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RESEARCH ARTICLE

Non-contracting groups generated by (3,2)-automata

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Abstract. We add to the classification of groups generated by 3-state automata over a 2-letter alphabet given by Bondarenko *et al.*, by showing that a number of the groups in the classification are non-contracting. We show that the criterion we use to prove a self-similar action is non-contracting also implies that the associated self-similarity graph introduced by Nekrashevych is non-hyperbolic.

Introduction

In [1] a list of automaton groups generated by 3-state automata over a 2-letter alphabet is given and a great deal of information is listed for each. Amongst the data given for each group was whether the group was contracting or non-contracting. For ten automata the classification did not determine whether or not the group was contracting. In the numbering system of [1, page 17] the ten automata are:

$$749, 861, 882, 887, 920, 969, 2361, 2365, 2402, 2427.$$

Later Muntyan [3] showed that three of these are conjugate to other groups in the classification, specifically $920 \cong 2401$, $2361 \cong 939$, and $2365 \cong 939$.

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The purpose of this note is to show that all of the automaton groups listed above are non-contracting. We first establish a criterion for a group to be non-contracting, and then apply it in each case.

We refer to [4] (Section 1.5) for the basic definitions of self-similar actions and automaton groups. A faithful action of a group G on the set of words X^* over a finite alphabet X is called *self-similar* if for every $x \in X, g \in G$ there exist $y \in X, h \in G$ so that $g(xw) = yh(w)$ for all $w \in X^*$. The element h is called the restriction of g to x and denoted $g|_x$. For $g \in G$ and a finite word $v \in X^*$, the restriction of g to v, denoted $g|_v$, is the element of G determined by the condition:

$$g(vw) = g(v)g|_v(w)$$

for all $w \in X^*$. We will make use of the following basic properties:

$$(gh)|_v = g|_{h(v)} h|_v \quad \text{and} \quad g|_{uv} = (g|_u)|_v .$$

We denote by X^ω the set of infinite words over X. The length of a word $v \in X^*$ is denoted $|v|$. The set X^* is naturally the vertex set of a binary rooted tree and X^ω corresponds to the set of ends of that tree. The action of G on X^* determines an action of G on X^ω. The group of all automorphisms of the rooted tree is denoted $\mathrm{Aut}(X^\omega)$.

A self-similar action of a group G is called *finite-state* if the set

$$\{g|_v : v \in X^*\}$$

is finite for every $g \in G$. An *automaton group* is a finite-state self-similar action of a group generated by the states of a finite state transducer with transitions from state g to state h labeled $x|y$ if $g(x) = y$ and $g|_x = h$ for all $x \in X$.

A $(3, 2)$-automaton group is an automaton group generated by an automaton with three states and a 2-letter alphabet $X = \{0, 1\}$. We label the states a, b, and c. The automata are represented by a *Moore diagram*, which is given below for each automaton (see for example Figure 1).

Recall the following definition from [4].

Definition 1. A self-similar group $G \leqslant \mathrm{Aut}(X^\omega)$ is called *contracting* if there exists a finite subset $\mathcal{N} \subseteq G$ such that for all $g \in G$ there exists $k \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all $v \in X^*$ with $|v| \geqslant k$. The minimal $\mathcal{N}$ is called the *nucleus* of the action.

Note that if a self-similar group is not finite-state, then it is non-contracting.

We make use of the following criterion, which was used in [1] to show that 744 is non-contracting, and many times after that.

Lemma 1. *Let* $G \leqslant \mathrm{Aut}(X^\omega)$ *be a self-similar action. Suppose that there exist* $g \in G$ *and* $v \in X^*$ *such that:*

1) $g|_v = g$,
2) $g(v) = v$,
3) g *has infinite order.*

Then G *is non-contracting.*

Proof. Assume for induction that $g|_{v^k} = g$ and $g(v^k) = v^k$ for $k \geq 1$. Then $g|_{v^{k+1}} = g|_{v^k v} = (g|_{v^k})|_v = g|_v = g$ and $g(v^{k+1}) = g(v)g|_v(v^k) = vg(v^k) = vv^k$.

Next assume for induction that $g^n|_{v^k} = g^n$ for $n \geq 1$ and fixed k. Then $g^{n+1}|_{v^k} = g|_{g^n(v^k)}\, g^n|_{v^k} = g|_{v^k}\, g^n = gg^n$.

It follows that a nucleus must contain g^n for infinitely many n and so, since g has infinite order, the action is not contracting.

Alternatively, though less directly, the lemma follows from Theorem 2 below and Theorem 3.8.6 of [4]. □

In the next section we apply this criterion to the ten automata listed above. In Section 2 we prove that a self-similar group satisfying this criterion has a non-hyperbolic self-similarity graph.

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1. The automata

Theorem 1. *The automaton groups generated by the automata*

$$749, 861, 882, 887, 920, 969, 2361, 2365, 2402, 2427$$

in [1] are non-contracting.

Proof. For each automaton group we give an element $g \in G$ and a word $v \in \{0,1\}^*$ with the (easily verifiable) property that $g(v) = v$ and $g|_v = g$. The Moore diagram of the automaton is given for reference. States that act non-trivially on $\{0,1\}$ are shaded in the diagram. We then prove that g has infinite order, so that the criterion of Lemma 1 applies. The approach to showing that g has infinite order is to find another string v' that is not fixed by any power of g. We found the candidates for suitable elements and strings using some simple computer code and observing various patterns.

It is convenient to introduce the equivalence relation on $\{0,1\}^\omega$ given by left shift equivalence, that is, $u \sim v$ if there are finite prefixes u' and v' of u and v respectively, and $w \in \{0,1\}^\omega$ such that $u = u'w$ and $v = v'w$.

For a finite word $u \in \{0,1\}^*$, we denote by u^∞ the element of $\{0,1\}^\omega$ formed by repeating u infinitely many times.

Automaton 749

We have $a^2bc(0100) = 0100$ and $(a^2bc)\,|_{0100} = a^2bc$.

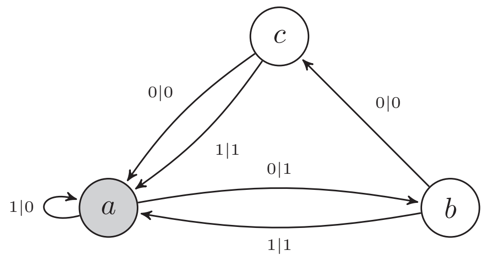

FIGURE 1. Automaton 749

To see that $g = a^2bc$ has infinite order we consider the string 0^∞. Observe that since $g\,|_{000} = babc$, $babc(000) = 101$, and $(babc)\,|_{000} = babc$, we have $g(0^\infty) = 001(101)^\infty$. Then note that $a\,|_{101} = b\,|_{101} = c\,|_{101} = a$ and $a^4(101) = 101$. It follows that for any $n \geqslant 1$, $g^n(0^\infty) = u_n(101)^\infty$ where $u_1 = 001$ and $u_n = g(u_{n-1}101)$. In other words $g^n(0^\infty)$ is left-shift equivalent to $(101)^\infty$. We now note that $g^{-1}(0^\infty)$ is not of this form, which establishes that g has infinite order. Observe that:

$$g^{-1}\,|_{0000} = a^{-1}b^{-1}a^{-2}, \quad g^{-1}(0000) = 0011,$$
$$a^{-1}b^{-1}a^{-2}\,|_{0000} = a^{-1}b^{-1}a^{-2}, \quad a^{-1}b^{-1}a^{-2}(0000) = 1011.$$

Therefore $g^{-1}(0^\infty) = 0011(1011)^\infty$.

Automaton 861

We have $c(010) = 010$ and $c|_{010} = c$.

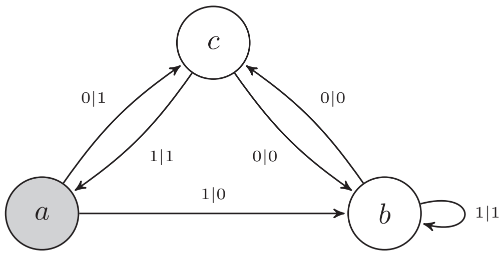

FIGURE 2. Automaton 861

Since $x|_{11} = b$ for any $x \in \{a, b, c\}$ and $b(1^\infty) = 1^\infty$, it follows that $c^n(1^\infty) \sim 1^\infty$ for any $n \geqslant 0$. But $c^{-1}(1^\infty) = (10)^\infty$, so c has infinite order.

Automaton 882

We have $acacbc(11) = 11$ and $(acacbc)|_{11} = acacbc$.

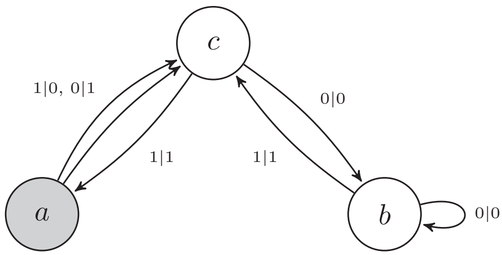

FIGURE 3. Automaton 882

To show that $g = acacbc$ has infinite order we claim that:

1) $g^{2^n}(0^{2n+1}) = 0^{2n+1}$,
2) $g^{2^n}|_{0^{2n+1}} = cacb$,
3) $cacb(0^\infty) = 110^\infty$.

from which it follows that $g^{2^n}(0^\infty) = 0^{2n+1}110^\infty$ for all $n \geqslant 1$.

We prove the first and second claims by induction on n. Note first that b^2 is the identity in the group, as can be seen from the automaton for b^2 in Figure 4.

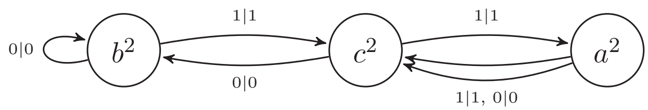

FIGURE 4. Action of b^2 in the group 882

We have $g(0) = 0$ and $g|_0 = cacbbb = cacb$. Then inductively,

$$\begin{aligned} g^{2^{n+1}}(0^{2n+3}) &= g^{2^n} g^{2^n}(0^{2n+1}00) \\ &= g^{2^n}(0^{2n+1} cacb(00)) = g^{2^n}(0^{2n+1}11) \\ &= 0^{2n+1} cacb(11) = 0^{2n+1}00, \\ g^{2^{n+1}}|_{0^{2n+3}} &= (g^{2^n} g^{2^n})|_{0^{2n+3}} = g^{2^n}|_{g^{2^n}(0^{2n+3})}\, g^{2^n}|_{0^{2n+3}} \\ &= g^{2^n}|_{0^{2n+1}11}\, g^{2^n}|_{0^{2n+3}} = (g^{2^n}|_{0^{2n+1}})|_{11}\, (g^{2^n}|_{0^{2n+1}})|_{00} \\ &= (cacb)|_{11}\, (cacb)|_{00} = bbcacbbb = cacb. \end{aligned}$$

For the third claim observe that $cacb(00) = 11, cacb|_{00} = cb^3$, and $cb^3(0^\infty) = 0^\infty$. Then

$$cacb(00^\infty) = cacb(00)(cacb)|_{00}\,(0^\infty) = 11cb^3(0^\infty) = 110^\infty.$$

Automaton 887

We have $bc(00) = 00$ and $(bc)|_{00} = bc$.

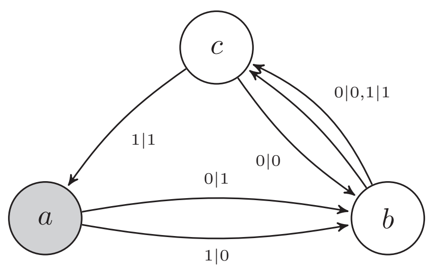

FIGURE 5. Automaton 887

To establish that bc has infinite order we show that for all $n \geqslant 1$, $(bc)^n(1^\infty) \neq 1^\infty$ as follows.

Since $bc(1) = 1$ and $(bc)|_1 = ca$, we have $(bc)^n(1^\infty) = 1(ca)^n(1^\infty)$ and it suffices to show that $(ca)^n(1^\infty) \neq 1^\infty$. We show that for $n \geqslant 2$:

1) $(ca)^{4n}(111) = 111$ and $(ca)^{4n}(110) = 110$,

2) $(ca)^{2^n}|_{111} = (ca)^{2^{n-1}}$,
3) $(ca)^{2^n}(1^\infty) = (111)^{n-1}(1010)1^\infty$.

It's clear that the third claim implies that $(ca)^n(1^\infty) \neq 1^\infty$ for all $n \geqslant 1$.

The first claim follows from $(ca)^4(111) = 111$ and $(ca)^4(110) = 110$.

For the second claim, note first that a, b and c all have order 2, as can be seen from the automaton for $\langle a^2, b^2, c^2 \rangle$ in Figure 6

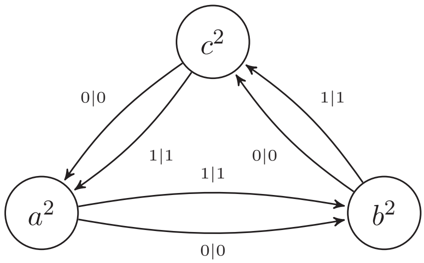

FIGURE 6. Action of the elements a^2, b^2, c^2 in the group 887

Then $(ca)|_{111} = aa = 1$ and $(ca)|_{110} = bb = 1$. Also,

$$(ca)^2|_{111} = (ca)|_{ca(111)}\,(ca)|_{111} = (ca)|_{011}\,(ca)|_{111} = ca,$$

and

$$(ca)^4|_{111} = (ca)^2|_{(ca)^2(111)}\,(ca)^2|_{111} = (ca)^2|_{101}\,ca = caaaca = caca.$$

Inductively, for $n \geqslant 3$,

$$\begin{aligned}(ca)^{2^n}|_{111} &= ((ca)^{2^{n-1}}(ca)^{2^{n-1}})|_{111} = (ca)^{2^{n-1}}|_{(ca)^{2^{n-1}}(111)}\,(ca)^{2^{n-1}}|_{111} \\ &= (ca)^{2^{n-1}}|_{111}\,(ca)^{2^{n-2}} = (ca)^{2^{n-2}}(ca)^{2^{n-2}} = (ca)^{2^{n-1}}.\end{aligned}$$

For the third claim, note that $ca(1^\infty) = 0(bb)(1^\infty) = 01^\infty$ and

$$\begin{aligned}(ca)^4(1^\infty) &= (ca)^4(111)(ca)^4|_{111}\,(1^\infty) = 111(ca)^2(1^\infty) \\ &= 111101(ca)^2|_{111}\,(1^\infty) = 111101(ca)(1^\infty) = 11110101^\infty.\end{aligned}$$

Then for $n \geqslant 3$

$$\begin{aligned}(ca)^{2^n}(1^\infty) &= (ca)^{2^n}(111)(ca)^{2^n}|_{111}\,(1^\infty) = 111(ca)^{2^{n-1}}(1^\infty) \\ &= 111(111)^{n-1}(1010)1^\infty = (111)^n(1010)1^\infty.\end{aligned}$$

Automaton 920

We have $b(1) = 1$ and $b|_1 = b$.

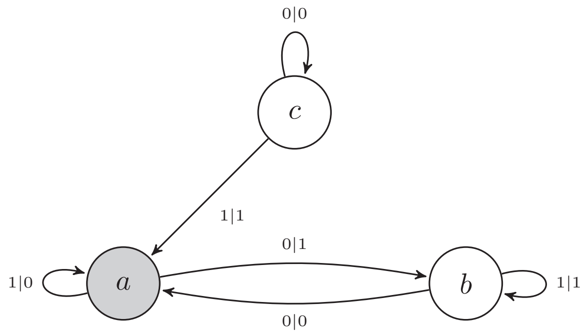

FIGURE 7. Automaton 920

Since $a^{-1}|_1 = b^{-1}|_1 = b^{-1}$ and $b^{-1}(1) = 1$, it follows that $b^{-n}(01^\infty) \sim 1^\infty$. But $b(01^\infty) = 0^\infty$, so b has infinite order.

Automaton 969

We have $c(0) = 0$ and $c|_0 = c$.

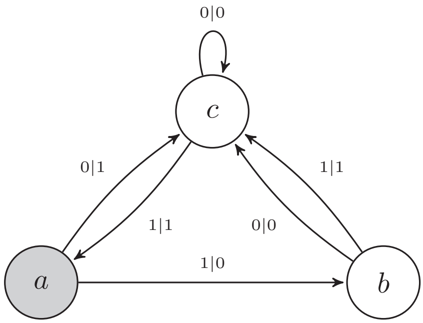

FIGURE 8. Automaton 969

To show that c has infinite order we claim that for $n \geqslant 1$,

$$c^n((101)^\infty) \sim \begin{cases} (100)^\infty & n \text{ even}, \\ (011)^\infty & n \text{ odd}. \end{cases}$$

To see this, note that $c((101)^\infty) = 11c((110)^\infty) = 11(100)^\infty$. If $u \sim (100)^\infty$, then $c(u) \sim (011)^\infty$. If $u \sim (011)^\infty$, then $c(u) \sim (100)^\infty$. Both statements follow from the observation that for any generator $x \in \{a, b, c\}$, $x|_{10} = c$.

Finally, observe that $c^{-1}((101)^\infty) = 1^\infty$, which proves that c has infinite order.

Automaton 2361

We have $c(0) = 0$ and $c|_0 = c$.

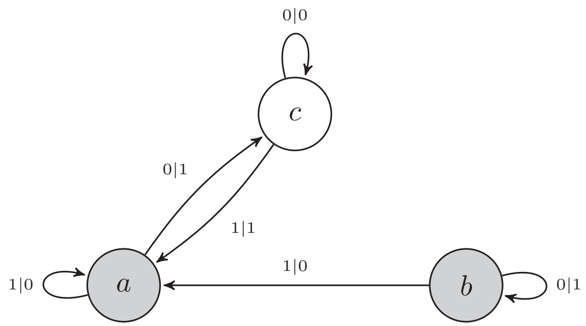

FIGURE 9. Automaton 2361

Observe that $a(0^\infty) = 10^\infty$ and $c(0^\infty) = 0^\infty$. Therefore, for all $n \geqslant 0$, $c^n(10^\infty) \sim 0^\infty$. Also, $c^{-1}(10^\infty) = 1^\infty$. It follows that c has infinite order.

Automaton 2365

We have $c(0) = 0$ and $c|_0 = c$.

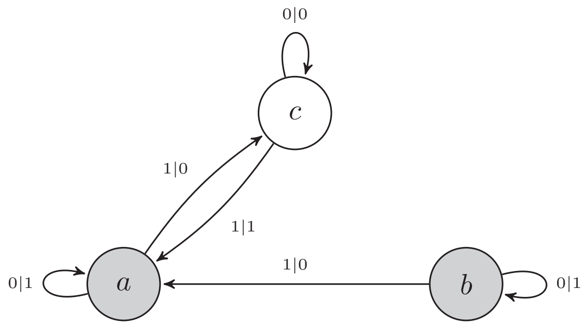

FIGURE 10. Automaton 2365

To see that c has infinite order, observe that $a^{-1}(0^\infty) = 10^\infty$ and $c^{-1}(0^\infty) = 0^\infty$. Therefore, for all $n \geqslant 0$, $c^{-n}(10^\infty) \sim 0^\infty$. As $c(10^\infty) = 1^\infty$, it follows that c has infinite order.

Automaton 2402

We have $c(0) = 0$ and $c|_0 = c$.

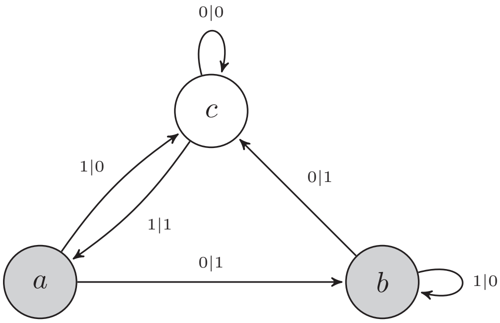

Figure 11. Automaton 2402

Note that $c^n(10^\infty) \sim 0^\infty$ since $x|_{00} = c$ for any $x \in \{a, b, c\}$. However $c^{-2}(10^\infty) = 101^\infty$. Therefore c has infinite order.

Automaton 2427

We have $c(0) = 0$ and $c|_0 = c$.

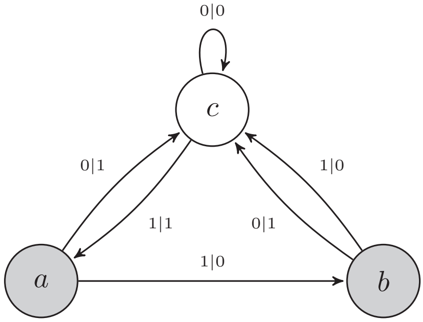

Figure 12. Automaton 2427

To see that c has infinite order note that $a((101)^\infty) = 01(101)^\infty$, $b((101)^\infty) = 00(101)^\infty$ and $c((101)^\infty) = 11(101)^\infty$. Therefore, for all $n \geqslant 1$, $c^n((101)^\infty \sim (101)^\infty$. However, $c^{-2}((101)^\infty) = (100)^\infty$. □

2. Non-hyperbolic self-similarity graphs

Nekrashevych introduced the notion of a self-similarity graph of a self-similar action. He proved that if a self-similar group is contracting, the corresponding self-similarity graph (endowed with the natural metric) is hyperbolic [4, Theorem 3.8.6]. The converse to this result is open.

Here we provide a partial converse to this fact, which applies to self-similar actions that satisfy the criterion of Lemma 1. We know of no automaton group that is non-contracting and doesn't satisfy the condition. An example (suggested by the referee) of a self-similar group that is non-contracting but does not satisfy the criterion is the infinite cyclic group generated by the element $a \in \mathrm{Aut}(X^\omega)$ determined by $a(0) = 1$, $a(1) = 0$, $a|_0 = a$ and $a|_1 = a^2$. This group is not finite-state. It can be shown that the self-similarity graph of this example is not hyperbolic.

Definition 2 ([4] Defn. 3.7.1)**.** The *self-similarity graph* $\Sigma(G, S, X)$ of a self-similar group G with generating set S acting on X^* is the graph with vertex set X^* and an edge $\{u, v\}$ whenever:

- $u = s(v)$ for some $s \in S$ — these are the *horizontal edges*,
- $u = xv$ for some $x \in X$ — these are the *vertical edges*.

Observe that horizontal edges connect strings in X^* of the same length, and vertical edges connect strings that differ in length by 1.

We use the characterization of hyperbolic geodesic metric spaces involving the divergence of geodesics, see [2, p.412].

Definition 3. Let Y be a geodesic metric space. A function $e : \mathbb{N} \to \mathbb{R}$ is called a *divergence function* if for all $y \in Y$, for all $R, r \in \mathbb{N}$ and for all geodesics $\alpha : [0, a] \to Y$ and $\beta : [0, b] \to Y$ with $\alpha(0) = \beta(0) = y$, $a > R + r$ and $b > R + r$ the following holds: if $d_Y(\alpha(R), \beta(R)) > e(0)$ then any path from $\alpha(R+r)$ to $\beta(R+r)$ that stays outside the open ball of radius $R + r$ about y has length at least $e(r)$.

Proposition 1 ([2, p.412])**.** *Let Y be a geodesic metric space. Then Y is hyperbolic if and only if it admits an exponential divergence function.*

Theorem 2. *Let G be a self-similar group with finite generating set S acting on X^*, and suppose that there exist $g \in G$ and $v \in X^*$ such that:*

1) $g|_v = g$,
2) $g(v) = v$,
3) *g has infinite order.*

Then the self-similarity graph $\Sigma(G, S, X)$ is non-hyperbolic.

Proof. The vertex in $\Sigma(G, S, X)$ corresponding to the empty string is labelled $\emptyset$. A vertex in the open ball based at $\emptyset$ of radius N corresponds to a string in X^* of length less than N. Note that an element of X^* uniquely defines a vertical geodesic emanating from $\emptyset$ whose length is equal to that of the word. Considering such geodesics, we show that $\Sigma(G, S, X)$ does not admit an exponential divergence function, and is therefore not hyperbolic.

Suppose for a contradiction that $e : \mathbb{N} \to \mathbb{R}$ is a divergence function for $\Sigma(G, S, X)$ and that it is increasing and unbounded. If the maximum size of an orbit of any $w \in X^*$ under g was N, then $g^{N!}(w) = w$ for all $w \in X^*$. Since g has infinite order, it follows that there are arbitrarily large orbits under its action on X^*. Vertices in $\Sigma(G, S, X)$ have uniformly bounded degree. It follows that there is a bound on the number of vertices in any metric ball of fixed radius, so we can choose $n \in \mathbb{N}$ and $w \in X^*$ such that $d_\Sigma(w, g^n(w)) > e(0)$. More explicitly, choose w so that its orbit under the action of g has size greater than the number of vertices in any ball in $\Sigma(G, S, X)$ of radius $e(0)$.

For an element $g \in G$, denote by $\|g\|_S$ the length of g with respect to the word metric on G determined by the generating set S. For all $k \in \mathbb{N}$ the vertices $v^k w$ and $g^n(v^k w) = v^k g^n(w)$ are connected by a horizontal path of length exactly $n\|g\|_S$, and this path lies outside the open ball of radius $|v^k w|$ centered at $\emptyset$. Choose $k \in \mathbb{N}$ such that $e(|v^k|) > n\|g\|_S$. Since e is a divergence function, any horizontal path connecting $v^k w$ and $v^k g^n(w)$ must have length at least $e(|v^k|)$. This contradiction establishes the result. □

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