Multirealisation of linear systems

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***Abstract*— For multiple model adaptive control systems, "multi-controller" architecture can be efficiently implemented (multirealised) by means of a "state-shared" parameter-dependent feedback system. Necessary and sufficient conditions for the multirealisation of a family of linear multivariable systems based on matrix fractional descriptions are presented. The problem of the minimal generic multirealisation of a set of linear systems is introduced and solved.**

***Index Terms*— System multirealisation, linear multivariable systems, switching systems, Multiple Model Adaptive Control.**

I. INTRODUCTION

JUST as one can consider a standard linear system realisation problem (given a transfer function, find a state-variable realisation), and a minimal realisation problem (ensure the state-variable realisation is of minimal degree), so for a finite collection of transfer functions one can consider a multirealisation problem and a minimal multirealisation problem. The original motivation for studying multirealisation problems comes from multiple model adaptive control (MMAC) algorithms [1] [2] [3] [4] [5] [6]. The implementation of "multi-controller" architecture is an important issue for MMAC applications. As argued in for example [4], because at any instant of time only one of the constituent controllers is to be applied to the plant, it is only necessary to generate one control signal at any time. Often this means significant simplification can be achieved if all control signals are capable of being generated by a single system. In other words, rather than implementing each of the controllers in the family as a separate dynamical system, one can often achieve the same results using a single controller with adjustable parameters (see Definition 1). Because the single controller state is, in effect, shared by the family of controllers, this implementation is termed a "state shared" multirealisation.

Almost all of the literature on linear system realisation deals with the implementation of a **single** linear time invariant (LTI) system [7] [8] [9] [10]. In contrast, Morse [4] presents some results for the **multirealisation** of **several** linear **scalar** systems in the context of examining MMAC for scalar plants. In this paper, we investigate the multirealisation of several linear **multiple input multiple output** (MIMO) systems; The results will be applicable to MMAC problems for MIMO plants.

Stability is an important issue for switched systems [1] [4]. In this paper, we assume that the time scale over which switchings occur is a longer time scale than the time scale for the dynamics of the various closed loop systems; this is virtually always the case in MMAC problems. Under this assumption, if each frozen closed loop system is stable then the switched system will be stable. Furthermore, the provided method can implement "bumpless" transfer between linear multivariable systems. It is well known that "bumpless" transfer is an effective way to improve poor transient response of switched systems [4]. One example is given here to show the main aim of this paper.

Consider two multivariable linear systems

$$\begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)(s+3)} \\ \frac{-6(s-2)}{(s-1)(s+3)} & \frac{s-2}{(s-1)(s+3)^2} \end{bmatrix} \quad and \quad \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{(s+2)} \end{bmatrix}.$$

A parameter dependent state space equation $\{A_0 + F_iC_0, B_i, C_0\}$ can be obtained by using Method 2 (at the end of Section III) to realise both these systems with only the parameters F_i and B_i system dependent:

$$A_0 = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 3 & -6 & 17 \\ 0 & 1 & 0 & 1 \end{bmatrix}^T, \quad B_2 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 2 & 1 & 1 \end{bmatrix}^T,$$

$$F_1 = \begin{bmatrix} 1 & 5 & -5 & 5 \\ 0 & 0 & 1 & 5 \end{bmatrix}^T, \quad and \quad F_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

It should be noted that A_0 is stable and the pair (C_0, A_0) is observable. When the transfer functions in question correspond to multiple controllers which may be switched serially, the multirealisation form $\{A_0 + F_iC_0, B_i, C_0\}$ can ensure that the output of the switched system remains continuous across switching instants, provided its input is reasonably well behaved, e.g. is piecewise continuous, i.e. "bumpless" transfer [4] is achieved. However, it is slightly more convenient to investigate the dual form $\{A_0 + B_0K_i, B_0, C_i\}$ because for this multirealisation form we can directly lift known results on the invariant description of linear multivariable systems [8] [11]. Corresponding results for the multirealisation form $\{A_0 + F_iC_0, B_i, C_0\}$ can be easily achieved by using the duality relationship (e.g. see Method 2).

The definition of the concept of a minimal stably based multirealisation is given as below:

Definition 1: Assume that there are given a number N of m-input p-output strictly proper real rational transfer function

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matrices P_i ($i \in \{1, \cdots, N\}$). A **multirealisation** of the set of systems P_i is a set of state variable realisations $\{A_0 + B_0K_i, B_0, C_i\}$ (with the pair (A_0, B_0) being controllable and adjustable parameter matrices C_i and K_i) realising all the systems P_i ($i \in \{1, \cdots, N\}$). If all eigenvalues of A_0 are in the left half plane, $\{A_0 + B_0K_i, B_0, C_i\}$ is termed a **stably based** multirealisation of the set of systems P_i ($i \in \{1, \cdots, N\}$). Furthermore, if the dimension of A_0 is the smallest of all such stably based multirealisations, then we call $\{A_0 + B_0K_i, B_0, C_i\}$ a **minimal** stably based multirealisation of the set of systems P_i ($i \in \{1, \cdots, N\}$).

Because of the assumption of controllability of the pair (A_0, B_0), it is evident that the requirement that the multirealisation be stably based poses no extra theoretical challenge. (If A_0 is not stable, find $\bar{K}$ so that $A_0 + B_0\bar{K}$ is stable, and replace K_i by $K_i - \bar{K}$). It is important in MMAC implementation for a multirealisation to be stably based, [4].

Standard concepts and notations, such as column degree and column reduced polynomial matrices, are defined as in [8]. A new operator ($\mathcal{D}_{hc}\{\cdot\}$) is introduced as below:

Definition 2: Given a polynomial matrix $D(s)$, it is always possible to write $D(s) = D^{hc}S(s) + D_{lc}\Psi(s)$. Where, $S(s) \triangleq diag\{s^{k_1}, s^{k_2}, \cdots, s^{k_m}\}$, k_i is the degree of the i-th column of $D(s)$, D^{hc} is a matrix formed from the coefficients of the highest degree polynomials in the columns of $D(s)$ (highest-degree-coefficient matrix),
$\Psi^T(s) \triangleq block\ diag\{[s^{k_1-1}, \cdots, s, 1], [s^{k_m-1}, \cdots, s, 1]\}$,
and D_{lc} is a matrix formed from the remaining coefficients of polynomials in the columns of $D(s)$ (lower-degree-coefficient matrix).

Define the operator $\mathcal{D}_{hc}(\cdot)$ as $\mathcal{D}_{hc}(D(s)) = D^{hc}S(s)$.

In the next section, necessary and sufficient conditions for multirealisation of multivariable systems is presented. Section III presents results for the minimal generic multirealisation problem for any given set of linear systems with compatible input and output dimensions.

II. Conditions of multirealisation

To derive conditions for the multirealisation of multivariable systems, we need to recall properties of the Popov form [8] [11] of polynomial matrices. The relationship between invariant Popov parameters α_{ijk} of a controllable pair (A, B) and the coefficients in a Popov form matrix $D_E(s)$ can be stated:

Lemma 1: For a Popov form polynomial matrix $D_E(s)$, if we denote

$$D_E(s) = [d_{ij}(s)] = [\sum_{l=0}^{k_j} d_{ijl}s^l] \tag{1}$$

then

$$d_{ijl} = \begin{cases} 1, & if\ l = k_j\ and\ i = p_j, \\ -\alpha_{p_jil} & \{l < k_j\}\ or \{l = k_j\ and\ i < p_j\}. \end{cases} \tag{2}$$

and d_{ijl} otherwise is zero. Here, p_j denote the j_{th} column pivot index of the polynomial matrix $D_E(s)$ [8] and the $\{\alpha_{ijk}\}$ are the Popov parameters of any controllable state variable realisation of $D_E^{-1}(s)$.

Proof: See equation (17) and associated statements on page 482 of [8]. ■

The following theorem relates the column degrees of a Popov polynomial matrix $D_E(s)$ and the controllability indices of a controllable pair (A, B) of a minimal state variable realisation of $D_E^{-1}(s)$. As far as the authors are aware, the following theorem has not been explicitly stated in the literature, though the ideas are probably known.

Theorem 1: Consider a strictly proper multivariable system $H(s)$ described by a right polynomial matrix fraction description (MFD), i.e. $H(s) = N_E(s)D_E^{-1}(s)$ where also $D_E(s)$ is a Popov polynomial matrix. Let k_i denote the i_{th} column degree of the Popov polynomial matrix $D_E(s)$, p_j denote the j_{th} column pivot index of the Popov polynomial matrix $D_E(s)$, and d_i denote the i_{th} controllability index of a controllable pair (A, B) of a state variable realisation of $D_E^{-1}(s)$. Then

i) $k_i = d_{p_i}$.

ii) The real matrix D_E^{hc}, the highest-degree-coefficient matrix of the columns of the polynomial matrix $D_E(s)$, is the identity matrix. i.e. $D_E^{hc} = I$, if and only if the i_{th} column pivot index of the polynomial matrix $D_E(s)$ is equal to i (That is equivalent, according to i) of this theorem, to the condition that $d_1 \leq d_2 \leq \cdots \leq d_m$).

Proof: i) Through post multiplication by a real matrix R, the columns of the Popov polynomial matrix $D_E(s)$ can be reordered so that the i_{th} column pivot index of the reordered polynomial matrix is equal to i. If we denote $\tilde{D}(s) = D_E(s)R$, and $\tilde{k}_i$ as the i_{th} column degree of the reordered polynomial matrix $\tilde{D}(s)$, then, we have

$$k_i = \tilde{k}_{p_i}. \tag{3}$$

It is easy to see that $\tilde{D}^{hc}$, the highest-degree-coefficient of matrix $\tilde{D}(s)$ is an upper triangular matrix. Then, we realise the right MFDs $H(s) = \tilde{N}(s)\tilde{D}_E^{-1}(s)$ by $\{A_c, B_c, C_c\}$, which is a controller form realisation by using the method in [8] (pp403-407). Considering that $\tilde{D}^{hc}$ is an upper triangular matrix, we can check that the controllability indices of the controllable pair (A_c, B_c) are $d_i = \tilde{k}_i$ according to [8] (see equation (8)-(10) in pp406-407 and the associated discussion). Thus, we have $k_i = d_{p_i}$.

ii) The necessity is obvious. We prove the sufficiency here. If for each i the i_{th} column pivot index of the polynomial matrix $D_E(s)$ is equal to i, then according to 2.c in [8] (p481, the description of a Popov form polynomial matrix), we conclude that D_E^{hc} is an upper triangular matrix. Furthermore, according to 2.b and 2.e in [8] of the description of a Popov form polynomial matrix (pp481-482, all entries in a row containing apart from the pivot element have degree lower than that of the pivot element), we conclude that $D_E^{hc} = I$. ■

Now, we present necessary and sufficient conditions for the existence of a multirealisation of given MIMO systems.

Theorem 2: (First Main Result) For a set of m-input p-output strictly proper systems $H_i(s)$ ($i \in \{1, \cdots, N\}$), there exists a controllable pair (A_0, B_0) ($dim\{A_0\} = n$), and appropriately dimensioned real matrices C_i and K_i (for $i \in \{1, \cdots, N\}$) such that A_0 is stable, and $\{A_0 + B_0K_i, B_0, C_i\}$ is a controllable realization of system $H_i(s)$, (for $i \in$

$\{1, \cdots, N\}$), if and only if, there exists a right polynomial MFD for each system $H_i(s)$ described by $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$ (where $D_{Ei}(s)$ is a Popov polynomial matrix with degree n, i.e. $deg\{D_{Ei}(s)\} = n, \forall i \in \{1, \cdots, N\}$) such that

i) $k_{il} = k_{jl}$ for $i, j \in \{1, \cdots, N\}$ and $l \in \{1, \cdots, m\}$, where k_{ij} is the j_{th} column degree of the matrix $D_{Ei}(s)$, and

ii) the matrices D_{Ei}^{hc}, which are the highest-degree-coefficient matrices of the $D_{Ei}(s)$, are identical (for $i \in \{1, \cdots, N\}$).

Proof: Assume first the existence of the controllable state variable multirealisation for $H_i(s)$ ($i \in \{1, \cdots, N\}$). It is standard that there exists a column reduced polynomial matrix $D_i(s)$ with $det[D_i(s)] = det(sI - A_0 - B_0K_i)$ and with column degrees corresponding (though possibly with different ordering) with the controllability indices d_i of (A_0, B_0) which are the same as those of $\{A_0 + B_0K_i, B_0\}$ (see [8] or [10]). Further, $D_i(s)$ is such that for any constant matrix $F \in \mathcal{R}^{p \times n}$, there exists an associated $N_F(s)_{p \times m}$ such that

$$F(sI - A_0 - B_0K_i)^{-1}B_0 = N_F(s)D_i^{-1}(s).$$

Conversely for any polynomial matrix $N_F(s)$ such that $N_F(s)D_i^{-1}(s)$ is strictly proper, there exists a real matrix F satisfying this equation.

Clearly, there exists a polynomial $N_i(s)$ such that $H_i(s) = N_i(s)D_i^{-1}(s)$. Without loss of generality, we can replace $D_i(s)$ by its Popov form $D_{Ei}(s)$, so that $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$. Further, the column degrees of each $D_{Ei}(s)$ are the controllability indices of (A_0, B_0) (though possibly with different ordering). In fact, with k_{ij} the column degree of the j_{th} column of $D_{Ei}(s)$, there holds $k_{ij} = d_{ip_j}$, by Theorem 1, where p_j is the j_{th} column pivot index.

By Theorem 3 of [11], the Popov parameters α_{ljd_l} of $\{A_0 + B_0K_i, B_0\}$ and $\{A_0, B_0\}$ are the same for $j \in \{1, \cdots, l-1\}$ (and $d_j > d_l$). (Equivalently, the parameters $\alpha_{p_ljd_{p_l}}$ are the same for $j < p_l$, and $d_{p_j} > d_{p_l}$.)

Now in $D_{Ei}(s)$, the j_{th} column for all i has maximum degree k_j by equation (2). Recalling (1), we see that the associated column of D_{Ei}^{hc} is

$$\begin{aligned} & [d_{1jk_j} \; d_{2jk_j} \; \cdots \; d_{p_jjk_j} \; 0 \cdots 0]^T \\ = \; & [-\alpha_{p_j1d_{p_j}} - \alpha_{p_j2d_{p_j}} \; \cdots - \alpha_{p_j,p_j-1,d_{p_j}} \; 1 \; 0 \cdots 0]^T \end{aligned}$$

which is the same for each $D_{Ei}(s)$. This proves claim ii).

Conversely, suppose there exist right polynomial MFDs $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$ where D_{Ei} is a Popov polynomial matrix of degree n for all i, and the other conditions of the theorem statement hold. Let (A_i, B_i) be a completely controllable pair in a state variable realisation of $D_{Ei}^{-1}(s)$. Lemma 1 and the hypothesis imply that the (A_i, B_i) pairs have the same controllability indices and the invariants α_{ljd_l} for $j < l$ are the same. Accordingly, by Theorem 3 of [11], linking any two pairs (A_i, B_i) and (A_j, B_j) there exists a nonsingular matrix T_{ij} and K_{ij} such that

$$A_i = T_{ij}(A_j + B_jK_{ij})T_{ij}^{-1}, and \; B_i = T_{ij}B_j.$$

Equivalently, there exists (A_0, B_0), K_i and C_i as in the Theorem statement. ■

III. Minimal generic multirealisation

In Section I, we introduced the concept of minimal stably based multirealisation problem (see Definition 1). It turns out that solving this problem is a difficult and intricate task (which we examine elsewhere), and there is another easier minimisation with practical value which we examine in this section, this being a form of **generic** dimension minimisation.

For a minimal stably based "generic" multirealisation, we aim to achieve a multirealisation, which is **independent** of all Popov real parameters of all multivariable systems defined by transfer function matrices $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$, $i \in \{1, \cdots, N\}$. Popov real parameters are determined by physical parameters, which are prone to vary in application. Popov integer parameters however are related to the number of integrators and their structure in the underlying physical system with transfer function matrix $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$, $i \in \{1, \cdots, N\}$. Because Popov integer parameters depend on the structure of the physical system rather than the particular real value of a physical parameter, they are relatively robust to modelling errors that arise due to parameter drift. So, the minimal stably based "generic" multirealisation has significant relevance in practical application.

Theorem 1 implies that if a controllable pair (A_i, B_i) of a minimal state variable realisation of each $D_i^{-1}(s)$ ($i \in \{1, 2, \ldots, N\}$) has the same increasingly ordered controllability indices (equivalent to $D_{Ei}^{hc} = I$), then the Popov real parameters $\{\alpha_{ljd_l}^i\}$ will be identical for each i. Thus, according to Theorem 2, if the controllability indices (Popov integer parameters) are increasingly ordered for each minimal realisation of $D_i^{-1}(s)$, then the minimal multi-realisation of the set of transfer functions $H_i(s)$ is independent of all the Popov real parameters $\{\alpha_{ljk}^i\}$. Based on this observation, we introduce the definition of the **minimal** generic multirealisation for a set of multivariable linear systems.

Definition 3: Assume that there are given a number N of m-input p-output strictly proper real rational transfer function matrices P_i ($i \in \{1, \cdots, N\}$). Any set of state variable realisations $\{A_0 + B_0K_i, B_0, C_i\}$ (with the pair (A_0, B_0) being controllable and having **increasingly ordered** controllability indices) that can realise all the systems P_i with adjustable parameters C_i and K_i, is termed a **generic** multirealisation of the set of systems P_i ($i \in \{1, \cdots, N\}$). If all eigenvalues of A_0 are in the left half plane, $\{A_0 + B_0K_i, B_0, C_i\}$ is termed a generic **stably based** multirealisation of the set of systems P_i ($i \in \{1, \cdots, N\}$). Furthermore, if the dimension of A_0 is the smallest of all such generic stably based multirealisations, then $\{A_0 + B_0K_i, B_0, C_i\}$ is termed a **minimal** generic stably based multirealisation of the set of systems P_i ($i \in \{1, \cdots, N\}$).

It can be proved (the proof is similar with that of Theorem 2) that the minimal generic stably based multirealisation problem is equivalent to the following "minimal generic common hc-(highest column degree) multiplier" problem:

Problem 1: Given a set of square ($m \times m$) column-reduced polynomial matrices $D_i(s)$, find nonsingular stable polynomial matrices $X_i(s)$ (that is, such that the zeros of $det(X_i(s))$ lie in the left half plane $Re(s) < 0$) such that there exists a column reduced polynomial matrix $\bar{D}_{min}(s)$ with the property that

$$\mathcal{D}_{hc}[D_i(s)X_i(s)] = \bar{D}_{min}(s), \quad \forall i \in \{1, \cdots, N\}, \quad (4)$$

with $\bar{D}^{hc}_{minE} = I$ and $\bar{D}_{min}(s)$ having the lowest possible degree. Here, the real matrix $\bar{D}^{hc}_{minE}$ is the highest-degree-coefficient matrix of $\bar{D}_{minE}(s)$ which is the Popov polynomial form of the matrix $\bar{D}_{min}(s)$.

In order to solve Problem 1, we introduce a new concept, $hc-$ (highest column degree) dependence on a set of polynomial vectors.

Definition 4: A polynomial vector $d_e(s)_{n\times 1}$ is hc-(highest column degree) dependent on a collection of polynomial vectors $d_i(s)_{n\times 1}$, $i = 1, 2, \cdots, m$ if there exists a set of scalar polynomials $r_i(s)$ such that

$$\mathcal{D}_{hc}\{d_e(s)\} = \mathcal{D}_{hc}\{\sum_1^m r_i(s)d_i(s)\}.$$

Theorem 3: (Conditions for hc-dependence) Assume there is given a collection of polynomial vectors $d_i(s)_{n\times 1}$, $i = 1, 2, \cdots, m$, such that their column degrees, k_i, are ordered as $k_1 \leq k_2 \leq \cdots k_m$.

Assume further that the matrix $[d_1(s)\, d_2(s) \cdots d_m(s)]$ is such that $D^{hc} = [d^{hc}_1 \;\; d^{hc}_2, \ldots, d^{hc}_m]$ has full column rank. Then a given polynomial vector $d_e(s)_{n\times 1}$ (with column degree k_e) is hc-dependent on the collection of polynomial vectors $d_i(s)$, $i = 1, 2, \cdots, m$ if and only if the real vector d^{hc}_e (the highest-(column)degree-coefficient vector of $d_e(s)$) is a linear combination of real vectors d^{hc}_1, d^{hc}_2, $\cdots$, d^{hc}_l where $l = \max_i\{\arg_i\{k_i \leq k_e\}\}$.

Proof: (Forward Implication)

If

$$\mathcal{D}_{hc}\{d_e(s)\} = \mathcal{D}_{hc}\{\sum_1^m r_i(s)d_i(s)\},$$

for some polynomial $r_i(s)$, then

$$d_e(s) + g(s) = \sum_1^m r_i(s)d_i(s),$$

where $g(s)$ is a polynomial vector with column degree less than k_e. According to Theorem 6.3-13 on p387 of [8], if $k_i > k_e$, we must have $r_i(s) = 0$, and the ordering of k_i and the definition of l imply that

$$d_e(s) + g(s) = \sum_1^l r_i(s)d_i(s). \quad (5)$$

If d^{hc}_e is not a linear combination of real vectors d^{hc}_1, d^{hc}_2, $\cdots$, d^{hc}_l, then $d_e(s), d_1(s), \cdots, d_l(s)$ are linearly independent. Considering that the column degree of $g(s)$ is less than k_e, equation (5) is impossible. Then, the necessity is proved.

(Reverse Implication)

If the real vector d^{hc}_e is a linear combination of real vectors d^{hc}_1, d^{hc}_2, $\cdots$, d^{hc}_l, then

$$d^{hc}_e = \Sigma^l_{i=1} r_i d^{hc}_i,$$

where r_i, for $i \in \{1, \cdots, l\}$ are real numbers.

It follows that

$$d^{hc}_e s^{k_e} = \Sigma^l_{i=1} r_i s^{k_e - k_i} d^{hc}_i s^{k_i} = \mathcal{D}_{hc}\{\Sigma^l_{i=1} r_i s^{k_e-k_i} d^{hc}_i s^{k_i}\}.$$

Therefore, setting $r_i(s) = r_i s^{k_e - k_i}$, we have

$$\mathcal{D}_{hc}\{d_e(s)\} = d^{hc}_e s^{k_e} = \mathcal{D}_{hc}\{\sum_1^m r_i(s)d_i(s)\}.$$

■

Now, we investigate Problem 1. Let us first indicate a simplification to Problem 1. If in the problem statement any $D_i(s)$ is replaced by $\tilde{D}_i(s) = D_i(s)U_i(s)$ where $U_i(s)$ is unimodular, but otherwise arbitrary, then the problem is effectively unchanged. In particular then, without loss of generality, we can assume $D_i(s)$ is a Popov form matrix $D_{Ei}(s)$, and seek a column ordered $\bar{D}_{min}(s)$.

We present a method to achieve a generic minimal common hc-multiplier for a set of polynomial matrices $D_i(s)$ ($i \in \{1, \cdots, N\}$).

Method 1: Step 1. By using column permutation, re-order the columns of each $D_{Ei}(s)$ to make the j_{th} column pivot index of the re-ordered matrix equal to j. Thus the ordered set of column degrees of the re-ordered matrix is equal to the ordered set of controllability indices (see Theorem 1). We define these indices as

$$\tilde{k}^i_1, \tilde{k}^i_2, \cdots, \tilde{k}^i_m, i \in 1, \cdots, N,$$

and denote the new polynomial matrix (which is not necessarily in Popov polynomial-echelon form) as $\tilde{D}_{Ei}(s)$.

Now set

$$\begin{cases} \gamma_1 &= \max_i\{\tilde{k}^i_1\} \\ \gamma_2 &= \max\{\gamma_1, \tilde{k}^1_2, \tilde{k}^2_2, \cdots, \tilde{k}^N_2\} \\ \vdots & \vdots \\ \gamma_m &= \max\{\gamma_{m-1}, \tilde{k}^1_m, \tilde{k}^2_m, \cdots, \tilde{k}^N_m\}. \end{cases} \quad (6)$$

Hence, $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m$ and $\gamma_j \geq \tilde{k}^i_j, \forall i \in 1, \cdots, N, \forall j \in 1, \cdots, m$.

Step 2. Let $\Lambda_i(s) = diag\{(s+a)^{\gamma_1 - \tilde{k}^i_1}, (s+a)^{\gamma_2 - \tilde{k}^i_2}, \cdots, (s+a)^{\gamma_n - \tilde{k}^i_n}\}$ for some $a > 0$.

Define $\bar{D}_{Ei}(s) = \tilde{D}_{Ei}(s)\Lambda_i(s)$, so that $\bar{D}_{Ei}(s)$ has ordered column indices $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m$. It follows that the $\bar{D}_{Ei}(s)$ are in Popov form, and according to Theorem 1, the highest-(column)degree-coefficient matrix for each $\bar{D}_{Ei}(s)$, $i \in \{1, \cdots, N\}$ is the identity matrix.

By rewriting $H_i(s) = N_i(s)D^{-1}_i(s)$ as $N_{Ei}(s)D^{-1}_{Ei}(s) = \tilde{N}_{Ei}(s)\tilde{D}^{-1}_{Ei}(s) = \tilde{N}_{Ei}(s)\Lambda_i(s) \;\; [\tilde{D}_{Ei}(s)\Lambda_i(s)]^{-1} = \bar{N}_{Ei}(s)\bar{D}^{-1}_{Ei}(s)$, it can be see that the necessary and sufficient conditions of Theorem 2 for the multirealisation of a set of multivariable systems are satisfied, and a generic multirealisation form $\bar{D}_m(s)$ can be achieved as $\mathcal{D}_{hc}(\bar{D}_{Ei}(s))$ $= diag\{s^{\gamma_1}, \cdots, s^{\gamma_m}\}$.

Method 1 presents a way to derive a generic common hc-multiplier of a set of square polynomial matrices. Theorem 4 below confirms that it is also a minimal generic common hc-multiplier. However, we require first a simple lemma.

Lemma 2: Denote the highest-(column) degree-coefficient vector of a polynomial vector $p(s)_{m\times 1}$ by a real vector $p^{hc}_{m\times 1}$. Suppose the elements of p^{hc} are structured as

$$p^{hc} = [p_1\, p_2\, \cdots\, p_{l-1}\, 1\, 0 \cdots 0]^T, \quad (7)$$

and define k as the column degree of the polynomial vector $p(s)$. For a Popov polynomial matrix $D_E(s)_{m\times m}$, denote the i-th column degree by k_i, and the i-th column pivot index by p_i. Further denote the t_{th} column pivot index by l, i.e. $p_t = l$. If the polynomial vector $p(s)_{m\times 1}$ is hc-dependent on the columns of the Popov polynomial matrix $D_E(s)$, then

$$k \geq k_t. \tag{8}$$

Proof: The polynomial vector $p(s)_{m\times 1}$ is hc-dependent on the columns of the Popov polynomial matrix $D_E(s)$. Let q be the number of columns of the matrix $D_E(s)$ whose degree is no more than k, i.e. $q = \max_i\{\arg_i\{k_i \leq k\}\}$ $(i \in \{1, \cdots, m\})$. Thus the q column degrees of the first q columns of $D_E(s)$ are less than or equal to k. According to the properties of hc-dependence (see Theorem 3), it follows that p^{hc} is in the range of $[d_{E1}^{hc}\, d_{E2}^{hc}\, \cdots\, d_{Eq}^{hc}]$ Considering equation (7), we conclude the column whose pivot index is equal to l must be one of these q columns. That is, $k \geq k_t$ with $p_t = l$. ∎

Theorem 4: (Second Main Result) The generic common hc-multiplier $\bar{D}_m(s)$ for a set of polynomial matrices $D_{Ei}(s)$ $(i \in \{1, \cdots, N\})$ (see Problem 1) achieved by using Method 1 is also a **minimal** generic common hc-multiplier.

Proof: For any generic common hc-multiplier $\bar{D}_{\bar{m}}(s)$ for the set of polynomial matrices $D_{Ei}(s)$ $(i \in \{1, \cdots, N\})$, we have

$$\mathcal{D}_{hc}\{D_{Ei}(s)X_i(s)\} = \bar{D}_{\bar{m}}(s), \tag{9}$$

and $\bar{D}_{\bar{m}}(s)$ has lowest possible degree. Denote

$$\begin{aligned} X_i(s) &= [x_{i1}(s)\, x_{i2}(s)\, \cdots\, x_{im}(s)], \\ \bar{D}_{\bar{m}}(s) &= [\bar{d}_{\bar{m}_1}(s)\, \bar{d}_{\bar{m}_2}(s)\, \cdots\, d_{\bar{m}_m}(s)]. \end{aligned} \tag{10}$$

From equation (9), we have

$$\mathcal{D}_{hc}\{D_{Ei}(s)x_{ij}(s)\} = \bar{d}_{\bar{m}_j}(s) = \mathcal{D}_{hc}\{\bar{d}_{\bar{m}_j}(s)\}.$$

That is each column of $\bar{D}_{\bar{m}}(s)$ is hc-dependent on the columns of each matrix $D_{Ei}(s)$. Note that the generic multiplier gives $\bar{D}_{\bar{m}}^{hc} = I$ so that the j_{th} column pivot index of the matrix $\bar{D}_{\bar{m}}(s)$ is equal to j. Now consider a fixed but arbitrary j. For each matrix $D_{Ei}(s)$, let column $t(i)$ have pivot index j. From Lemma 2, we conclude that if the j-th column of $\bar{D}_{\bar{m}}(s)$ has degree $\bar{k}_{\bar{m}_j}$, then

$$\bar{k}_{\bar{m}_j} \geq k_{t(i)}, \forall i \in \{1, \cdots, N\},$$

where $k_{t(i)}$ is the $t(i)_{th}$ column degree of each matrix $D_{Ei}(s)$. By considering equation (3) of Theorem 1, we can easily see that

$$\bar{k}_{\bar{m}_j} \geq k_{t(i)} = \tilde{k}_j^i, \forall i \in \{1, \cdots, N\},$$

where $\tilde{k}_j^i$ is defined as in Method 1. Further, because $\bar{D}_{\bar{m}}^{hc} = I$ Theorem 1 implies $\bar{k}_{\bar{m}_1} \leq \bar{k}_{\bar{m}_2} \leq \cdots, \leq \bar{k}_{\bar{m}_m}$ and so we have

$$\left\{\begin{array}{lcl} \bar{k}_{\bar{m}_1} \geq \gamma_1 & = & \max_i\{\tilde{k}_1^i\} \\ \bar{k}_{\bar{m}_2} \geq \gamma_2 & = & \max\{\gamma_1, \tilde{k}_2^1, \tilde{k}_2^2, \cdots, \tilde{k}_2^N\} \\ & \vdots & \vdots \\ \bar{k}_{\bar{m}_m} \geq \gamma_m & = & \max\{\gamma_{m-1}, \tilde{k}_m^1, \tilde{k}_m^2, \cdots, \tilde{k}_m^N\}. \end{array}\right. \tag{11}$$

∎

Based on above results and the dual relationship of the multirealisation forms $\{A_0 + B_0K_i, B_0, C_i\}$ and $\{A_0 + F_iC_0, B_i, C_0\}$, a generic minimal multirealisation $\{A_0 + F_iC_0, B_i, C_0\}$ which ensure **bumpless** transfer can then be constructed according to the following method.

Method 2: 1. Find a right irreducible MFD for each $H_i^T(s)$ $i \in \{1, \cdots, N\}$, and transfer them to Popov MFDs. That is $H_i^T(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$.

2. According to Method 1, construct a **minimal** generic common hc-multiplier $\bar{D}_m(s) = diag\{s^{\gamma_1}, \cdots, s^{\gamma_m}\}$ for the set of Popov polynomial matrices $D_{Ei}(s)$ $i \in \{1, \cdots, N\}$. Each $H_i^T(s)$ can be rewritten as $N_{Ei}(s)D_{Ei}^{-1}(s) = \tilde{N}_{Ei}(s)\tilde{D}_{Ei}^{-1}(s) = \tilde{N}_{Ei}(s)\Lambda_i(s)\ [\tilde{D}_{Ei}(s)\Lambda_i(s)]^{-1} = \bar{N}_{Ei}(s)\bar{D}_{Ei}^{-1}(s)$ (See Step 2 of Method 1).

3. Construct a **stable** polynomial matrix $\bar{D}_{ms}(s)$ such that $\mathcal{D}_{hc}[\bar{D}_{ms}(s)] = \bar{D}_m(s)$. By using the method in [8] (pp403-407), a controller form realisation $\{A_{c0}, B_{c0}, C_{c0}\}$ of $\bar{D}_{ms}^{-1}(s)$ can be found with the pair (A_{c0}, B_{c0}) controllable and A_{c0} stable. Let $C_{ci} = \bar{N}_{Eilc}$ and $K_i = \bar{D}_{mslc} - \bar{D}_{Eilc}$. A generic minimal multirealisation for the set of linear multivariable systems $H_i^T(s)$ $i \in \{1, \cdots, N\}$ is $\{A_{c0} + K_iB_{c0}, B_{c0}, C_{ci}\}$.

4. Denote $A_0 = A_{c0}^T$, $B_i = C_{ci}^T$, $C_0 = B_{c0}^T$ and $F_i = K_i^T$. Then, $\{A_0 + F_iC_0, B_i, C_0\}$ is a generic minimal stably based multirealisation for the set of linear multivariable systems $H_i(s)$ $i \in \{1, \cdots, N\}$.

IV. Conclusion

This paper provides necessary and sufficient conditions for the multirealisation of a family of linear multivariable systems based on matrix fraction descriptions. By introducing the concept of hc-dependence, the minimal generic stably based multirealisation problem has been solved.

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