Design of $\mathscr{H}_2$ ($\mathscr{H}_\infty$)-based optimal structured and sparse static output feedback gains

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Abstract

This paper is devoted to the problem of designing an $\mathscr{H}_2$ ($\mathscr{H}_\infty$)-based optimal sparse static output feedback (SOF) controller for continuous linear time invariant systems. Incorporating an extra term for penalising the number of non-zero entries of the static output (state) feedback gain into the optimisation objective function, we propose an explicit scheme and an iterative process in order to identify the desired sparse structure of the feedback gain. In doing so, the so-called reweighted ℓ_1-norm, which is known as a convex relaxation of the ℓ_0-norm, is exploited to make a convex problem through an iterative process rather than the original NP-hard problem. This paper will also show that this problem reformulation allows us to incorporate additional constraints, such as regional pole placement constraints which provide more control over the satisfactory transient behavior and closed-loop pole location, into the designing problem. Then using the obtained structural constraints, we solve the structural $\mathscr{H}_2$ ($\mathscr{H}_\infty$) SOF problem. Illustrative examples are presented to show the effectiveness of the proposed approaches.

1 Introduction

Control systems that utilise spatially distributed components have been studied and researched for a while. In early control systems, the information of the distributed sensors was transmitted to a central station (controller) through direct hard-wired links and then the generated control commands were sent to the spatially distributed actuators. With the recent advances in communication technology, efficient communication networks have been used in control systems, which has opened a new research area to consider the influences of the communication networks on control systems. Spatially distributed control systems which exploit communication networks in their loop have been regarded as networked control systems (NCSs) [23].

On the other hand, decentralised or distributed control architectures have been proposed and used in the literature [37, 30, 2, 16]. The general idea behind the decentralised control scheme is to use only the local state information in order to control the subsystems and thus there is no control network. This can be effective only when the interconnections between the subsystems are not strong [33], [28]. In other words, when the interconnections are strong, utilising distributed control frameworks has been considered. In this strategy, each subsystem can exploit local state as well as the state of some other subsystems. Hence, compared to the decentralised control scheme, distributed control scheme can ensure the stability of the overall system in the presence of stronger subsystem interconnections [31]. Meantime, it also has less complexity and improved computational aspects compared to the centralised control scheme.

Some work in the literature considers the problem of designing centralised and distributed control systems, with imposing *a priori* constraints on communication network structure. It is shown that the problem of designing the structured optimal controllers can be set as a convex optimisation problem [25, 24]. Furthermore, another arising research problem, for the centralised or distributed control systems, is to design the control network with the minimum number of communication links while satisfying a global control objective [23]. Indeed, a trade off between the control performance and sparsity of the feedback gain matrices should be considered [20, 38, 14]. It is worth noting that several algorithms have been proposed in the field of sparse signal reconstruction and representation to gain a quick reconstruction and a reduced requirement on the number of measurements compared to the classical ℓ_1 approach (see [19, 32, 6, 7]), and among them reweighted

ℓ_1 (REL1) and reweighted iterative support detection (RISD) algorithms are employed in this paper for sparse pattern recognition purposes.

A major drawback in most of the aforementioned references is the assumption of the availability of all the system states. Hence, it is natural to use e.g. the static output feedback (SOF) scheme to control the system by directly employing the measured system outputs [29, 15, 12, 11, 8]. The main difficulty for the SOF problem is that, in general, the output feedback gain matrix cannot be directly obtained. Therefore, roughly speaking, most of the methods in the literature to solve (e.g. $\mathscr{H}_2$) SOF problems utilise iterative processes [29, 15, 12], and their solutions and more importantly their convergence depend quite significantly on the initial conditions. Exceptionally, several explicit expressions for SOF are presented in [36, 34, 35]. However, these explicit expressions are not desirable for structured feedback problems. Recently, in order to address this issue, a novel scheme is proposed in [26] for $\mathscr{H}_\infty$ SOF problems employing a similar method for SF problems. Motivated by this method, this paper aims to extend it to different control problems such as the *structured optimal SOF problems* and, more importantly, *optimal sparse SOF problems*.

The method presented in [20] solves the $\mathscr{H}_2$ problem, by incorporating a sparsity promoting penalty function to its objective function, to obtain a sub-optimal sparse *state-feedback* (SF) controller. This paper uses a similar strategy for penalising the number of nonzero entries of the feedback gain. However, this paper's contributions are essentially different from those presented in [20]; we develop an alternative explicit formulation which not only can address the *structured and sparse SF* controller design problem, but also enables us to extend it to the *structured and sparse SOF* control problem. Besides, this scheme allows us to impose more convex restrictions on the closed-loop dynamics, e.g. pole-placement can be incorporated within standard convex regions of the complex plane (e.g. circles, cones and strips) [9].

In order to penalise the communication links in the feedback matrix, we firstly incorporate the ℓ_0-norm (cardinality function) of the feedback gain matrix into the objective function of the optimisation problem. As the ℓ_0-norm in the optimisation objective function results in a non-convex problem, the so-called reweighted ℓ_1-norm [3], which is known as a convex relaxation of the cardinality function, will be used here to make a convex optimisation problem. Employing the reweighted ℓ_1-norm in the objective function, we then propose two iterative processes in order to find the desired sparse structure with respect to the given sparsity regularisation parameter exploited in the objective function. Finally, using the identified structure we design the structured static output (or state) feedback for the underlying system.

In summary, the major focus of this paper is on the development of an approach for optimal selection of a subset of available communication links while minimising the variance amplification (i.e. $\mathscr{H}_2$ or $\mathscr{H}_\infty$ norm of the closed-loop system). In order to deal with this issue, this paper includes several novel initiatives as follows:

- This paper develops a novel framework for the design of a $\mathscr{H}_2$ ($\mathscr{H}_\infty$) sparse SOF. To the authors' best knowledge, this is a new technology which uses the system's output only and, in the meantime, can satisfy structural constraints on the feedback gain as well as performance specifications and regional pole placements constraints.
- Further, as the approach in this paper does not employ an iterative method to find the SOF gain, it is very attractive for the second purpose of this paper which is to identify sparse subsets of communication links while minimising the performance degradation.
- This paper also includes two schemes (reweighted ℓ_1 norm (REL1) and reweighted iterative support detection (RISD)) for the identification of favourable sparse patterns for the SOF.
- Roughly speaking, most of the proposed methods in the literature, for identifying the most sparse feedback gain, are not able to provide control over the transient behaviour and closed-loop pole locations. To address this issue, we propose augmenting the optimisation problems exploited for the control synthesis by a set of LMI constraints to guarantee the poles of the closed-loop system be located in a suitable subregion.

Notation: $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\left[\Sigma_{ij}\right]_{q\times q}$ is a block matrix with block entries $\Sigma_{ij},\ i=1,\ \cdots,\ q,\ j=1,\ \cdots,\ q$. diag$\left[\Sigma_i\right]_{i=1}^{q}$ is a block-diagonal matrix with block entries $\Sigma_i,\ i=1,\ \cdots,\ q$. $\{\circ\}$ denotes an operator for $\Xi=[\xi_{ij}]_{h\times h}$ in which $\xi_{ij}\in\mathbb{R}$ and $W=[W_{ij}]_{h\times h}$ in which $W_{ij}\in\mathbb{R}^{r_i\times s_j}$ such that $\Xi\circ W=[\xi_{ij}W_{ij}]_{h\times h}$.

2 Problem Formulation and Preliminaries

Consider the following overall LTI system:

$$\begin{cases} \dot{x}(t)=Ax(t)+B_2u(t)+B_1f(t), \\ z(t)=C_zx(t)+D_zu(t), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^q$ are the state vector, control input vector and performance output vector of the overall system, respectively. The matrices in (1) are constant and of appropriate dimensions. Moreover, B_1, B_2, C_z and D_z in (1) are block diagonal matrices as $\text{diag}[B_{1,i}]_{i=1}^h$, $\text{diag}[B_{2,i}]_{i=1}^h$, $\text{diag}[C_{z,i}]_{i=1}^h$ and $\text{diag}[D_{z,i}]_{i=1}^h$, respectively, in which h denotes the number of subsystems and $B_{1,i} \in \mathbb{R}^{n_i \times r_i}$, $B_{2,i} \in \mathbb{R}^{n_i \times m_i}$, $C_{z,i} \in \mathbb{R}^{q_i \times n_i}$ and $D_{z,i} \in \mathbb{R}^{q_i \times m_i}$. In addition, matrix A is an overall matrix that contains sub-systems' dynamics and their mutual interactions, i.e. $A = \text{diag}[A_i]_{i=1}^h + \left[A_{ij}\right]_{h \times h}$, with $A_i \in \mathbb{R}^{n_i \times n_i}$ and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, where $A_{ij} \neq 0$ if the sub-system j influences directly the sub-system i. Without loss of generality, it is also assumed that $m \leq q \leq n$ and $\text{rank}(B_2) = m$. Also

$$C_z = \begin{bmatrix} Q_s^T & 0 \end{bmatrix}^T, \; D_z = \begin{bmatrix} 0 & R^{\frac{1}{2}} \end{bmatrix}^T,$$

where $C_z^T C_z = Q_s^T Q_s \triangleq Q \geq 0$, with $Q_s \in \mathbb{R}^{s \times n}$ is a full rank matrix in which $0 < s = q - m \leq n$, and $D_z^T D_z = R > 0$. $f(t)$ is the external disturbance of system. It is also assumed that (A, B_2) is stabilisable.

Our main objective in this paper is to design an $\mathscr{H}_2$-based optimal distributed SOF gain, exploiting feedback from (some of) other subsystems, to stabilise the overall system in (1) through a sparse control network. Notice that the $\mathscr{H}_2$-based sparse SOF gains, in this paper, will be attained by solving an explicit LMI optimisation problem, derived from the associated SF LMI formulation through LMI variables transformations. Hence, firstly, we consider the $\mathscr{H}_2$-based sparse SF gain design problem and then generalise the problem to the case that only the underlying system outputs are available.

Definition 1. *A matrix is said to be structure matrix if its elements are either 0 or 1. The structure matrix of a block matrix* $Y = [Y_{ij}]_{m \times n}$ *with* $Y_{ij} \in \mathbb{R}^{r_i \times s_j}$ *is* $\mathcal{S}(Y) \triangleq [s_{ij}]_{m \times n}$ *with*

$$s_{ij} = \begin{cases} 0 & \text{if } Y_{ij} = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2. *Two matrices* Y_1 *and* Y_2 *are said to have the same structure if* $\mathcal{S}(Y_1) = \mathcal{S}(Y_2)$.

Definition 3. *The matrix* Y_1 *with* $\mathcal{S}(Y_1) \triangleq [s_{ij}^1]_{m \times n}$ *is said to be structurally subset of* Y_2 *with* $\mathcal{S}(Y_2) \triangleq [s_{ij}^2]_{m \times n}$ *while* $s_{ij}^2 - s_{ij}^1 \geq 0$. *We denote this as* $\mathcal{S}(Y_1) \subseteq \mathcal{S}(Y_2)$.

2.1 $\mathscr{H}_2$-based structured SF design

This subsection aims at the design of the $\mathscr{H}_2$-based SF, subject to structural constraints. Let the structure matrix $\Gamma = [\gamma_{ij}]_{h\times h}$ denotes *a priori* specified structure for the control gain. Assume that there exists a gain $F \in \mathbb{R}^{m\times n}$, with $\mathcal{S}(F) \subseteq \Gamma$, making the closed loop system $A + B_2F$ stable while minimising the following cost functional,

$$J(F) = \text{trace}(B_1^T \bar{X} B_1), \tag{2}$$

where $\bar{X}$ is the closed-loop observability Gramian

$$\bar{X} = \int_0^\infty e^{(A+B_2F)^T t}(Q + F^T RF)e^{(A+B_2F)t} dt. \tag{3}$$

It is also known that $\bar{X}$ can solve the following Lyapunov equation,

$$(A + B_2F)^T \bar{X} + \bar{X}(A + B_2F) + Q + F^T RF = 0. \tag{4}$$

Letting $X = \bar{X}^{-1}$ and pre and post multiplying X to (4), we have

$$X(A + B_2F)^T + (A + B_2F)X + XQX + XF^T RFX = 0. \tag{5}$$

By relaxing the equality in (5) to the following inequality

$$X(A + B_2F)^T + (A + B_2F)X + XQX + XF^T RFX < 0, \tag{6}$$

and letting $X = \bar{X}^{-1}$ be a block diagonal matrix variable, that is, $X = \text{diag}[X_i]_{i=1}^h$, and $X_i \in \mathbb{R}^{n_i\times n_i}$, rather than the minimisation problem explained in (2), we consider a sub-optimal problem utilising the LMI approach,

$$\text{minimise} \ \ \text{trace}\,(Z) \ \ \text{subject to} \tag{SSF}$$

$$\begin{bmatrix} AX + XA^T + B_2Y + Y^T B_2^T & \star & \star \\ Q_s X & -I & \star \\ R^{\frac{1}{2}} Y & 0 & -I \end{bmatrix} < 0, \tag{7}$$

$$\begin{bmatrix} -Z & \star \\ B_1 & -X \end{bmatrix} < 0 \tag{8}$$

where $Y = \Gamma \circ \bar{Y}$ with $\bar{Y} \in \mathbb{R}^{m\times n}$ and Z is a slack variable. Thus, the structural state-feedback would be obtained as $F = YX^{-1}$. Notice that it is a usual technique to treat FX as a new variable; e.g. Y, in order to have a convex representation. However, as recovering the SF matrix $\boldsymbol{F}$ requires a reverse transformation, $\boldsymbol{F} = \boldsymbol{Y}\boldsymbol{X}^{-1}$ would not necessarily regain the structure of Γ, unless a structural assumption (block diagonal pattern) is made on the decision matrix $\boldsymbol{X}$.

Remark 1. *It is also easy to realise that*

$$\mathcal{S}(\boldsymbol{X}^{-1}) = \mathcal{S}(\boldsymbol{X}) = \boldsymbol{I},$$

and since $\mathcal{S}(\boldsymbol{Y}) \subseteq \Gamma$*, thus*

$$\mathcal{S}(\boldsymbol{Y}\boldsymbol{X}^{-1}) \subseteq \Gamma.$$

This means that the SF gain $\boldsymbol{F}$ *obtained from* $\boldsymbol{Y}\boldsymbol{X}^{-1}$ *would have the desired structure* Γ.

2.2 $\mathcal{H}_2$-based sparse SF design

Previous subsection has studied the problem of designing $\mathcal{H}_2$-based structured SF with imposing *a priori* constraints directly on the control gain. Here, instead, we search for sparse feedback structures without imposing any a priori structure on the sparsity patterns of the matrix. We now consider an optimisation framework in which the sparsity of the feedback gain is directly incorporated into the objective function [20]. This problem can be formulated as:

$$\begin{aligned} &\text{minimise} \quad \text{trace}\,(\boldsymbol{Z}) + \eta\,\mathbf{card}(\mathcal{S}(\boldsymbol{Y})), \\ &\text{subject to (7), (8) and } \mathcal{S}(\boldsymbol{X}) = \boldsymbol{I}, \end{aligned} \tag{9}$$

where $\boldsymbol{Y} \in \mathbb{R}^{m \times n}$ in (7) does not have any preset structure, $\mathbf{card}(\cdot)$ denotes the cardinality function (the number of nonzero elements of a matrix) and $\eta > 0$ is the regularisation parameter that implies the emphasis on the sparsity of the SF matrix $\boldsymbol{F}$; i.e. a larger η will result in a more sparse $\boldsymbol{F}$ and inversely $\eta = 0$ converts the minimisation problem (9) to a standard $\mathcal{H}_2$ problem. Owing to the existence of the quasi-norm $\mathbf{card}(\cdot)$ in the objective functional in (9), this problem is indeed a combinatorial problem. The ℓ_1-norm and weighted ℓ_1-norm are used in the literature as a convex relaxation of the cardinality function [14, 3, 20]. However, the weighted ℓ_1-norm is not implementable for this problem as the weights depend on the feedback gain. Alternatively, an iterative scheme referred to as reweighted ℓ_1 algorithm can be used to deal with the problem. In this scheme the weights obtained in the previous iteration are exploited in the current iteration; e.g. see [5, 21]. We now propose a reweighted ℓ_1 algorithm (see Algorithm A.1 in Appendix) for finding an $\mathscr{H}_2$ sparse SF. Now the optimisation problem (9) can be cast as

$$\begin{aligned}&\text{minimise } \operatorname{trace}(\boldsymbol{Z}) + \eta \, \|\boldsymbol{W} \circ \boldsymbol{Y}\|_{\ell_1} \\ &\text{subject to (7), (8) and } \mathcal{S}(\boldsymbol{X}) = \boldsymbol{I},\end{aligned} \tag{SPSF}$$

where $\boldsymbol{W}$ is a given weighting matrix with the same dimension of $\mathcal{S}(\boldsymbol{F})$. We denote the favourable structure, obtained from solving the minimisation problem in (SPSF), as $\mathcal{S}(\boldsymbol{F}) \triangleq \Gamma_{\boldsymbol{F}}$. Now we turn to the minimisation problem introduced in (SSF), by letting $\Gamma = \Gamma_{\boldsymbol{F}}$, in order to find the $\mathscr{H}_2$ structured SF.

Remark 2. *It should be noted that the value of the $\mathscr{H}_2$ cost obtained from* (SSF) *is not the true one, due to the conservatism introduced by assuming the block diagonal structure for $\boldsymbol{X}$. Nevertheless, the true value can be computed by solving the following Lyapunov equation*

$$\boldsymbol{X}_{true}(\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{F}) + (\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{F})^T \boldsymbol{X}_{true} + \boldsymbol{Q} + \boldsymbol{F}^T \boldsymbol{R} \boldsymbol{F} = 0. \tag{10}$$

Then one can find the $\mathscr{H}_2$ cost as $\sqrt{trace(\boldsymbol{B}_1^T \boldsymbol{X}_{true} \boldsymbol{B}_1)}$.

Remark 3. *Notice that although in the methods proposed in [20, 13] no diagonal structural assumption needs to be imposed on the matrix decision variables, the downside is that the iterative processes utilised in [20, 13] are more computationally intensive and moreover have no convergence guarantees.*

3 $\mathscr{H}_2$-Based Sparse SOF Design

Let the system in (1) has the measured output vector $y(t) \in \mathbb{R}^p$, and

$$y(t) = \boldsymbol{C}x(t), \tag{11}$$

where $\boldsymbol{C} \in \mathbb{R}^{p \times n}$ is a full row rank matrix, i.e. $\operatorname{rank}(\boldsymbol{C}) = p$. In this section, we seek for the $\mathscr{H}_2$ sparse SOF of the form

$$u(t) = \boldsymbol{F}_y y(t), \tag{12}$$

where $\boldsymbol{F}_y \in \mathbb{R}^{m\times p}$. It can be shown that

$$u(t) = \boldsymbol{F}_y \boldsymbol{C} x(t). \tag{13}$$

Now the $\mathscr{H}_2$-based sparse SOF problem can be considered as a constrained sparse SF problem in which the SF matrix $\boldsymbol{F}$ should satisfy the constraint $\boldsymbol{F} = \boldsymbol{F}_y \boldsymbol{C}$.

3.1 $\mathscr{H}_2$-based structured SOF design

The problem of designing an $\mathscr{H}_2$-based structured SOF can be cast as the following optimisation problem:

$$\begin{aligned} &\text{minimise} \;\; \text{trace}\,(\boldsymbol{Z}) \\ &\text{subject to (7), (8) and } \boldsymbol{Y}\boldsymbol{X}^{-1} = \boldsymbol{F}_y\boldsymbol{C} \text{ and } \mathcal{S}(\boldsymbol{F}_y) = \Gamma_y, \end{aligned} \tag{14}$$

where Γ_y is a certain structure matrix. Notice that an effective scheme to address a similar non-convex optimisation problem, for the design of an $\mathscr{H}_\infty$ SOF, is proposed in [26]. This scheme indeed introduces specific LMI decision variable ($\boldsymbol{X}$ and $\boldsymbol{Y}$) transformations as

$$\begin{aligned} \boldsymbol{X} &= \boldsymbol{N}\boldsymbol{X}_N\boldsymbol{N}^T + \boldsymbol{M}\boldsymbol{X}_M\boldsymbol{M}^T, \\ \boldsymbol{Y} &= \boldsymbol{Y}_M\boldsymbol{M}^T, \end{aligned} \tag{15}$$

where $\boldsymbol{X}_N \in \mathbb{R}^{(n-p)\times(n-p)}$ and $\boldsymbol{X}_M \in \mathbb{R}^{p\times p}$ are symmetric matrices, and $\boldsymbol{Y}_M \in \mathbb{R}^{m\times p}$. Besides, $\boldsymbol{N} = \text{null}(\boldsymbol{C}) \in \mathbb{R}^{n\times(n-p)}$ and $\boldsymbol{M} \in \mathbb{R}^{n\times p}$ is any matrix that satisfies $\boldsymbol{C}\boldsymbol{M} = \boldsymbol{I}$. In general form, $\boldsymbol{M}$ can be considered as $\boldsymbol{M} = \boldsymbol{C}^\dagger + \boldsymbol{N}\boldsymbol{D}$, where $\boldsymbol{D} \in \mathbb{R}^{(n-p)\times p}$ is a given matrix and $\boldsymbol{C}^\dagger = \boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{C}^T)^{-1}$. Now by letting the LMI variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ to be as (15), the SOF gain would be simply obtained through the following lemma.

Lemma 3.1 ([26]). *Let $\boldsymbol{X} = \boldsymbol{N}\boldsymbol{X}_N\boldsymbol{N}^T + \boldsymbol{M}\boldsymbol{X}_M\boldsymbol{M}^T$ and $\boldsymbol{Y} = \boldsymbol{Y}_M\boldsymbol{M}^T$, then $\boldsymbol{X}$ is invertible if and only if $\boldsymbol{X}_M$ is invertible. Besides we have $\boldsymbol{Y}\boldsymbol{X}^{-1} = \boldsymbol{F}_y\boldsymbol{C}$ with $\boldsymbol{F}_y = \boldsymbol{Y}_M\boldsymbol{X}_M^{-1}$.*

Now the $\mathscr{H}_2$-based structured SOF problem can be set as an optimisation problem by exploiting LMI approach,

$$\text{minimise} \;\; \text{trace}\,(\boldsymbol{Z}) \text{ subject to} \tag{SSOF}$$

$$\begin{bmatrix} \boldsymbol{A}\boldsymbol{N}\boldsymbol{X}_N\boldsymbol{N}^T+\boldsymbol{N}\boldsymbol{X}_N\boldsymbol{N}^T\boldsymbol{A}^T+\boldsymbol{A}\boldsymbol{M}\boldsymbol{X}_M\boldsymbol{M}^T+\boldsymbol{M}\boldsymbol{X}_M\boldsymbol{M}^T\boldsymbol{A}^T+\boldsymbol{B}_2\boldsymbol{Y}_M\boldsymbol{M}^T+\boldsymbol{M}\boldsymbol{Y}_M^T\boldsymbol{B}_2^T & \star & \star \\ \boldsymbol{Q}_s(\boldsymbol{N}\boldsymbol{X}_N\boldsymbol{N}^T+\boldsymbol{M}\boldsymbol{X}_M\boldsymbol{M}^T) & -\boldsymbol{I} & \star \\ \boldsymbol{R}^{\frac{1}{2}}\boldsymbol{Y}_M\boldsymbol{M}^T & 0 & -\boldsymbol{I} \end{bmatrix} < 0, \tag{16}$$

$$\begin{bmatrix} -\boldsymbol{Z} & \star \\ \boldsymbol{B}_1 & -\boldsymbol{N}\boldsymbol{X}_N\boldsymbol{N}^T-\boldsymbol{M}\boldsymbol{X}_M\boldsymbol{M}^T \end{bmatrix} < 0, \tag{17}$$

where $\boldsymbol{X}_M$ is defined as a block diagonal matrix variable, $\boldsymbol{Y}_M = \Gamma_y \circ \bar{\boldsymbol{Y}}_M$ with $\bar{\boldsymbol{Y}}_M \in \mathbb{R}^{m\times p}$ and $\boldsymbol{Z}$ is a slack variable. The structural SOF is obtained as $\boldsymbol{F}_y = \boldsymbol{Y}_M\boldsymbol{X}_M^{-1}$. Besides, notice that $\mathcal{S}(\boldsymbol{X}_M^{-1}) = \mathcal{S}(\boldsymbol{X}_M)$, and thus $\mathcal{S}(\boldsymbol{Y}_M\boldsymbol{X}_M^{-1}) \subseteq \mathcal{S}(\boldsymbol{Y}_M) \subseteq \Gamma_y$.

Remark 4. *1) The scheme proposed here is a framework for the design of structured $\mathcal{H}_2$ SOF for LTI systems. This novel scheme is of importance as most existing methods [29, 15, 12] i) employ iterative processes for addressing SOF design problem and so their solutions and their convergence depend quite significantly on the initial conditions, ii) have shortcomings in terms of imposing structural constraints on the feedback gains.*
2) As stated in [26], the solution of the problem in (SSOF) *may not necessarily be an optimal solution of* (14)*. Nevertheless, it is an effective way to minimise the cost function on a set that satisfies all the constraints in* (14)*.*
3) Matrix $\boldsymbol{D}$ plays an important role in this strategy. A simplified choice is $\boldsymbol{D} = 0$. An advanced choice of $\boldsymbol{D}$ is also presented in [22] which is $\boldsymbol{D} = (\boldsymbol{N}^T\boldsymbol{N})^{-1}\boldsymbol{N}^T\boldsymbol{X}\boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{X}\boldsymbol{C}^T)^{-1}$. This choice, however, requires solving the SF optimisation problem in advance. A benefit of this choice is that if no solution can be attained by the ideal SF problem, no further efforts is required for the output feedback problem.

3.2 $\mathcal{H}_2$-based sparse SOF design

While the previous section considered the problem of designing the $\mathcal{H}_2$-based structured SOF having a certain structure constraint for the control gain, this section explores favourable sparse SOF gains without imposing a priori structure constraint on the sparsity patterns of the control gain. Similar to the SF case, an optimisation framework, in which the sparsity of the feedback gain is directly incorporated into the objective function [20], is considered here. This problem can be formally formulated as:

$$\begin{aligned}&\text{minimise} \quad \text{trace}\,(\boldsymbol{Z})+\eta_y\,\mathbf{card}(\mathcal{S}(\boldsymbol{Y_M})),\\&\text{subject to (16), (17) and } \mathcal{S}(\boldsymbol{X_M})=\boldsymbol{I},\end{aligned} \tag{18}$$

where $\boldsymbol{Y_M}$ here is a full decision matrix that does not have any a priori structure and $\eta_y > 0$ is the regularisation parameter. A reweighted ℓ_1 algorithm (see Algorithm B.1 in Appendix) can be exploited again to design an $\mathscr{H}_2$ sparse SOF gain. The optimisation problem (18) can be cast as

$$\begin{aligned}&\text{minimise trace}(\boldsymbol{Z})+\eta_y\left\|\boldsymbol{W_o}\circ\boldsymbol{Y_M}\right\|_{\ell_1}\\&\text{subject to (16), (17) and } \mathcal{S}(\boldsymbol{X_M})=\boldsymbol{I},\end{aligned} \tag{SPSOF}$$

where $\boldsymbol{W_o}$ is a known weighting matrix with the same dimension of $\mathcal{S}(\boldsymbol{F_y})$. We denote the obtained structure of the minimisation problem (MSOP), as $\mathcal{S}(\boldsymbol{F_y}) \triangleq \Gamma_{\boldsymbol{F_y}}$. Eventually, in order to find the $\mathscr{H}_2$ structured SOF with given $\Gamma_y = \Gamma_{\boldsymbol{F_y}}$, we turn to the minimisation problem in (SSOF).

Remark 5. *Notice that in the case of dealing with distributed systems, the sparsification procedure is implemented at the level of system components rather than the subsystems, and thus, the matrix decision variable $\boldsymbol{X_M}$ would be a diagonal matrix, that is, $\boldsymbol{X_M} = diag[X_{M,i}]_{i=1}^{n}$, where $\boldsymbol{n}$ denotes the order of system and $\boldsymbol{X_{M,i}} \in \mathbb{R}$. Accordingly, for the SF case $\boldsymbol{X} = diag[X_i]_{i=1}^{n}$, where $\boldsymbol{X_i} \in \mathbb{R}$. Besides, penalising the nonzero elements of the feedback gains should be implemented at the level of individual components. Hence, the reweighted process given for the design of static output (state) feedback gains should be revised by replacing $\left\|(Y_M)_{ij}\right\|_F$ (or $\left\|Y_{ij}\right\|_F$), where $\|\cdot\|_F$ denotes the Frobenius norm, with $|(Y_M)_{ij}|$ (or $|Y_{ij}|$) (see Algorithm B.1, A.1 in Appendix).*

3.3 Reweighted iterative support detection (RISD) algorithm for sparsity promoting penalty function

One drawback of reweighted ℓ_1 algorithms compared to the traditional ℓ_1 ones is their slow execution. This issue is clearly of great importance, since one of the main challenges of the numerical optimisation-based schemes for real-time pattern identification for controller structure is computational complexity and computation time. In order to speed up the reweighted algorithms, [19] develops a scheme referred to as reweighted iterative support detection (RISD) algorithm. Indeed, the iterative support detection (ISD) scheme (see, e.g., [32]) aims to improve the existing signal recovery methods by incorporating prior information, e.g., support (the locations of the nonzero elements). Assume $\mathtt{S}$ is the support of $\mathrm{col}\big(\mathrm{col}(\boldsymbol{Y}_{M,ij})_{j=1}^{p}\big)_{i=1}^{m}$ (in our case) with the subset of Λ, which denotes the known information, that is,

$$\forall\ i,j \in \Lambda \quad \boldsymbol{Y}_{M,ij} \neq 0.$$

Hence, for recovering $\boldsymbol{Y}_M$ we only need to minimise $\sum_{i,j\in\Lambda^C} |\boldsymbol{Y}_{M,ij}|$ (Λ^C is the absolute complement of Λ) subject to the involved control specifications, and there is no need to penalise the non-zero entries whose locations are known. This idea referred to as *nonuniform sparsity promoting* utilises prior information [19]. The prior information here may include two distinct types of knowledge: i) the knowledge of some very unattractive communication channels or components (e.g., due to the high execution cost or faulty situation); i.e., those components can be added in the subset Λ^C, ii) the knowledge of the previous feedback gain structure before, e.g., a fault happening; i.e., those non-faulty components can be added in the subset Λ.

Without prior information, ISD and its reweighted form [32, 19] still sparsify the solution by incorporating useful information from the current iteration into the next iteration. The subset Λ will be detected and updated automatically between iterations, that is,

$$\Lambda_{l+1} = \left\{ i,j \;\middle|\; |\boldsymbol{Y}^{l}_{M,ij}| > \epsilon_l \right\},$$

where l denotes the iteration number and ϵ_l is a threshold obtained iteratively according to the so-called first jump rule; see, e.g., [32, 19]. The proposed RISD algorithm is given in the Appendix. As can be seen in this scheme, there is a higher probability of remaining nonzero for the elements in Λ_l. Choosing the smaller ϵ_l will increase the reliability of Λ^C. Compared to the REL1 algorithm, this scheme does not require a regularisation parameter.
Moreover, μ_l here, with $\rho = 1$, can be regarded as the average jump in $\mathrm{col}\left(\mathrm{col}(|\boldsymbol{Y}^{l}_{M,ij}|)_{j=1}^{p}\right)_{i=1}^{m}$, and the parameter ρ can be considered as a design freedom which can be tuned for different applications. One may also imagine a variety of different alternatives for μ_l; see, e.g., [19] and [32]. Our extensive numerical tests show that the threshholding system proposed here is effective for most of the problems.

Notice again that the above algorithm identifies the favourable sparse pattern and, in order to design the $\mathscr{H}_2$ structured static output feedback with the obtained structure, one needs to turn to the minimisation problem in (SSOF) (polishing step).

3.4 Augmenting the problem with regional pole placement constraints

As argued before, the proposed method here for the design of structured and sparse static output (state) feedback gains makes it possible to introduce extra convex constraints, with appropriate LMI representations, on the closed-loop dynamics. A satisfactory transient response can be ensured by placing the closed-loop system poles in a predetermined region [9]. Roughly speaking, the proposed methods in the literature, for identifying the most sparse feedback gains, are not able to provide control over the transient behavior and closed-loop pole location. Without having additional constraints to control the closed-loop transient behaviour, a very sparse structure might be identified, whereas an unsatisfactory time response and closed-loop damping may occur. Hence, the objective now is not only to solve the optimisation problem described in (MSOP), but also to guarantee that the poles of the overall closed-loop system are located in a suitable subregion. In brief, an LMI region is a subset $\mathscr{D}$ of the complex plane as

$$\mathscr{D} := \{z \in \mathbb{C} : f_{\mathscr{D}}(z) \triangleq \Xi + z\Pi + \bar{z}\Pi^T < 0\} \tag{19}$$

in which $\Xi = \Xi^T$ and Π are real matrices. $f_{\mathscr{D}}(z)$ is referred to as the characteristic equation of the region $\mathscr{D}$.

Definition 4 ([10])**.** *A real matrix $\mathscr{A}$ is said to be $\mathscr{D}$-stable if all its eigenvalues lie within the LMI region $\mathscr{D}$.*

Lemma 3.2 ([10])**.** *A real matrix $\mathscr{A}$ is $\mathscr{D}$-stable if and only if a symmetric matrix $X_{\mathscr{D}} > 0$ exists such that*

$$\Xi \otimes X_{\mathscr{D}} + \Pi \otimes (\mathscr{A} X_{\mathscr{D}}) + \Pi^T \otimes (X_{\mathscr{D}} \mathscr{A}^T) < 0, \tag{20}$$

where $\otimes$ denotes the Kronecker product.

Now, the $\mathscr{H}_2$-based structured SOF problem with pole placement constraint can be set as

$$\begin{aligned} &\text{minimise} \quad \operatorname{trace}(\boldsymbol{Z}) \\ &\text{subject to (7), (8), (15), } \mathscr{S}(\boldsymbol{X_M}) = \boldsymbol{I},\ \mathscr{S}(\boldsymbol{Y_M}) \subseteq \Gamma_y, \text{ and} \\ &\Xi \otimes X_{\mathscr{D}} + \Pi \otimes (\mathscr{A}_{cl} X_{\mathscr{D}}) + \Pi^T \otimes (X_{\mathscr{D}} \mathscr{A}_{cl}^T) < 0, \end{aligned} \tag{21}$$

where $\mathscr{A}_{cl} = \boldsymbol{A} + \boldsymbol{B}_2 \boldsymbol{F}_y \boldsymbol{C}$. The above problem is not convex in the variables $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{X}_{\mathscr{D}}$. However, the convexity can be obtained by enforcing $\boldsymbol{X}_{\mathscr{D}} = \boldsymbol{X}$. As an illustration, we aim now to confine the closed-loop poles to the region $\mathsf{S}(\alpha, r, \theta)$ (see [9]) which can ensure a minimum decay rate α, a minimum damping ratio $\zeta = \cos\theta$, and a maximum undamped natural frequency $\omega_d = r\sin\theta$, while minimising the $\mathscr{H}_2$-norm of the closed-loop transfer function from f to z, and satisfying the structural constraint $\mathscr{S}(\boldsymbol{Y}) \subseteq \Gamma_y$. Notice that $\mathsf{S}(\alpha, r, \theta)$ is indeed the intersection of three elementary LMI regions: an α-stability region with the following LMI characterisation,

$$\mathscr{A}_{cl} X_{\mathscr{D}_1} + \mathscr{A}_{cl}^T X_{\mathscr{D}_1} + 2\alpha X_{\mathscr{D}_1} < 0, \quad X_{\mathscr{D}_1} > 0,$$

a disk with a radius r characterised in terms of LMIs as,

$$\begin{bmatrix} -rX_{\mathscr{D}_2} & \star \\ (\mathscr{A}_{cl} X_{\mathscr{D}_2})^T & -rX_{\mathscr{D}_2} \end{bmatrix} < 0, \quad X_{\mathscr{D}_2} > 0,$$

and the conic sector with the LMI characterisation as,

$$\begin{aligned} (W_\theta \otimes \mathscr{A}_{cl})\operatorname{diag}(X_{\mathscr{D}_3}, X_{\mathscr{D}_3}) + \operatorname{diag}(X_{\mathscr{D}_3}, X_{\mathscr{D}_3})(W_\theta \otimes \mathscr{A}_{cl})^T < 0, \\ X_{\mathscr{D}_3} > 0, \end{aligned}$$

with $W_\theta = \left[\begin{smallmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{smallmatrix}\right]$. When the desired region is the intersection of several elementary LMI regions, as discussed in [10], the synthesis problem would not be convex when different Lyapunov matrix is used for each constraint. One alternative to address this issue is to use the same matrix decision variable for all the LMIs involved in the problem; i.e. $\boldsymbol{X} = \boldsymbol{X}_{\mathscr{D}_1} = \boldsymbol{X}_{\mathscr{D}_2} = \boldsymbol{X}_{\mathscr{D}_3}$, at the expense of additional conservatism. However, as stated in [10], the conservatism emanated from this limitation is modest in most applications. Finally, we augment the optimisation problem described in (MSOP) with the three elementary LMI regions as

$$\text{minimise} \quad \text{trace}(Z) + \eta_y \left\| W_o \circ Y_M \right\|_{\ell_1}, \qquad \text{(SPRPP)}$$

subject to (15), (16), (17), $\mathcal{S}(X_M) = I$, and

$$AX + XA^T + B_2Y + Y^T B_2^T + 2\alpha X < 0 \qquad (22)$$

$$\begin{bmatrix} (AX + XA^T + B_2Y + Y^T B_2^T)\sin\theta & \star \\ (AX - XA^T + B_2Y - Y^T B_2^T)^T \cos\theta & (AX + XA^T + B_2Y + Y^T B_2^T)\sin\theta \end{bmatrix} < 0 \qquad (23)$$

$$\begin{bmatrix} -rX & \star \\ (AX + B_2Y)^T & -rX \end{bmatrix} < 0 \qquad (24)$$

Then the poles of $A + B_2F_yC$, in which F_y is a sparse SOF obtained from the optimisation problem in (SPRPP), lie within the region $\mathsf{S}(\alpha, r, \theta)$.

4 $\mathscr{H}_\infty$-Based Sparse SOF Design With Regional Pole Placement Constraints

This section would alternatively consider a tradeoff between the number of communication links, i.e. the number of nonzero entries in the feedback gain, and the achievable $\mathscr{H}_\infty$ performance of the system. Consider the following output performance signal

$$z_\infty(t) = C_\infty x(t) + D_\infty u(t), \qquad (25)$$

where matrices C_∞ and D_∞ are constant and of appropriate dimensions. The standard H_∞ SF control problem can be solved through the following LMI [4]:

$$\begin{bmatrix} AX_\infty + X_\infty A^T + B_2Y_\infty + Y_\infty^T B_2^T + \gamma^{-2} B_1 B_1^T & \star \\ C_\infty X_\infty + D_\infty Y_\infty & -I \end{bmatrix} < 0, \qquad (26)$$

$$X_\infty > 0, \qquad (27)$$

where the decision matrices $X_\infty > 0$ and Y_∞ are of appropriate dimensions, $\gamma > 0$ is a scalar parameter which indeed bounds the H_∞-norm of the closed-loop transfer function from f to z_∞. In this case the SF gain is obtained as $F_\infty = Y_\infty X_\infty^{-1}$. One may search for the smallest value of γ under static output-feedback through the following optimisation problem

$$\text{minimise } \gamma \text{ subject to (26), (27), } Y_\infty X_\infty^{-1} = F_y^\infty C,$$

where $F_y^\infty \in \mathbb{R}^{m\times p}$. Notice that the above optimisation problem is not convex in the current form. By introducing $\lambda = \gamma^{-2}$, we can alternatively make the following convex optimisation problem

$$\text{minimise } -\lambda \text{ subject to (26), (27), } Y_\infty X_\infty^{-1} = F_y^\infty C.$$

Now the H_∞ structured SOF problem can be cast as the following optimisation problem

$$\begin{gathered}\text{minimise } -\lambda \text{ subject to (26), (27),} \\ Y_\infty X_\infty^{-1} = F_y^\infty C,\ \mathcal{S}(Y_\infty) \subseteq \Gamma_y,\ \mathcal{S}(X_\infty) = I.\end{gathered} \tag{HI}$$

This problem will be solved via similar LMI decision variable (X_∞ and Y_∞) transformations introduced in Section 3 as

$$\begin{aligned} X_\infty &= N X_\infty^N N^T + M X_\infty^M M^T, \\ Y &= Y_\infty^M M^T, \end{aligned} \tag{28}$$

where $X_\infty^N \in \mathbb{R}^{(n-p)\times(n-p)}$ and $X_\infty^M \in \mathbb{R}^{p\times p}$ are symmetric matrices, and $Y_\infty^M \in \mathbb{R}^{m\times p}$. Moreover, N and M are introduced in Section 3. In addition the minimisation problem (HI) may be augmented by the regional constraints similarly as in (22), (23) and (24). Then (HI) will be converted to

$$\begin{gathered}\text{minimise } -\lambda \text{ subject to (26), (27), (28),} \\ \mathcal{S}(Y_\infty^M) \subseteq \Gamma_y,\ \mathcal{S}(X_\infty^M) = I,\ \text{(22), (23) and (24),} \\ \text{by letting } X = X_\infty,\ Y = Y_\infty.\end{gathered} \tag{SOHI}$$

Accordingly, the problem of designing the sparse $\mathscr{H}_\infty$ SOF gains can be set as the following optimisation problem,

$$\begin{gathered}\text{minimise } -\lambda + \eta_y \left\| W_o \circ Y_\infty^M \right\|_{\ell_1} \\ \text{subject to (26), (27), (28), } \mathcal{S}(X_\infty^M) = I,\ \text{(22), (23) and (24),} \\ \text{by letting } X = X_\infty,\ Y = Y_\infty,\end{gathered} \tag{SPOHI}$$

where $\eta_y > 0$ is the regularisation parameter and W_o is a known weighting matrix with the same dimension of $\mathsf{S}(F_y^\infty)$. A similar reweighted ℓ_1 algorithm (obtained by replacing trace(Z) with $-\lambda$ in Algorithm B.1) can be used then to identify a pattern for $\mathscr{H}_\infty$ sparse SOF gain regarding the specific value of η_y. Following to the identified pattern, we build it into the structure matrix Γ_y and solve (SOHI) to obtain the $\mathscr{H}_\infty$ structured SOF gain.

5 Numerical examples

5.1 Example 1: Sparse SF design with pole placement

As the first example, a decentralised interconnected system, presented in [27] and [18] that consists of three subsystems with two states, is considered. We have the partitioned matrices:

$$A = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0.5 & 1 & 0.6 & 0 \\ -2 & -3 & 1 & 0 & 0 & 1 \\ \hline 0 & 2 & 0.5 & 1 & 1 & 0.5 \\ 1 & 3 & 0 & 0.5 & 0 & -0.5 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 \\ -3 & -4 & 0 & 0.5 & 0.5 & 0 \end{array}\right], \; B_2 = \left[\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ \hline 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right],$$

$$B_1 = I, \; Q = I, \; R = I.$$

As can be seen, this system is unstable and fully coupled. Solving the convex problem in (SSF), by assuming a block diagonal structure for matrix decision variable X as diag$[X_i]_{i=1}^3$ with $X_i \in \mathbb{R}^{2\times 2}$, would result in a true value $\mathscr{H}_2$ cost of 4.2295. Utilising Algorithm A.1 in Appendix with the revisions explained in Remark 5 and $\kappa = 0.01$, $\epsilon = 0.1$, and by increasing η from zero, the number of nonzero off-diagonal blocks of the SF gain decreases; see Fig. 1. When the sparsity structures of controllers are identified for different η, the obtained patterns are used to solve (SSF) and obtain the optimal structured controllers. Notice that in the case of fully decentralised feedback the $\mathscr{H}_2$ cost is 4.4682 which is only about 6% worse than that of the centralised feedback gain. Notice that the most sparse structure reported by [27] for the controller is not decentralised structure, whereas this reference does not even penalise the level of control effort in the associated objective function; i.e. $z(t) = Ix(t)$. In other words, according to the control structure sparsification scheme proposed in our paper, we may stabilise the system under study with a

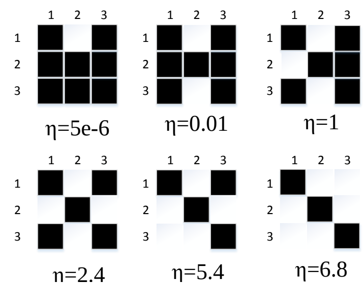

Figure 1: Structure of $\mathscr{H}_2$ sparse SF gains for different values of η for Example 1

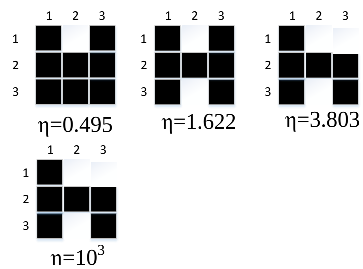

Figure 2: Structure of $\mathscr{H}_2$ sparse SF gains for different values of η with additional pole placement constraints for Example 1

decentralised SF controller and, in such a case, the performance degradation remains in an acceptable range.
Now we aim to assign the closed-loop poles within the intersection of the half-plane $x < -\alpha < -1$ with the disk of radius $r = 15$ centered at the origin and the sector centred at the origin making an angle of $\theta = \pi/3$. Once again we use Algorithm A.1 in the Appendix by imposing additional pole placement constraints introduced in (22), (23) and (24) and assuming $\kappa = 1\mathrm{e}{-6}$, $\epsilon = 0.01$ and increasing η from zero to a very large value. In this case, however, the rate of truncation of the off-diagonal blocks is slower compared with the previous case. Besides as can be seen in Fig. 2, even by increasing η to a very large number, the most sparse structure that could satisfy the optimisation problem in (SPRPP) (with $C = I_n$) is not the decentralised feedback gain. Fig. 5 demonstrates the closed-loop pole locations using the decentralised feedback gain obtained from the structured optimisation problem in (SSF) that does not involve any regional pole-placement constraints and the most sparse structure obtained in the second case with regional constraints on the

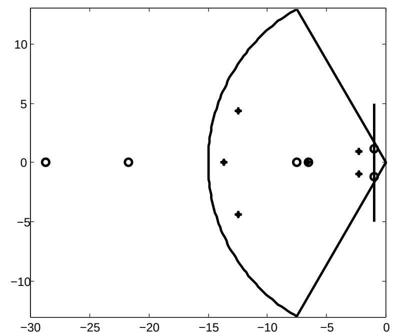

Figure 3: Closed-loop pole locations using the most sparse structure obtained in the second case ($*$), the decentralised structure for the first case (o) and $\mathsf{D}(1, 15, \frac{\pi}{3})$ for Example 1

nominal closed-loop poles.

5.2 Example 2: Sparse SOF design

Consider the system given in [26] with the following matrices; which is element-wise without any partition:

$$A = \begin{bmatrix} -4 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & -1 \\ 0 & -2 & 0 & -1 & 0 \\ 3 & 0 & -2 & 0 & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T, \; Q = I, \; R = I.$$

Besides, we select

$$M = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; N^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; D = 0.$$

5.2.1 Sparse $\mathscr{H}_2$ SOF

With imposing no structure on $\boldsymbol{Y_M}$ and $\boldsymbol{X_M}$, the $\mathscr{H}_2$ cost is 2.0903. However, this value will be 2.1504 for the case of $\mathcal{S}(\boldsymbol{X_M}) = \boldsymbol{I}$. Exploiting Algorithm B.1 in Appendix with $\kappa_y = 0.001$, $\epsilon_y = 0.0001$ and by increasing η_y from zero, the number of nonzero off-diagonal entries of the SOF gain decreases. The most sparse structure is identified by $\eta_y \geq 0.752$. With the identified structure we solve (SSOF) to obtain a suboptimal structured SOF controller as

$$\boldsymbol{F}_y^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3.1264 \end{bmatrix}.$$

In this case the $\mathscr{H}_2$ cost is 2.3148. It means that only about 8% performance degradation happens compared to the centralised feedback gain.
Now let us repeat this example to ensure that the complex conjugate pairs of poles of the closed-loop system have a damping ratio greater than 0.7660. To this end, we enforce the closed-loop poles to lie in the sector centred at the origin making an angle of $\pm 2\pi/9$ relative to the negative real axis. In order to ensure this, we solve the optimisation problem in (MSOP) together with the constraint in (23). The most sparse pattern is achieved by $\eta_y \geq 0.297$. The SOF controller is then

$$\boldsymbol{F}_y^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.3409 & 0 \\ 0 & 0 & -3.1498 \end{bmatrix}.$$

The $\mathscr{H}_2$ cost is 2.4959 and the closed-loop poles are $\{-4.6607 \pm 1.2063i, -0.8283, -1.5000 \pm 1.0335i\}$ which satisfy the damping ratio constraint.

5.2.2 Sparse $\mathscr{H}_\infty$ SOF

Let us once again consider the system given in this example but considering the $\mathscr{H}_\infty$ performance. In doing so, we set $\boldsymbol{C}_\infty = \boldsymbol{C}_z$ and $\boldsymbol{D}_\infty = \boldsymbol{D}_z$ and recall the optimisation problem in (HI) without imposing any structure on $\boldsymbol{X}_M^\infty$ and $\boldsymbol{Y}_M^\infty$ to obtain the $\mathscr{H}_\infty$ performance level of the closed loop with centralised controller, which is 1.2077. Besides, this value will increase to 1.2364 for the case of $\mathcal{S}(\boldsymbol{X}_M^\infty) = \boldsymbol{I}$. We now exploit Algorithm B.1 in Appendix, which is revised by replacing trace($\boldsymbol{Z}$) with $-\lambda$. With $\kappa_y = 0.001$, $\epsilon_y = 0.01$ and by increasing η_y from zero, it can be seen that the number of nonzero off-diagonal

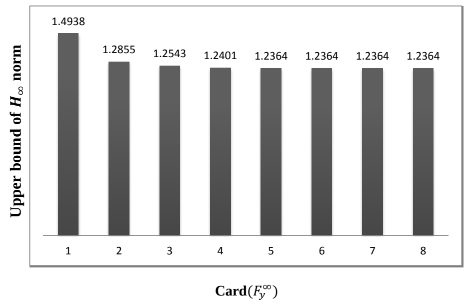

Figure 4: Upper bound of the $\mathscr{H}_\infty$ performance of closed loop versus different sparsity levels of the feedback gain

entries of the SOF gain decreases. Fig. 4 shows the upper bound of $\mathscr{H}_\infty$ norm of the closed-loop system versus different sparsity level of the feedback gains. As can be seen, almost no remarkable performance degradation is visible between the centralised controller with $\mathcal{S}(X_M^\infty) = I$ and a distributed controller up to $\mathbf{card}(F_y^\infty) = 2$. Note that $\mathbf{card}(F_y^\infty) = 2$ corresponds to the entries $(1,1)$ and $(3,3)$, and $\mathbf{card}(F_y^\infty) = 1$ corresponds to the entry $(3,3)$. It is shown in Fig. 4 that the entry $(1,1)$ is significant for the closed-loop control performance as there is a large performance degradation when removing it from the controller gain.

5.3 Example 3: Sparse $\mathscr{H}_2$ SF design

As the last example we consider a first-order system with $\mathcal{N} = 5$ nodes distributed over a circle. This example is studied in [38] by using the alternating direction method of multipliers (ADMM) as a tool for solving the problem numerically. Here, we show the effectiveness of our LMI-based algorithm. The dynamics of the system are defined by

$$
\begin{aligned}
A &= \text{Toeplitz}([-2,\ 1,\ \overbrace{0,\ \cdots,\ 0}^{\mathcal{N}-3},\ 1]),\\
B_2 &= B_1 = I_{\mathcal{N}},\ Q = I_{\mathcal{N}},\ R = I_{\mathcal{N}}.
\end{aligned}
$$

The centralised gain leads to an $\mathscr{H}_2$ cost of 1.3853. Also letting the decision matrix X in (SSF) to be a diagonal one with the full decision matrix Y will result in an $\mathscr{H}_2$ cost of 1.4614, which shows 5% conservatism. By increasing

η from zero to 5 in Algorithm A.1 (in Appendix) with $\kappa = 0.001$, $\epsilon = 0.1$, the number of nonzero off-diagonal entries of the SF gain decreases, such that when $\eta \geq 1.8 \times 10^{-6}$ the identified pattern is diagonal for the SF. Solving the $\mathscr{H}_2$ optimal structured problem in (7) and (8) by imposing the diagonal structure on both $\boldsymbol{X}$ and $\boldsymbol{Y}$ would result in the following SF

$$\boldsymbol{F} = -\boldsymbol{I}_{\mathcal{N}}.$$

Moreover, the $\mathscr{H}_2$ cost, in this case, is 1.5076, which is about 9% worse than that of the centralised feedback. This performance degradation, introduced by the diagonal structure of the SF into the optimisation problem, roughly speaking, is same as the one reported in [38].

5.4 Example 4: Flight control

In over-actuated systems, there is the possibility to select a subset of available actuators in order to, for example, minimise power or fuel consumption and/or actuator wear and tear, etc [17]. The optimisation problem proposed in (9) (or its relaxed version in (SPSF)), achieved by incorporating a secondary cost function into the main cost function, can be revised so that it can select a subset of available actuators/effectors in an optimal manner. This problem can be formulated as:

$$\begin{aligned} \text{minimise} \quad & \operatorname{trace}(\boldsymbol{Z}) + \eta \, \|\boldsymbol{Y}\|_{\text{row}-\ell_0}, \\ & \text{subject to (7) and (8)}, \end{aligned} \tag{ROW0}$$

where $\boldsymbol{X} > 0$ and $\boldsymbol{Y}$ are full decision matrices that do not have any a priori structures, the row-ℓ_0 norm is a quasi-norm that counts the number of non-zero rows of $\boldsymbol{Y}$, and $\eta > 0$ is the regularisation parameter that implies the emphasis on the row-sparsity of $\boldsymbol{Y}$, and thus the SF gain $\boldsymbol{F}$. Indeed, the row-sparse structure of the feedback gain defines the redundant components of the control inputs or the redundant actuators in the system. However, the variable selection, as proposed earlier in this paper, amounts to the selection of important individual variables (elements in the feedback gain) rather than the important groups of variables (rows).

Let us now recast the optimisation problem (ROW0) as

$$\begin{aligned} \text{minimise} \quad & \operatorname{trace}(\boldsymbol{Z}) + \eta f(\boldsymbol{Y}) \\ & \text{subject to (7) and (8)}, \end{aligned} \tag{MSOP}$$

where $f(\cdot)$ denotes the relaxed row-sparsity promoting function for which a choice can be proposed as:

$$f(Y) = \sum_{i,j} W_i |Y_{ij}|. \tag{29}$$

Let the update rule be as

$$W_i^l = \frac{1}{\sum_j |Y_{ij}^{(l-1)}| + \epsilon}, \tag{30}$$

where l denotes the current iteration and $0 < \epsilon \ll 1$ is used to provide stability and to ensure that a zero valued entry in Y does not strictly prevent a non-zero value at the next step. The weighting matrix can be formed as $W = \text{diag}[W_i]_{i=1}^m$. In other words, each entry of the diagonal weighting matrix will be updated inversely proportional to the ℓ_1-norm of its corresponding row in the feedback gain obtained at the previous iteration. Notice that one can also imagine a variety of possible norms in place of (30), e.g., ℓ_2-norm and ℓ_∞-norm. Finally, Algorithm A.1 can be revised according to the above-mentioned formulations so that it can identify a row sparse SF.

Now we consider the B747 aircraft [1] whose 12 rigid body states can be split into two separate axes: 6 longitudinal axis states and 6 lateral and directional axes states. Same as in [1], we only consider the first four states of the lateral axis which are the roll rate p, yaw rate r, sideslip angle β, and roll angle ϕ. Considering an operating condition of 263,000 Kg, 92.6 m/s true airspeed, and 600 m altitude at 25.6 % of maximum thrust and at a 20° flap position, a linear model can be obtained. In this case, the lateral system, about the trim condition, is represented as:

$$A = \begin{bmatrix} -1.0579 & 0.1718 & -1.6478 & 0.0004 \\ -0.1186 & -0.2066 & 0.2767 & -0.0019 \\ 0.1014 & -0.9887 & -0.0999 & 0.1055 \\ 1.0000 & 0.0893 & 0 & 0 \end{bmatrix}, \tag{31}$$

$$B = \begin{bmatrix} -0.0832 & 0.0832 & -0.2285 & 0.2285 & -0.2625 & -0.0678 & 0.0678 \\ -0.0154 & 0.0154 & -0.0123 & 0.0123 & -0.0180 & -0.0052 & 0.0052 \\ 0 & 0 & 0 & 0 & 0.0017 & 0.0006 & -0.0006 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix.$$
$$\begin{matrix} 0.2625 & 0.1187 & 0.0246 & 0.0140 & -0.0140 & -0.0246 \\ 0.0180 & -0.2478 & 0.1269 & 0.0724 & -0.0724 & -0.1269 \\ -0.0017 & 0.0174 & 0.0005 & 0.0005 & -0.0005 & -0.0005 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the lateral control surfaces are

$$\delta_{lat} = \begin{bmatrix} \delta_{air} & \delta_{ail} & \delta_{aor} & \delta_{aol} & \delta_{sp1-4} & \delta_{sp5} & \delta_{sp8} \\ & & & & & & \delta_{sp9-12} & \delta_r & e_1 & e_2 & e_3 & e_4 \end{bmatrix}. \quad (32)$$

denoting aileron deflection (right and left - inner and outer)(rad), spoiler deflections (left: 1-4 and 5, right: 8 and 9-12) (rad), rudder deflection (rad) and lateral contributions to the engine pressure ratios (EPR), respectively. Let the system outputs be sideslip angle β and roll angle ϕ, and thus the output distribution matrix is

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (33)$$

A tracking facility can be included in the problem, by exploiting an integral action. Defining

$$\dot{\xi}(t) = \mathsf{r}(t) - y(t), \quad (34)$$

where $\mathsf{r}(t)$ is the input reference to be tracked by $y(t) = Cx(t) \in \mathbb{R}^p$, and ξ represents the integral of the tracking error, i.e. $\mathsf{r}(t) - y(t)$, and introducing $\tilde{x} := \left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$, an augmented system can be derived as:

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t) + \tilde{B}_1 f(t) + B_{\mathsf{r}}\mathsf{r}(t) \\ \tilde{z}(t) &= \tilde{C}_z\tilde{x}(t) + \tilde{D}_z u(t), \end{aligned} \quad (35)$$

with

$$\tilde{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \ \tilde{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} I_4 \\ 0 \end{bmatrix}, \ B_{\mathsf{r}} = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad (36)$$
$$\tilde{C}_z = \begin{bmatrix} \mathrm{diag}(10, 10, 1, 1, 0.1, 0.1) \\ 0_{13\times 6} \end{bmatrix}, \ \tilde{D}_z = \begin{bmatrix} 0_{6\times 13} \\ \mathrm{diag}(\sqrt{0.5}I_8, \sqrt{2}I_5) \end{bmatrix}.$$

Note that the last two nonzero terms of $\tilde{C}_z$ are associated with the integral action and are less heavily weighted. Further, the first and second nonzero terms of $\tilde{C}_z$ are more strongly weighted in comparison with the third and fourth nonzero terms to provide an adequate quick closed-loop response in terms of the angular acceleration in roll and yaw. Note also that if the matrix pair (A, B_2) is controllable and the matrix triplet (A, B_2, C) has no zeros at the origin, it can be shown that $(\tilde{A}, \tilde{B})$ is controllable [1]. It is assumed the whole system states are available to the controller. Now the control law can be considered as:

$$u(t) = \begin{bmatrix} F & F_{\mathrm{r}} \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} \triangleq \mathscr{F}\tilde{x}(t), \tag{37}$$

where $F \in \mathbb{R}^{m\times n}$ is the SF gain, $F_{\mathrm{r}} \in \mathbb{R}^{m\times p}$ is the feed-forward gain due to the reference signal $\mathsf{r}(t)$. It is also aimed to assign the closed-loop poles in the half-plane $x < -\alpha < -0.1$. Imposing no structure on Y, solving the optimisation problem in (MSOP), which is augmented with the constraint in (22), the upper bound of the $\mathscr{H}_2$-norm is 10.5907. We then use Algorithm A.1 in the Appendix with $\kappa = 0.01$, $\epsilon = 0.0001$ and exploit the row-sparsity promoting function (29) and the update rule in (30). By increasing η from zero, the number of non-zero rows of the SF gain decreases; e.g., letting $\eta = 2000$ the most row-sparse is suggested by the algorithm, i.e. exploiting only δ_{sp1-4} and δ_r. With the identified structure for Y; i.e. $\mathcal{S}(Y) \subseteq \Gamma$, we turn to the problem in (SSF), by letting $X > 0$ be a full decision matrix, to obtain a suboptimal structured SF controller. In this case, the upper bound of $\mathscr{H}_2$-norm of the closed-loop system is 12.4986, which is about 18% worse than that obtained by the non-structured feedback gain.

Now, we run the RISD algorithm (Algorithm C.1), revised for the row-wise sparsity pattern recognition by letting $\Lambda_{l+1} = \left\{ i \;\middle|\; \sum_{j=1}^{p} |Y^l_{M,ij}| > \epsilon_l \right\}$, replacing $\mathrm{col}\big(\mathrm{col}(|Y_{M,ij}|)_{j=1}^{p}\big)_{i=1}^{m}$ by $\mathrm{col}\big(\sum_{j=1}^{p} |Y_{M,ij}|\big)_{i=1}^{m}$, defining

$$\mu_l = \frac{\mathrm{range}\left(\mathrm{col}\left(\sum_{j=1}^{p} |Y^l_{M,ij}|\right)_{i=1}^{m}\right)}{\rho \cdot (m-1)}, \tag{38}$$

letting the weighting matrix be diagonal, i.e. $W_0 = I_{m\times m}$, and changing the element-wise multiplication by matrix multiplication, i.e. $W_o Y_M$. We also use the given parameters above in addition to $\rho = 1$ and $\eta = 2000$. It can be found that this algorithm promotes the same row-sparse structure as the

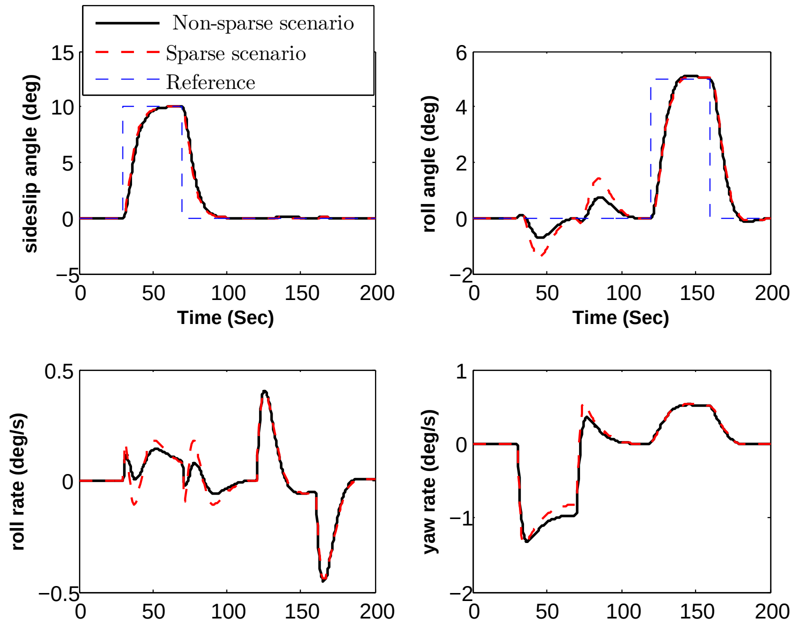

Figure 5: System states for lateral control

one suggested by the REL1 algorithm. However, it only takes 3 iterations for the RISD to satisfy the termination criterion, while the REL1 method terminates in 4 iterations. As can be seen, the RISD algorithm achieves the solution faster than REL1, which leads to a less computational time. It is worth noting that although the RISD algorithm gains slight improvements over REL1 in terms of computation time in this specific example, it has the potential to improve the computation time in different applications.
Considering a step change of 10 degrees for β during 30 to 70 s as well as a step change of 5 degrees for ϕ during 120 to 160 s , Figs. 5 and 6 show the tracking responses of the system when the controller governs 1) all the control surfaces (i.e. δ_{lat} in (32)), and 2) only δ_{sp1-4} and δ_r.

6 Conclusions

This paper proposes a framework for addressing the issue of designing $\mathscr{H}_2$ ($\mathscr{H}_\infty$) optimal (block) sparsely distributed control by utilising only system

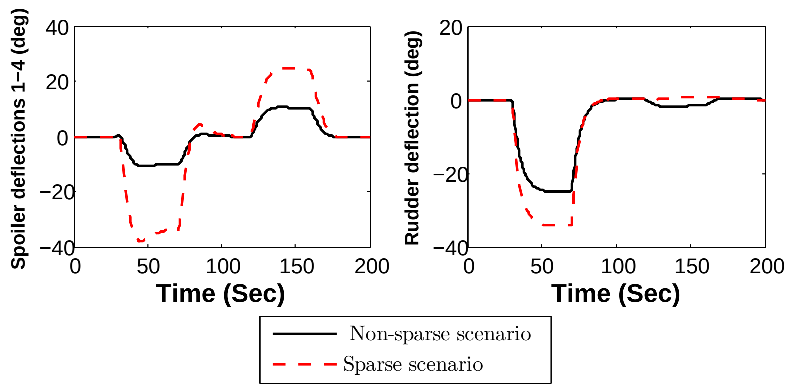

Figure 6: Control efforts for lateral control

sensors' signals. Besides it is discussed that this framework is capable of incorporating additional regional pole placement constraints on the feedback gain matrix as well as the structural constraints. This framework includes an explicit scheme and an iterative process in order to identify the desired sparse structure of the feedback gain. To this end, the so-called reweighted ℓ_1-norm, which is known as a convex relaxation of the ℓ_0-norm, is utilised to make a convex problem. Following this, the $\mathscr{H}_2$ ($\mathscr{H}_\infty$) structured static output (state) feedback design problem is solved using the achieved structure for the feedback gain. The simulation results are given to show the effectiveness of our proposed approach. Comparing to the existing literature, while our framework has the possibility of performing the SOF control design with additional constraints, the performance of our approach is quite acceptable when applied to the normal sparse SF gain design problem.

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Appendices

A Reweighted ℓ_1 minimisation method for the sparse SF problem

Using the reweighted ℓ_1 norm as a promoting sparsity has been considered in e.g. [14, 3]. Define the matrix $\boldsymbol{V}$ which has the same dimension as $\boldsymbol{Y}$, and all its entries equal to one. It can be shown that (e.g. see [14, 3]) the optimisation problem in (SPSF) is equivalent to

$$\begin{aligned}&\text{minimise trace}(\boldsymbol{Z})+\eta\,\text{trace}(\boldsymbol{V}^T\boldsymbol{G})\\&\text{subject to (7), (8), } \mathcal{S}(\boldsymbol{X})=\boldsymbol{I} \text{ and}\\&-\boldsymbol{G}\le \boldsymbol{W}\circ\boldsymbol{Y}\le\boldsymbol{G},\end{aligned}\tag{39}$$

where $0<\boldsymbol{X}\in\mathbb{R}^{n\times n}$, $\boldsymbol{Y}\in\mathbb{R}^{m\times n}$, $\boldsymbol{Z}$ is a slack variable, $\boldsymbol{W}$ denotes the weighting matrix and the last inequality is element-wise with $\boldsymbol{G}\in\mathbb{R}^{m\times n}$ whose entries are nonnegative. Then the algorithm to solve the above optimisation problem is as the following,

Algorithm A.1. 1) *With given $\epsilon>0$, $\kappa>0$ and $\eta>0$, initialise $\boldsymbol{W}=\mathbb{1}_{h\times h}$, $l=1$ and $\boldsymbol{Y}^l=0$.*

2) *Solve the minimisation problem* (39) *to obtain* $\boldsymbol{F}^{\star} = \boldsymbol{Y}^{\star}\boldsymbol{X}^{\star^{-1}}$.

3) *Update* $\boldsymbol{W}_{ij} = \frac{1}{\left\|Y_{ij}^{\star}\right\|_F + \epsilon}$ *and form* $\boldsymbol{W} = [W_{ij}]_{h\times h}$.

5) *If* $\left\|\boldsymbol{Y}^{\star} - \boldsymbol{Y}^{l}\right\| \leq \kappa$ *go to Step 6, else* $\boldsymbol{Y}^{l} = \boldsymbol{Y}^{\star}$, $l = l+1$ *and return to Step 2.*

6) *Return* $\boldsymbol{F}^{\star} = \boldsymbol{Y}^{\star}\boldsymbol{X}^{\star^{-1}}$.

Solving Algorithm A.1 gives the most effective sparse structure of $\boldsymbol{F}$. Then, by ignoring the unnecessary entries in $\boldsymbol{F}$ we find the structure matrix Γ.

B Reweighted ℓ_1 minimisation method for the sparse SOF problem

The optimisation problem in (MSOP) is equivalent to

$$
\begin{aligned}
&\text{minimise } \operatorname{trace}(\boldsymbol{Z}) + \eta_y \operatorname{trace}(\boldsymbol{V}_o^T \boldsymbol{G}_o) \\
&\text{subject to (17), (16), } \mathcal{S}(\boldsymbol{X}_M) = \boldsymbol{I} \text{ and} \\
&-\boldsymbol{G}_o \leq \boldsymbol{W}_o \circ \boldsymbol{Y}_M \leq \boldsymbol{G}_o,
\end{aligned}
\tag{40}
$$

where $\boldsymbol{W}_o$ denotes the weighting matrix, the last inequality is element-wise with $\boldsymbol{G}_o \in \mathbb{R}^{m\times p}$ whose entries are nonnegative and $\boldsymbol{V}_o \in \mathbb{R}^{m\times p}$ whose all entries are equal to one. Besides, to solve the above optimisation problem, the following algorithm is utilised,

Algorithm B.1. 1) *With given* $\epsilon_y > 0$, $\kappa_y > 0$ *and* $\eta_y > 0$, *initialise* $\boldsymbol{W}_o = \mathbb{1}_{h\times h}$, $l = 1$ *and* $\boldsymbol{Y}_M^l = 0$.

2) *Solve the minimisation problem* (40) *to obtain* $\boldsymbol{F}_y^{\star} = \boldsymbol{Y}_M^{\star}\boldsymbol{X}_M^{\star^{-1}}$.

3) *Update* $(\boldsymbol{W}_o)_{ij} = \frac{1}{\left\|(Y_M^{\star})_{ij}\right\|_F + \epsilon_y}$ *and form* $\boldsymbol{W}_o = [(W_o)_{ij}]_{h\times h}$.

5) *If* $\left\|\boldsymbol{Y}_M^{\star} - \boldsymbol{Y}_M^{l}\right\| \leq \kappa_y$ *go to Step 6, else* $\boldsymbol{Y}_M^{l} = \boldsymbol{Y}_M^{\star}$, $l = l+1$ *and return to Step 2.*

6) *Return* $\boldsymbol{F}_y^{\star} = \boldsymbol{Y}_M^{\star}\boldsymbol{X}_M^{\star^{-1}}$.

Solving Algorithm B.1 gives the most effective sparse structure of $\boldsymbol{F}_y$. By letting the unnecessary entries of $\boldsymbol{F}_y$ as zero, we achieve the structure matrix Γ_y.

C RISD method for the sparse SOF problem at component level

Algorithm C.1. 1) *With given* $\kappa > 0$ *and* $\eta > 0$*, initialise* $\Lambda_0 = \{\}$*,* $\boldsymbol{W}_o = \mathbb{1}_{m\times p}$*,* $l = 1$ *and* $\boldsymbol{Y}_M^l = 0$*.*

2) *Solve the minimisation problem* (40) *to obtain* $\boldsymbol{F}_y^\star = \boldsymbol{Y}_M^\star \boldsymbol{X}_M^{\star^{-1}}$*.*

3) *Update* ϵ_l *according to the first jump rule.*

   i) Sort $col\left(col(|Y_{M,ij}^l|)_{j=1}^p\right)_{i=1}^m \to y^l$ *in ascending order and let* y_h^l *be the magnitude of the* h*-th largest element of* y^l*.*

   ii) Find the smallest h *such that* $\left|y_{h+1}^l - y_h^l\right| > \mu_l$*, with*

$$\mu_l = \frac{range\left(col\left(col(|Y_{M,ij}^l|)_{j=1}^p\right)_{i=1}^m\right)}{\rho\cdot(mp-1)}, \tag{41}$$

   with ρ *some constants, which is discussed in Subsection 3.3.*

   iii) Set $\epsilon_l = y_h^l$*.*

4) *Update the detected support*

$$\Lambda_l = \left\{ i,j \;\middle|\; |Y_{M,ij}^l| > \epsilon_l \right\}.$$

5) *Update weights as*

$$W_{ij}^l = \begin{cases} \frac{1}{|Y_{M,ij}^l|} & if \quad i \in \Lambda_l \\ \frac{1}{\epsilon_l} & if \quad i \in \Lambda_l^C, \end{cases} \tag{42}$$

and form $\boldsymbol{W}^l = [W_{ij}^l]_{m\times p}$*.*

6) *If* $\left\| \boldsymbol{Y}_{\boldsymbol{M}}^{\star} - \boldsymbol{Y}_{\boldsymbol{M}}^{l} \right\| \leq \kappa$ *go to Step 7, else* $\boldsymbol{Y}_{\boldsymbol{M}}^{l} = \boldsymbol{Y}_{\boldsymbol{M}}^{\star}$, $l = l + 1$ *and return to Step 2.*

7) *Let the unnecessary rows of* $\boldsymbol{Y}_{\boldsymbol{M}}^{\star}$ *be zero and return* $\Gamma^{\star} = \mathcal{S}(\boldsymbol{F}_{y}^{\star}) = \mathcal{S}(\boldsymbol{Y}_{\boldsymbol{M}}^{\star})$.