,u_i,y_i), W_a(p_j)\},\\
\forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_\varepsilon, \mathrm{support}p), p_0=p, u=K_\varepsilon(y).
\end{gathered}$$

hence

$$\begin{aligned}
&W_a(p)+\varepsilon\\
\geq &\sup_{y_{0,j-1}\in\mathcal{Y}_{0,j-1}(K_\varepsilon,\mathrm{support}p)}\left\{\max\{\max_{0\leq i\leq j-1} G(p_i,u_i,y_i), W_a(p_j)\}: p_0=p, u=K_\varepsilon(y)\right\}\\
= &\sup_{y_{0,j-1}\in\mathcal{Y}_{0,j-1}}\left\{\max\{\max_{0\leq i\leq j-1} G(p_i,u_i,y_i), W_a(p_j)\}: p_0=p, u=K_\varepsilon(y)\right\}
\end{aligned}$$

certainly

$$W_a(p)+\varepsilon \geq \inf_{K\in\mathcal{K}_{mf}}\sup_{y_{0,j-1}\in\mathcal{Y}_{0,j-1}}\{\max\{\max_{0\leq i\leq j-1} G(p_i,u_i,y_i), W_a(p_j)\}: p_0=p, u=K(y)\}$$

since $\varepsilon$ is arbitrary, we have

$$W_a(p) \geq \inf_{K\in\mathcal{K}_{mf}}\sup_{y_{0,j-1}\in\mathcal{Y}_{0,j-1}}\{\max\{\max_{0\leq i\leq j-1} G(p_i,u_i,y_i), W_a(p_j)\}: p_0=p, u=K(y)\}. \tag{4.35}$$

To prove the opposite inequality, for $p \in \mathrm{dom}W_a, j \geq 0$, define

$$R(p) = \inf_{K\in\mathcal{K}_{mf}}\sup_{y_{0,j-1}\in\mathcal{Y}_{0,j-1}}\{\max\{\max_{0\leq i\leq j-1} G(p_i,u_i,y_i), W_a(p_j)\}: p_0=p, u=K(y)\}. \tag{4.36}$$

From (4.35), $R(p) \leq W_a(p) < +\infty$. Moreover, $R(p) \geq \breve{G}(p) > -\infty$, So $R(p)$ is a finite number.

Let $\varepsilon > 0$, choose $K^1$ such that

$$\begin{aligned} & R(p) \\ \geq & \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}} \left\{ \max\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j)\} : p_0 = p, u = K^1(y) \right\} - \varepsilon \\ = & \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K^1, \text{support}p)} \left\{ \max\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j)\} : p_0 = p, u = K^1(y) \right\} - \varepsilon. \end{aligned} \tag{4.37}$$

(Therefore we have $p_j \in \text{dom}W_a, \forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K^1, \text{support}p), u = K^1(y)$.)

$\forall q \in \text{dom}W_a$, $\exists K_q^2$ such that

$$W_a(q) \geq \bar{J}_q(K_q^2) - \varepsilon.$$

Define $K^3$ by

$$K^3(y_{0,i-1}) = \begin{cases} K^1(y_{0,i-1}), & 0 \leq i \leq j-1 \\ K_{p_j}^2(y_{j,i-1}), & i \geq j \end{cases} \tag{4.38}$$

Then $\forall k \geq j, \forall y_{0,k} \in \mathcal{Y}_{0,k}(K^3, \text{support}p)$, we have $y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K^1, \text{support}p)$ and

$$\begin{aligned} R(p) & \geq \{\max\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j)\} : p_0 = p, u = K^1(y)\} - \varepsilon \\ & \geq \{\max\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), \bar{J}_{p_j}(K_{p_j}^2)\} : p_0 = p, u = K^1(y)\} - 2\varepsilon \\ & \geq \{ \sup_{0 \leq i \leq k} G(p_i, u_i, y_i) : p_0 = p, u = K^3(y)\} - 2\varepsilon. \end{aligned} \tag{4.39}$$

So we have

$$R(p) \geq \bar{J}_p(K^3) - 2\varepsilon$$

and hence

$$W_a(p) \leq \bar{J}_p(K^3) \leq R(p) + 2\varepsilon.$$

Since $\varepsilon$ is arbitrary, we have

$$W_a(p) \leq R(p). \tag{4.40}$$

From (4.35) and (4.40), the proof is completed. □

### 4.4.2 Information State Controllers

In state feedback synthesis of Section 3.2, the optimal controller was obtained by minimizing the RHS of the dynamic programming equation or inequality over $u$, to yield a static state feedback controller. We follow the same procedure in the measurement feedback case (next subsection), and obtain an optimal controller defined in terms of a static function of the information state. This gives a dynamic controller, a causal function of the measurements, of the type we now describe.

Let $\mathbf{u}$ be a function

$$\mathbf{u} : \tilde{\chi} \to \mathbf{R}^m. \tag{4.41}$$

For $p_0 \in \tilde{\chi}$, define

$$u_k = \mathbf{u}(p_k), \quad k \geq 0, \tag{4.42}$$

where $p_{i+1} = F(p_i, u_i, y_i)$, $\forall i \geq 0$. Since $p_{k+1} = F(p_k, \mathbf{u}(p_k), y_k)$, the information state $p_{k+1}$ can be regarded as a function of $y_{0,k}$ and $u_k$ is a function of $y_{0,k-1}$. This leads us to define the measurement feedback controller $K^{\mathbf{u}}_{p_0} \in \mathcal{K}_{mf}$ by

$$K^{\mathbf{u}}_{p_0}(y.)_k = K^{\mathbf{u}}_{p_0}(y_{0,k-1}) = \mathbf{u}(p_k). \tag{4.43}$$

This *information state feedback controller* is a special kind of measurement feedback controller.

### 4.4.3 Sufficiency

We will show how to obtain the optimal information state controller from dynamic programming inequality.

**Definition 4.11** *Given a function $W : \tilde{\chi} \to \tilde{\mathbf{R}}$ and a nonempty set $\tilde{S} \subset \mathrm{dom}W \subset \tilde{\chi}$, the pair $(W, \tilde{S})$ is called a* good *solution of the dynamic programming inequality (4.25) if*

*(i) $(W, \tilde{S})$ is a solution of (4.25) and there exists $\mathbf{u}^* : \tilde{S} \to \mathbf{U}$ such that $\forall p \in \tilde{S}$,*

$$\begin{aligned} &\sup_{y \in \mathbf{Y}} \max\{G(p, \mathbf{u}^*(p), y), W(F(p, \mathbf{u}^*(p), y))\} \\ =\ &\inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{G(p, u, y), W(F(p, u, y))\}. \end{aligned} \tag{4.44}$$

*(ii) $\tilde{S}$ is an invariant set under the closed-loop dynamics when the controller is $\mathbf{u}^*(p), \forall p \in \tilde{S}$. i.e. $\forall p \in \tilde{S}, \forall y \in \mathcal{Y}_{0,0}(K^{\mathbf{u}^*}_p, \mathrm{support}p)$, $F(p, \mathbf{u}^*(p), y) \in \tilde{S}$ (here $K^{\mathbf{u}^*}_p$ is defined in (4.43) and $\mathcal{Y}_{0,k}(K, B)$ is defined by (4.28)).*

**Theorem 4.12** *Assume that $(W, \tilde{S})$ is a good solution of the dynamic programming inequality (4.25) and $p_0 \in \tilde{S}$. Then the closed-loop system with the controller defined by*

$$u_k = \mathbf{u}^*(p_k) \tag{4.45}$$

*satisfies $\forall x_0 \in \mathrm{support}p_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}$,*

$$z_k \leq W(p_0) - p_0(x_0). \tag{4.46}$$

*Moreover, if $B_0 \subset \mathrm{support}p_0$, then the closed-loop system is LIB dissipative with respect to $B_0$ where $\beta(x_0) = W(p_0) - p_0(x_0)$.*

Proof. Let $K^{\mathbf{u}^*}_{p_0}$ denote the information state controller obtained by (4.45). (Notice that $K^{\mathbf{u}^*}_{p_0}$ depends on the initial information state $p_0$ though $\mathbf{u}^*(\cdot)$ doesn't.) Now from (4.25) and (4.44), we have

$$W(p) \geq \sup_{y \in \mathbf{Y}} W(F(p, \mathbf{u}^*(p), y)), \forall p \in \tilde{S}.$$

Hence $\forall j \geq 0, \forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K^{\mathbf{u}^*}_{p_0}, \text{support} p_0)$,

$$W(p_i) \geq W(p_{i+1}), \quad 0 \leq i \leq j-1, p_0 = p_0, u = K^{\mathbf{u}^*}_{p_0}(y)$$

and certainly

$$W(p_j) \leq W(p_0).$$

Also from (4.25) and (4.44), we have

$$G(p_j, \mathbf{u}^*(p_j), y_j) \leq W(p_j),$$

hence

$$J_{p_0}(K^{\mathbf{u}^*}_{p_0}) = \bar{J}_{p_0}(K^{\mathbf{u}^*}_{p_0}) = \sup_{j \geq 0} \sup_{y_{0,j}} G(p_j, \mathbf{u}^*(p_j), y_j) \leq W(p_0).$$

For the closed-loop system, $\forall x_0 \in \text{support} p_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}$,

$$p_0(x_0) + z_k \leq p_k(\xi_k) + g(\xi_k, \mathbf{u}^*(p_k), w_k) \leq G(p_k, \mathbf{u}^*(p_k), y_k) \leq J_{p_0}(K^{\mathbf{u}^*}_{p_0}) \leq W(p_0)$$

and hence (4.46) holds. Moreover, when $B_0 \subset \text{support} p_0$,

$$z_k \leq W(p_0) - p_0(x_0), \quad \forall x_0 \in B_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}.$$

Therefore the closed-loop system is LIB dissipative with respect to $B_0$ where $\beta(x_0) = W(p_0) - p_0(x_0)$. □

**Corollary 4.13** *If $(W, \tilde{S})$ is a good solution of the dynamic programming inequality (4.25), then we have*

$$\tilde{S} \subset \text{dom} W_a, \quad W_a(p) \leq W(p), \forall p \in \tilde{S}. \tag{4.47}$$

*where $W_a$ is the measurement feedback value function defined in (4.27).*

PROOF. From the proof of Theorem 4.12, $\forall p_0 \in \tilde{S}$,

$$\bar{J}_{p_0}(K^{\mathbf{u}^*}_{p_0}) = J_{p_0}(K^{\mathbf{u}^*}_{p_0}) \leq W(p_0).$$

Hence

$$W_a(p_0) \leq \bar{J}_{p_0}(K^{\mathbf{u}^*}_{p_0}) \leq W(p_0) < +\infty.$$

Since we also have $W_a(p_0) \geq \breve{G}(p_0) > -\infty$, (4.47) holds. □

**Remark 4.14** We know from Theorem 4.10 that $(W_a, \text{dom} W_a)$ is a solution to the dynamic programming inequality (4.25), so it follows that if $(W_a, \text{dom} W_a)$ is in fact a *good solution*, then the controller $K^*_a(y.)_k = \mathbf{u}^*_a(p_k)$ obtained from it achieves the best LIB performance possible (i.e. it achieves the smallest bound $\beta(x_0) = W_a(p_0) - p_0(x_0)$ possible in (2.3) (see Definition 2.1). If the LIB dissipation control problem is solvable by *some* measurement feedback controller, then it is also solvable by an information state feedback controller whenever a good solution to the dynamic programming inequality (4.25) exists. $(W_a, \text{dom} W_a)$ will be a solution, and its "goodness" depends on the attainment of the infimum in item (i) of Definition 4.11. Similar comments can be made for the state feedback case. □

**Remark 4.15** It will be of particular interest to find good solutions $(W, \tilde{S})$ with $\tilde{S}$ finite dimensional (see Sections 5.1 and 5.2). □

## 4.5 Special Case: Uniform LIB

In [28], the authors formulate the problem of obtaining *uniform* bounds on the LIB performance. In this section, we consider this uniform LIB case. The results of Section 4.4 simplify, and the connection with the Shamma-Tu separation structure [28] is given.

**Problem:** Given $B_0 \subset \mathbf{R}^n$, find a measurement feedback controller $K \in \mathcal{K}_{mf}$ such that the closed-loop system is *uniform LIB dissipative with respect to* $B_0$, i.e. there exists a $\beta \in \mathbf{R}$ such that

$$z_k \leq \beta, \quad \forall x_0 \in B_0, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0. \tag{4.48}$$

For this special problem, since $\beta$ does not depend on $x_0$, we can constrain the information states in the subset

$$\bar{S} \stackrel{\triangle}{=} \{\delta_X : X \subset \mathbf{R}^n\} \subset \tilde{\chi}, \tag{4.49}$$

where $\delta_M$ is defined in (4.8). Notice that $\bar{S}$ is an invariant set under the recursion (4.18). Write

$$\bar{S}' \stackrel{\triangle}{=} \{\text{subsets of } \mathbf{R}^n\}. \tag{4.50}$$

Then it is easy to check that $F(\delta_X, u, y) \in \bar{S}$ for all $X \in \bar{S}'$, $u \in \mathbf{U}$ and $y \in \mathbf{Y}$.

Now we choose the initial information state

$$p_0(x) = \delta_{X_0}(x)$$

where $X_0 \subset \mathbf{R}^n$ (i.e. $X_0 \in \bar{S}'$). Then we have

$$p_k(x) = \delta_{X_k}(x)$$

where

$$X_k = \text{support} p_k \tag{4.51}$$

and $x \in X_k$ if and only if there exists a trajectory $\xi_\cdot$ of (4.1) with $\xi_0 \in X_0$ that is consistent with the given signals $u$, $y$ (satisfying (4.1)). This is the *set-valued observer* in [28, page 259].

The set-valued state estimate $X_k$ can be computed from the recursion

$$X_{k+1} = \hat{F}(X_k, u_k, y_k) \stackrel{\triangle}{=} F(\delta_{X_k}, u_k, y_k) \tag{4.52}$$

where, given $X \in \bar{S}'$, $u \in \mathbf{U}$, $y \in \mathbf{Y}$, $x \in \hat{F}(X, u, y)$ if and only if there exists $x' \in X$, $w \in \mathbf{W}$ such that $f(x', u, w) = x$ and $h(x', u, w) = y$ (cf. [28, equation (6)]).

These considerations allow us to restrict our attention to the "smaller" space $\bar{S}' \approx \bar{S}$, in place of $\tilde{\chi}$. Any function $W : \tilde{\chi} \to \tilde{\mathbf{R}}$ projects to (or defines) a function $\hat{W} : \bar{S}' \to \tilde{\mathbf{R}}$ via

$$\hat{W}(X) \stackrel{\triangle}{=} W(\delta_X)$$

for $X \in \bar{S}'$. Now, denote

$$\hat{G}(X, u, y) \stackrel{\triangle}{=} G(\delta_X, u, y), \tag{4.53}$$

and

$$\breve{\hat{G}}(X) \stackrel{\triangle}{=} \breve{G}(\delta_X). \tag{4.54}$$

Let $\hat{W} : \bar{S}' \to \tilde{\mathbf{R}}$, and define

$$\mathrm{dom}\hat{W} \triangleq \left\{ X \in \bar{S}' : -\infty < \hat{W}(X) < +\infty \right\}. \tag{4.55}$$

Let $\hat{S} \subset \mathrm{dom}\hat{W} \subset \bar{S}'$. The measurement feedback dynamic programming equation (4.24) becomes

$$\hat{W}(X) = \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{\hat{G}(X,u,y), \hat{W}(\hat{F}(X,u,y))\}, \quad \forall X \in \hat{S}, \tag{4.56}$$

The measurement feedback dynamic programming inequality (4.25) becomes

$$\hat{W}(X) \geq \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{\hat{G}(X,u,y), \hat{W}(\hat{F}(X,u,y))\}, \quad \forall X \in \hat{S}. \tag{4.57}$$

The results of Section 4.4 become the following.

**Lemma 4.16** *The closed-loop system (with controller $K \in \mathcal{K}_{mf}$) is uniform LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$ if and only if there exists $\beta \in \mathbf{R}$ such that*

$$\hat{G}(X_k, u_k, y_k) \leq \beta, \quad \forall k \geq 0, \tag{4.58}$$

*where $X_k$ is defined by (4.52) with initial state $X_0 = B_0$, $u = K(y)$ and $y$ is the measurement output of (4.1) for any initialization $x_0 \in B_0$ and disturbance $w$.*

We define the value function $\hat{W}_a : \bar{S}' \to \tilde{\mathbf{R}}$ by

$$\hat{W}_a(X) \triangleq \inf_{K \in \mathcal{K}_{mf}} \sup_{k \geq 0} \sup_{y_{0,k} \in \mathcal{Y}_{0,k}} \left\{ \hat{G}(X_k, u_k, y_k) : \ X_0 = X, u = K(y) \right\}. \tag{4.59}$$

**Theorem 4.17** *(Necessity) Assume that there exists a controller $K_0$ such that the closed-loop system is uniform LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$. Then the value function $\hat{W}_a(X)$ defined by (4.59) satisfies:*

*(i)* $\mathrm{dom}\hat{W}_a$ *is nonempty,* $B_0 \in \mathrm{dom}\hat{W}_a$*;*

*(ii)* $\hat{W}_a(X) \geq \breve{\hat{G}}(X), \forall X \subset \mathbf{R}^n$*, where* $\breve{\hat{G}}(X)$ *is defined in (4.54).* $\hat{W}_a(X) = -\infty \Leftrightarrow X = \emptyset$*;* $X_1 \subset X_2 \Rightarrow \hat{W}_a(X_1) \leq \hat{W}_a(X_2)$*, if* $X_1 \subset X_2, X_1 \neq \emptyset$ *and* $X_2 \in \mathrm{dom}\hat{W}_a$*, then* $X_1 \in \mathrm{dom}\hat{W}_a$*;*

*(iii)* $\hat{W}_a(X)$ *satisfies the dynamic programming relation*

$$\hat{W}_a(X) = \inf_{K \in \mathcal{K}_{mf}} \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}} \{\max\{\max_{0 \leq i \leq j-1} \hat{G}(X_i, u_i, y_i), \hat{W}_a(X_j)\} : X_0 = X, u = K(y)\}, \quad \forall X \in \mathrm{dom}\hat{W}_a, \forall j \geq 0. \tag{4.60}$$

*i.e.* $\hat{W}_a(X)$ *is a solution of the dynamic programming equation (4.56) with* $\hat{S} = \mathrm{dom}\hat{W}_a$*.*

**Definition 4.18** *Given a function* $\hat{W} : \bar{S}' \to \tilde{\mathbf{R}}$ *and a nonempty set* $\hat{S} \subset \mathrm{dom}\hat{W} \subset \bar{S}'$, *the pair* $(\hat{W}, \hat{S})$ *is said to be a* good solution *of the dynamic programming inequality (4.57) provided*

*(i)* $(\hat{W}, \hat{S})$ *is a solution of (4.57) and* $\forall X \in \hat{S}$, *there exists* $\hat{\mathbf{u}}^* : \hat{S} \to \mathbf{U}$ *such that*

$$\begin{aligned}&\sup_{y\in\mathbf{Y}} \max\{\hat{G}(X, \hat{\mathbf{u}}^*(X), y), \hat{W}(\hat{F}(X, \hat{\mathbf{u}}^*(X), y))\} \\ =\ &\inf_{u\in\mathbf{U}} \sup_{y\in\mathbf{Y}} \max\{\hat{G}(X, u, y), \hat{W}(\hat{F}(X, u, y))\}, \forall X \in \hat{S}.\end{aligned} \tag{4.61}$$

*(ii)* $\hat{S}$ *is an invariant set under the closed-loop dynamics when the controller is* $\hat{\mathbf{u}}^*(X), \forall X \in \hat{S}$. *i.e.* $\forall X \in \hat{S}, \forall y \in \mathcal{Y}_{0,0}(K_X^{\hat{\mathbf{u}}^*}, X)$, $\hat{F}(X, \hat{\mathbf{u}}^*(X), y) \in \hat{S}$ *(here* $K_X^{\hat{\mathbf{u}}^*}$ *is defined similarly as (4.43) and* $\mathcal{Y}_{0,k}(K, B)$ *is defined by (4.28)).*

**Theorem 4.19** *(Sufficiency) Assume that* $(\hat{W}, \hat{S})$ *is a good solution of the dynamic programming inequality (4.57) and* $X_0 \in \hat{S}$. *Then the closed-loop system with the controller defined by*

$$u_k = \hat{\mathbf{u}}^*(X_k) \tag{4.62}$$

*satisfies* $\forall x_0 \in X_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}$,

$$z_k \leq \hat{W}(X_0). \tag{4.63}$$

*Moreover, if* $B_0 \subset X_0$, *then the closed-loop system is uniform LIB dissipative with respect to* $B_0$ *and* $\beta = \hat{W}(X_0)$.

**Corollary 4.20** *If* $(\hat{W}, \hat{S})$ *is a good solution of the dynamic programming inequality (4.57), then we have*

$$\hat{S} \subset \mathrm{dom}\hat{W}_a, \quad \hat{W}_a(X) \leq \hat{W}(X), \forall X \in \hat{S}. \tag{4.64}$$

*where* $\hat{W}_a$ *is the value function defined in (4.59).*

**Remark 4.21** The above results are consistent with [28, Theorem 4.1] which asserts, in our terminology, that if there exists a measurement feedback controller achieving the uniform LIB dissipation property, then there exists a separation structure controller that feeds back the set-valued observer state and also achieves uniform LIB dissipation. □

# 5 Examples

## 5.1 Example 1 - A System with Linear Dynamics

Consider one-dimensional discrete-time system with linear dynamics:

$$\begin{cases} \xi_{k+1} = a\xi_k + bu_k + w_k \\ z_k = |\xi_k| \\ y_k = \xi_k + w_k \end{cases} \tag{5.1}$$

where $\xi_k, w_k, u_k, y_k, z_k \in \mathbf{R}, 1 < a \leq 2, b > 0$. Notice that when $u_k = w_k = 0, k \geq 0$, the open-loop system is unstable. We consider the uniform LIB dissipation problem described in Section 4.5. Suppose $B_0 = [a_{0l}, a_{0r}] \subset \mathbf{R}, \mathbf{W} = [-d, d], d > 0, \mathbf{U} = [-\delta, \delta], \delta > 0$.

We first consider state feedback synthesis. Assume that

$$\delta \geq \frac{ad}{b}; \quad \max\{|a_{0l}|, |a_{0r}|\} \leq \frac{b\delta - d}{a-1}. \tag{5.2}$$

The state feedback value function $V_a(x)$ is given by

$$V_a(x_0) = \max\{|x_0|, d\}, \quad \forall x_0 \in \mathrm{dom} V_a = [-\frac{b\delta - d}{a-1}, \frac{b\delta - d}{a-1}], \tag{5.3}$$

and the corresponding optimal state feedback controller is

$$\mathbf{u}^*(x) \triangleq \begin{cases} \delta, & \text{if } x \in (-\infty, -\frac{b\delta}{a}) \\ -\frac{a}{b}x, & \text{if } x \in [-\frac{b\delta}{a}, \frac{b\delta}{a}] \\ -\delta, & \text{if } x \in (\frac{b\delta}{a}, +\infty) \end{cases} \tag{5.4}$$

It is easy to prove that this $V_a$ is a solution of (3.5) (such equations could have multiple solutions).

The dynamic programming equation (3.5) was also solved numerically ($a = 2, b = 1, d = 0.5, \delta = 2$). These numerical results correspond to the analytical solution above for the value function $V_a$ and corresponding controller. The results are shown in (a), (c) Figure 5.1.

Theorem 3.5 asserts that the closed-loop system should be LIB dissipative. Indeed, it is not difficult to verify the uniform LIB dissipation with respect to $B_0 = [a_{0l}, a_{0r}]$ and $\beta = \max\{|a_{0l}|, |a_{0r}|, d\}$, provided that the assumption (5.2) holds. This is illustrated in (b), (d) Figure 5.1 ($B_0 = [-1.5, 1.5], \beta = 1.5, x_0 = 1.5, V_a(x_0) = 1.5$).

We turn now to measurement feedback synthesis. Assume that

$$\delta \geq \frac{ad}{b}; \quad \max\{|a_{0l}|, |a_{0r}|\} \leq \frac{b\delta - d}{a-1}; \quad a_{0r} - a_{0l} \leq \frac{2(b\delta - ad)}{a(a-1)}. \tag{5.5}$$

We set

$$X_0 = B_0 = [a_{0l}, a_{0r}]. \tag{5.6}$$

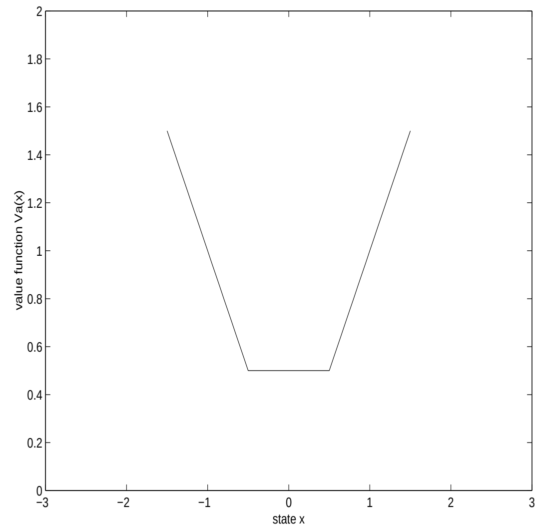
Then the set-valued state estimate is an interval given by

$$X_k = [a_{kl}, a_{kr}], k \geq 0. \tag{5.7}$$

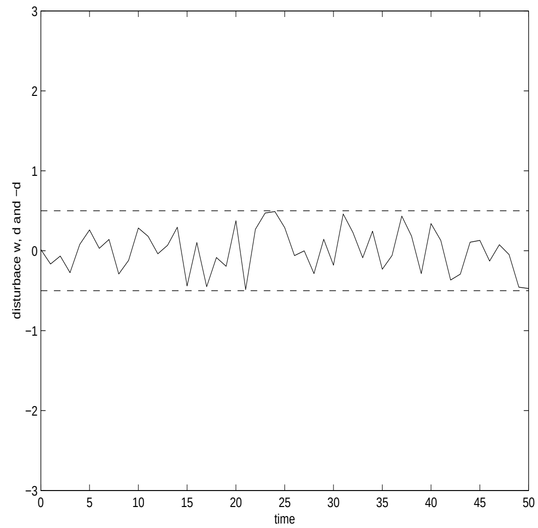
where

$$\begin{aligned} a_{(k+1)l} &= y_k + bu_k + (a-1)\max\{a_{kl}, y_k - d\}, \\ a_{(k+1)r} &= y_k + bu_k + (a-1)\min\{a_{kr}, y_k + d\}. \end{aligned} \tag{5.8}$$

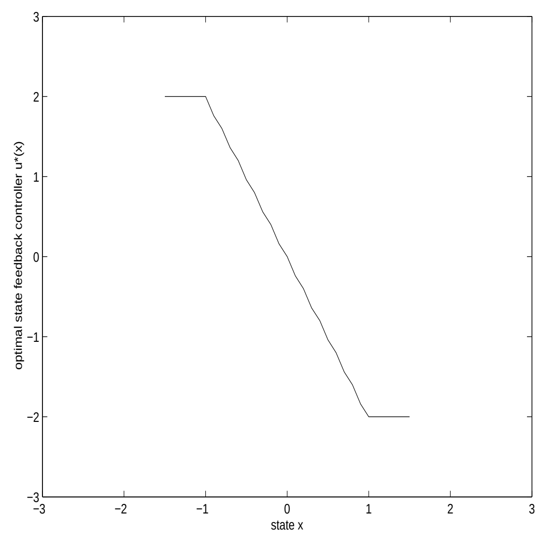
This means that when $X_0$ is of the form (5.6), the results of Section 4.5 apply on a two-dimensional space $\hat{S} \subset \bar{S}'$.



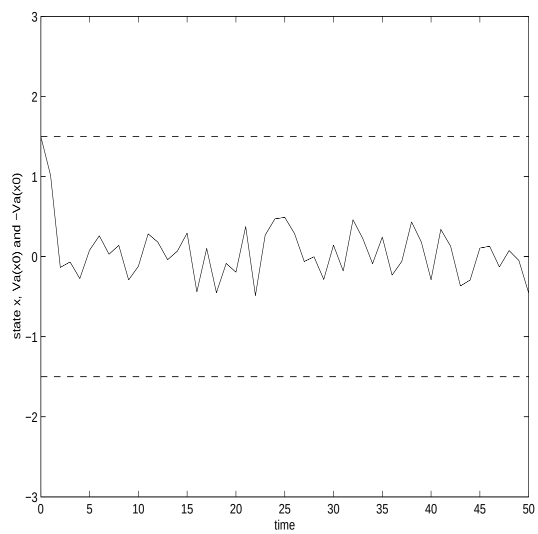

(a) Value function $V_a$. Note that when $x < -1.5$ or $x > 1.5$, $V_a(x) = +\infty$




(b) Trajectory of disturbance $w$.




(c) Optimal state feedback controller.




(d) State trajectory of close-loop system: $a_{0l} = -1.5, a_{0r} = 1.5, \beta = 1.5, x_0 = 1.5, V_a(x_0) = 1.5$.

Figure 5.1: Example 1 - system with linear dynamics - state feedback synthesis.

The measurement feedback value function $\hat{W}_a(X)$ is

$$\hat{W}_a(X) \triangleq W_a(\delta_X) = \hat{W}_a(a_l, a_r) = \max\{|a_l|, |a_r|, d + \frac{a}{2}(a_r - a_l)\}, \forall X = [a_l, a_r] \in \hat{S}$$

where

$$\hat{S} \triangleq \left\{[a_l, a_r] : [a_l, a_r] \subset [-\frac{b\delta - d}{a-1}, \frac{b\delta - d}{a-1}], a_r - a_l \leq \frac{2(b\delta - ad)}{a(a-1)}\right\},$$

and the corresponding optimal information state controller is given by

$$u_k = \hat{\mathbf{u}}^*(X_k) = \hat{\mathbf{u}}^*(a_{kl}, a_{kr}) \triangleq \begin{cases} \delta; & \text{if } (a_{kl} + a_{kr}) < -\frac{2b\delta}{a} \\ -\frac{a}{2b}(a_{kl} + a_{kr}); & \text{if } |a_{kl} + a_{kr}| \leq \frac{2b\delta}{a} \\ -\delta; & \text{if } (a_{kl} + a_{kr}) > \frac{2b\delta}{a} \end{cases} \tag{5.9}$$

It can be proved that this $\hat{W}_a(a_l, a_r)$ is a solution of (4.56) (such equations could have multiple solutions). If $[a_{0l}, a_{0r}] \notin \hat{S}$, then $\hat{W}_a(a_{0l}, a_{0r}) = +\infty$, so the uniform LIB problem is not solvable when the assumption (5.5) does not hold.

The dynamic programming equation (4.56) was also solved numerically ($a = 2, b = 1, d = 0.5, \delta = 2$). These numerical results correspond to the analytical solution above for the value function $\hat{W}_a$ and corresponding controller. The results are shown in (a), (c) Figure 5.2.

Theorem 4.12 asserts that the closed-loop system should be LIB dissipative. Indeed, when the assumption (5.5) holds, it is easy to verify the uniform LIB dissipation with respect to $B_0 = [a_{0l}, a_{0r}]$ and $\beta = \max\{|a_{0l}|, |a_{0r}|, \frac{a}{2}(a_{0r} - a_{0l}) + d\}$. This is illustrated in (b), (d) Figure 5.2 ($B_0 = [-0.5, 0.5], \beta = 1.5$).

**Remark 5.1** In contrast to the $H_\infty$ problem, the certainty equivalence principle [1], [21], [15] can not be usefully applied for the linear system given in this example. Indeed, $W_0(p) = \langle p + V_a \rangle$ (where $V_a$ is the state feedback value function defined by (5.3)) only satisfies the dynamic programming equation on a very special set

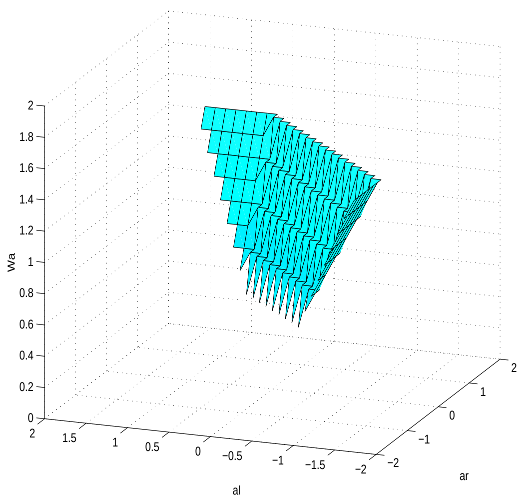
$$\tilde{\tilde{S}} = \left\{[a_l, a_r] : a_l = a_r \in [-\frac{b\delta - d}{a-1}, \frac{b\delta - d}{a-1}]\right\}.$$

The set $\tilde{\tilde{S}}$ is too small to be used for the measurement feedback synthesis. In fact, for any $0 < \varepsilon \leq \frac{b\delta - ad}{a(a-1)}$, $W_0([-\varepsilon, \varepsilon]) = \max\{\varepsilon, d\} < \hat{W}_a([-\varepsilon, \varepsilon]) = \max\{\varepsilon, a\varepsilon + d\}$, so it is impossible for $W_0(p)$ to be an measurement feedback value function. □

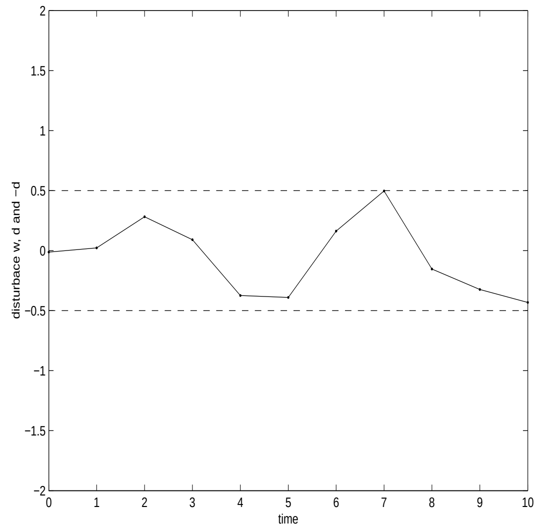
## 5.2 Example 2 - A System with Bilinear Dynamics

Consider one-dimensional system with bilinear dynamics

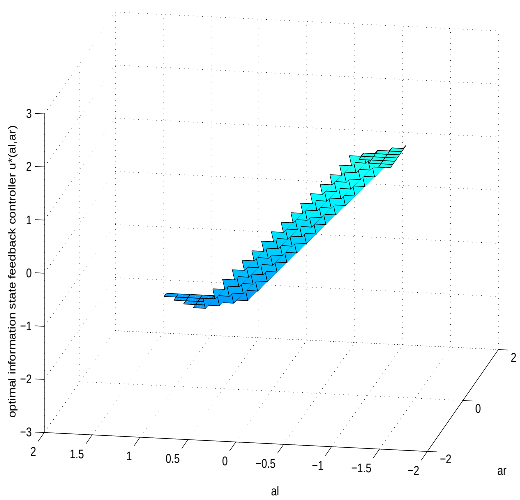
$$\begin{cases} \xi_{k+1} = \xi_k + b_1 \xi_k u_k + b_2 u_k + w_k \\ z_k = |\xi_k| \\ y_k = \xi_k + w_k \end{cases} \tag{5.10}$$



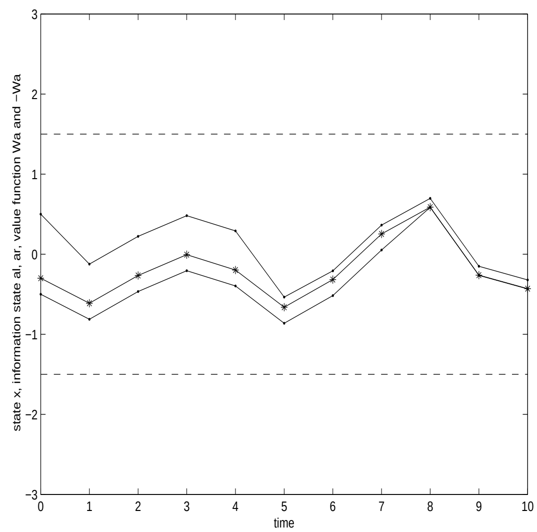

(a) Value function $\hat{W}_a$. Note that when $a_l < -1.5$ or $a_r > 1.5$ or $a_r - a_l > 1$, $\hat{W}_a(a_l, a_r) = +\infty$




(b) Trajectory of disturbance $w$.




(c) Optimal measurement feedback controller.




(d) State and estimate trajectory of close-loop system: $a_{0l} = -0.5, a_{0r} = 0.5, \beta = 1.5, x_0 = -0.3$.

Figure 5.2: Example 1 - system with linear dynamics - measurement feedback synthesis.

where $\xi_k, w_k, u_k, y_k, z_k \in \mathbf{R}, b_1 > 0, b_2 > 0$. We again consider the problem of uniform LIB synthesis. We suppose $B_0 = [a_{0l}, a_{0r}] \subset \mathbf{R}, \mathbf{W} = [-d, d], d > 0, \mathbf{U} = [-\delta, \delta], \delta > 0$. Assume that

$$\delta > \frac{1}{b_1}; \quad d \leq \frac{b_2 \delta}{b_1 \delta + 1}. \tag{5.11}$$

We first consider state feedback synthesis. We do not have an analytical solution to the dynamic programming equation (3.5). The dynamic programming equation (3.5) was solved numerically ($b_1 = 1, b_2 = 1,\ d = 0.5, \delta = 2$). The results are shown in (a), (c) Figure 5.3.

A simulation of the closed-loop system is illustrated in (c), (d) Figure 5.3 ($b_1 = 1, b_2 = 1,\ d = 0.5, \delta = 2$), consistent with the uniform LIB dissipation with respect to $B_0 = [-1.2, 1.2]$ and $\beta = 1.5$ ($x_0 = -1.2, V_a(x_0) = 1.5$).

We now consider measurement feedback synthesis. As with Example 1, we can explicitly solve for the information state in terms of a set-valued state estimate, an interval.

Choose $X_0 = [a_{0l}, a_{0r}] = B_0$. The set-valued state estimate is given by

$$X_k = [a_{kl}, a_{kr}], k \geq 0. \tag{5.12}$$

where when $b_1 u_k \geq 0$,

$$\begin{aligned} a_{(k+1)l} &= y_k + b_2 u_k + b_1 u_k \max\{a_{kl}, y_k - d\}, \\ a_{(k+1)r} &= y_k + b_2 u_k + b_1 u_k \min\{a_{kr}, y_k + d\}. \end{aligned}$$

and when $b_1 u_k < 0$

$$\begin{aligned} a_{(k+1)l} &= y_k + b_2 u_k + b_1 u_k \min\{a_{kr}, y_k + d\}, \\ a_{(k+1)r} &= y_k + b_2 u_k + b_1 u_k \max\{a_{kl}, y_k - d\}. \end{aligned}$$

We do not have an analytical solution to the dynamic programming equation (4.56). The dynamic programming equation (4.56) was solved numerically ($b_1 = 1, b_2 = 1,\ d = 0.5, \delta = 2$). The results are shown in (a), (c) Figure 5.4.

A simulation of the closed-loop system is illustrated in (b), (d) Figure 5.4, consistent with the uniform LIB dissipation with respect to $B_0 = [-2, 2]$, $\beta = 2$.

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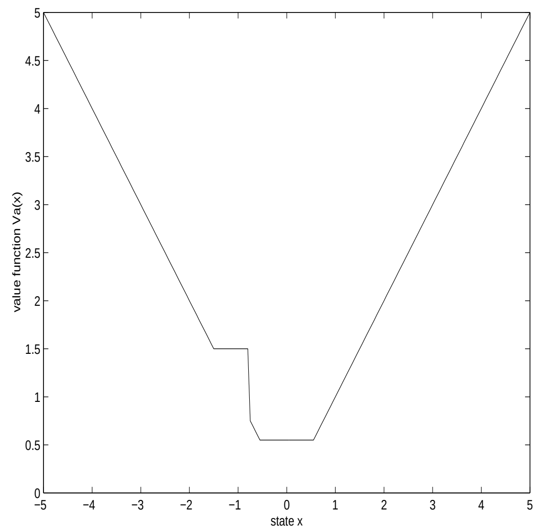
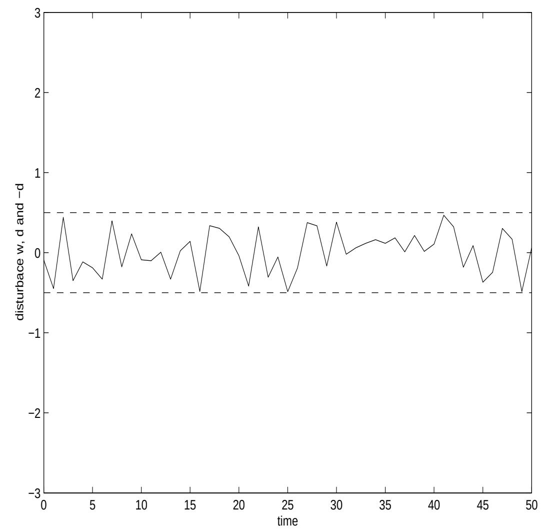
# References

[1] T. Basar and P. Bernhard, "$H^\infty$-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach", 2nd Ed., Birkhauser, Boston, 1995.

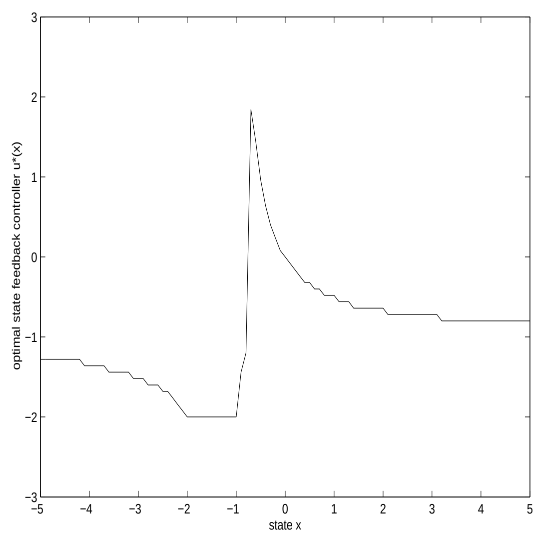
[2] E.N. Barron and H. Ishii, *The Bellman equation for minimizing the maximum cost*, Nonlinear Analysis, Theory, Methods and Applications, **13**(9), 1067-1090, 1989.



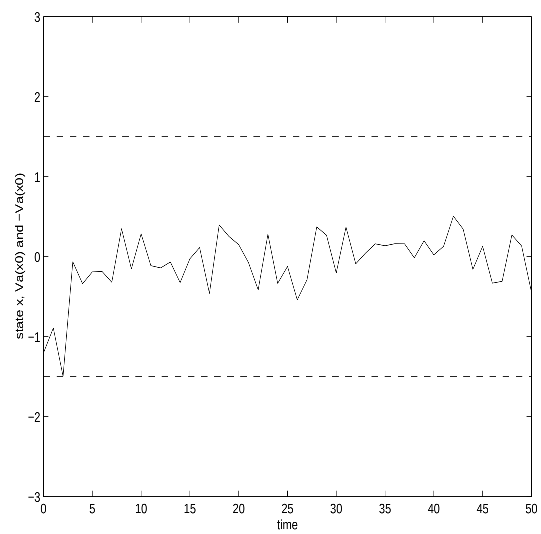

(a) Value function $V_a$.




(b) Trajectory of disturbance $w$.

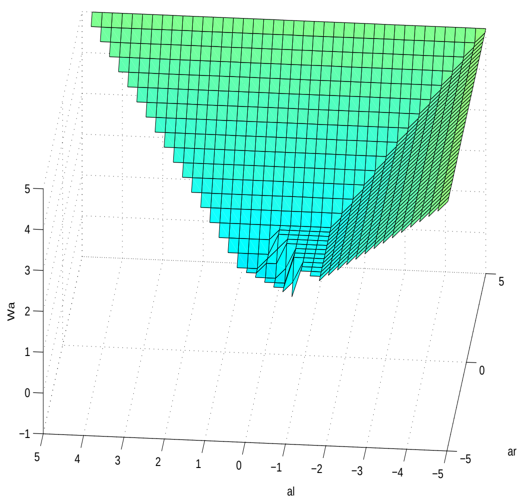



(c) Optimal state feedback controller.

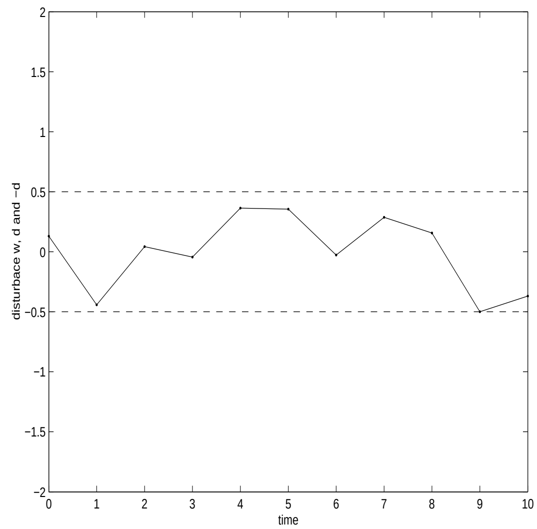



(d) State trajectory of close-loop system: $a_{0l} = -1.2, a_{0r} = 1.2, \beta = 1.5, x_0 = -1.2, V_a(x_0) = 1.5$.

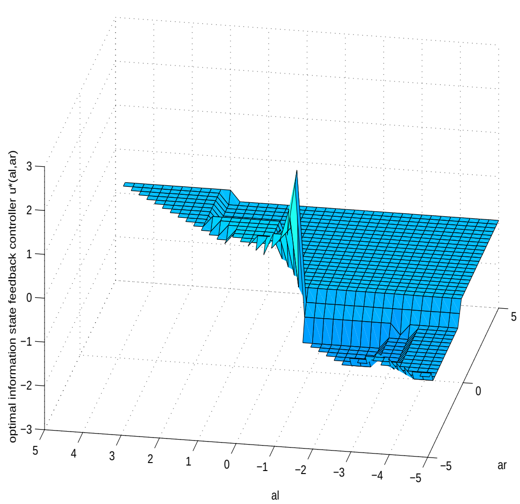
Figure 5.3: Example 2 - system with bilinear dynamics - state feedback synthesis.



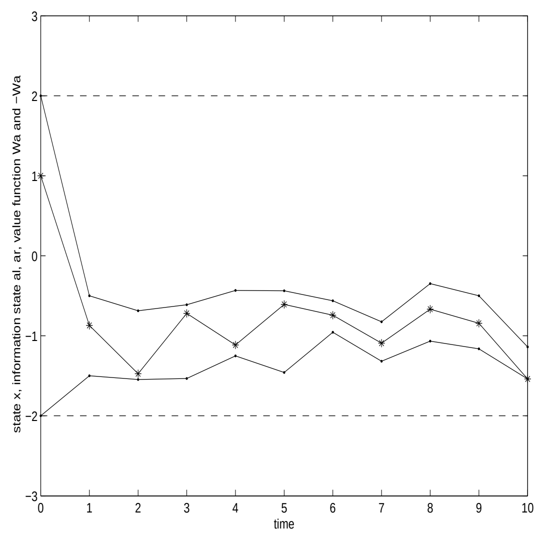

(a) Value function $\hat{W}_a$.




(b) Trajectory of disturbance $w$.




(c) Optimal measurement feedback controller.




(d) State and estimate trajectory of close-loop system: $a_{0l} = -2$, $a_{0r} = 2, \beta = 2, x_0 = 1$.

Figure 5.4: Example 2 - system with bilinear dynamics - measurement feedback synthesis.

[3] D.P. Bertsekas and I.B. Rhodes, *On the minimax reachability of target sets and target tubes*, Automatica, **7**, 233-247, 1971.

[4] D.P. Bertsekas and I.B. Rhodes, *Recursive state estimation for a set-membership description of uncertainty*, IEEE Trans. Automat. Contr., **16**(2), 117-128, 1971.

[5] D.P. Bertsekas, *Infinite-time reachability of state-space regions by using feedback control*, IEEE Trans. Automat. Contr., **17**(5), 604-613, 1972.

[6] D.P. Bertsekas, "Dynamic Programming and Optimal Control", Athena Scientific, Boston, 1995.

[7] E.N. Barron, *Differential games with maximum cost*, Nonlinear Analysis, Theory, Methods and Applications, **14**(11), 971-989, 1990.

[8] M.A. Dahleh and I.J. Diaz-Bobillo, *Minimization of the maximum peak-to-peak gain: the general multiblock problem*, IEEE Trans. Automat. Contr., **38**(10), 1459-1482, 1993.

[9] M.A. Dahleh and I.J. Diaz-Bobillo, "Control of Uncertain Systems: A Linear Programming Approach", Prentice-Hall, Englewood Cliffs, NJ, 1995.

[10] M.A. Dahleh and J.B. Pearson, *$l^1$-optimal feedback controllers for MIMO discrete-time systems*, IEEE Trans. Automat. Contr., **32**(4), 314-322, 1987.

[11] M.A. Dahleh and J.B. Pearson, *Optimal rejection of persistent disturbances, robust stability, and mixed sensitivity minimization*, IEEE Trans. Automat. Contr., **33**(8), 722-731, 1988.

[12] N. Elia and M.A. Dahleh, *Minimization of the worst case peak-to-peak gain via dynamic programming: state feedback case*, IEEE Trans. Automat. Contr., **45**(4), 687-701, 2000.

[13] I.J. Fialho and T.T. Georgiou, *Worst case analysis of nonlinear systems*, IEEE Trans. Automat. Contr., **44**(6), 1180-1196, 1999.

[14] M. Green and D. J. N. Limebeer, "Linear Robust Control", Prentice-Hall, Englewood Cliffs, N. J., 1995.

[15] J.W. Helton and M.R. James, "Extending $H^\infty$ Control to Nonlinear Systems: Control of Systems to Achieve Performance Objectives", Advances in Design and Control, SIAM, 1999.

[16] D.J. Hill and P.J. Moylan, *The stability of nonlinear dissipative systems*, IEEE Trans. Automat. Contr., **21**, 708-711, 1976.

[17] D.J. Hill and P.J. Moylan, *Dissipative dynamical systems: basic input–output and state space properties*, J. Franklin Inst., **309**, 327-357, 1980.

[18] S. Huang, M.R. James, D. Nesic and P.M. Dower, *Analysis of input to state stability for discrete-time nonlinear systems via dynamic programming*, submitted to Automatica, Dec. 2002.

[19] S. Huang, M.R. James, D. Nesic and P.M. Dower, *On input to state stabilization for nonlinear discrete-time systems: a dynamic programming approach*, European Control Conference 2003 (to appear).

[20] M.R. James, *$L^\infty$ bounded robustness: state feedback analysis and design*, NOLCOS 2001.

[21] M.R. James, J.S. Baras and R.J. Elliott, *Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems*, IEEE Trans. Automat. Contr., **39**, 780-792, 1994.

[22] M.R. James and J.S. Baras, *Robust $H^\infty$ output feedback control for nonlinear systems*, IEEE Trans. Automat. Contr., **40**, 1007-1017, 1995.

[23] J.P. Lasalle, "The Stability of Dynamical Systems", SIAM, Philadelphia, 1976.

[24] W.M. Lu, *Rejection of persistent $L_\infty$-bounded disturbances for nonlinear systems*, IEEE Trans. Automat. Contr., **43**(12), 1692-1702, 1998.

[25] M. Sznaier, R. Suarez, S. Miani, and J. Alvarez-Ramirez, *Optimal $l^\infty$ disturbance attenuation and global stabilization of linear systems with bounded control*, Int. J. Robust and Nonlinear Control, **9**, 659-675, 1999.

[26] J.S. Shamma, *Nonlinear state feedback for $l^1$ optimal control*, Systems and Control Letters, **21**, 265-270, 1993.

[27] J.S. Shamma, *Optimization of the $l^\infty$-induced norm under full state feedback*, IEEE Trans. Automat. Contr., **41**(2), 533-544, 1996.

[28] J.S. Shamma and K.-Y. Tu, *Set-valued observers and optimal disturbance rejection*, IEEE Trans. Automat. Contr., **42**(2), 253-264, 1999.

[29] M. Vidyasagar, *Optimal rejection of persistent bounded disturbances*, IEEE Trans. Automat. Contr., **31**(6), 527-534, 1986.

[30] A.J. van der Schaft, "$L_2$-Gain and Passivity Techniques in Nonlinear Control", Springer Verlag, 1996.

[31] J.C. Willems, *Dissipative dynamical systems, Part I: general theory*, Arch. Rational Mech. Anal. **45**, 321-351, 1972.