

FINITENESS AND THE FALSIFICATION BY FELLOW TRAVELER PROPERTY

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ABSTRACT. We prove that groups enjoying the falsification by fellow traveler property are of type F_3 , and have at most an exponential second order isoperimetric function.

1. INTRODUCTION

The intriguing properties of almost convexity and the falsification by fellow traveler property have been introduced by [1] and [6] respectively. It has been shown that both properties are generating set dependent [8], [6]; that groups enjoying the falsification by fellow traveler property are almost convex; that both imply a finite presentation; that almost convexity groups have at most an exponential isoperimetric function and groups with the falsification by fellow traveler property at most a quadratic isoperimetric function. The classes of groups that enjoy these properties include hyperbolic groups, virtually abelian groups [6], and Coxeter groups [2], [7]. In this paper we are interested in higher dimensional finiteness for groups with these properties.

A group is said to be of type F_n if it has an Eilenberg MacLane space with finite n -skeleton. It follows that a group is finitely generated if and only if it is of type F_1 and finitely presented if and only if it is of type F_2 , so F_n is a natural generalization of these finiteness properties.

Gersten showed that asynchronously combable groups with departure function (hence asynchronously automatic groups) are of type F_3 , by constructing the universal cover of an Eilenberg MacLane space having finitely many types of 3-cells [5]. His result can easily be extended to all dimensions. It also follows that such groups have at most an exponential second order isoperimetric function [9].

We will modify Gersten's proof for groups with the falsification by fellow traveler property, to show they are of type F_3 . It is unclear how

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to extend our results to higher dimensions, nor to the larger class of almost convex groups.

2. FELLOW TRAVELING AND ALMOST CONVEXITY

Suppose G is a group with finite generating set X . A word in X^* represents a path in the Cayley graph based at any vertex. Paths can be parameterized by non-negative $t \in \mathbb{R}$ by defining $w(t)$ as the point distance t along the path if $t < |w|$ and $w(t) = \bar{w}$ if $t \geq |w|$, where \bar{w} is the endpoint of w . Paths w and u are said to k -fellow travel if $d(w(t), u(t)) \leq k$ for each $t \in \mathbb{R}$ with $t \geq 0$. The two paths are *asynchronous* k -fellow travelers if there is a non-decreasing proper continuous function $\phi : [0, \infty) \rightarrow [0, \infty)$ such that $d(w(t), u(\phi(t))) \leq k$. This means that any point on w is within k of some point on u and vice versa. We imagine the two paths traveling at different speeds (but not backtracking) to keep within k of each other.

A language $L \subset X^*$ enjoys the (*asynchronous*) *fellow traveler property* if there is a constant k such that for each $w, u \in L$ with $d(\bar{w}, \bar{u}) \leq 1$ in $\Gamma_X(G)$, w and u (asynchronously) k -fellow travel. We say a group is (*asynchronously*) *combable* if there is a language having the (asynchronous) fellow traveler property and surjecting to it.

A related property is the falsification by fellow traveler property, defined as follows.

Definition 1. (G, X) has the (asynchronous) falsification by fellow traveler property if there exists a constant k so that for any non-geodesic word $w \in X^*$ there exists $u \in X^*$ so that $u =_G w$, u and w (asynchronously) k -fellow travel and $|u| < |w|$ (Figure 1).

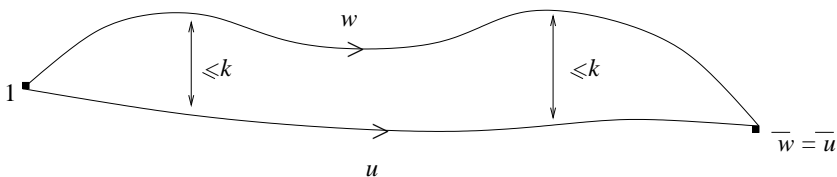


FIGURE 1. The falsification by fellow traveler property

Lemma 1. If w and u are geodesics in (G, X) and asynchronously k -fellow travel then they synchronously $2k$ -fellow travel (Figure 2).

Proof: Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing proper function such that $d(w(t), u(\phi(t))) \leq k$ for all $t \in [0, \infty)$.

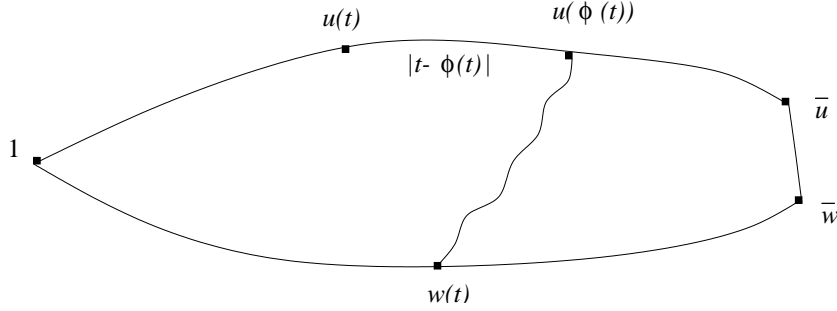


FIGURE 2. Geodesics asynchronously k -fellow traveling

Now $u(\phi(t))$ is within k of the sphere $S(t)$ of radius t about 1 (namely the point $w(t)$) and $u(t)$ is a closest point of $S(t)$ to $u(\phi(t))$, so $d(u(\phi(t)), u(t)) \leq k$. Thus $d(w(t), u(t)) \leq 2k$ for all $t \in [0, \infty)$. \square

Corollary. *The asynchronous falsification by fellow traveler property and the synchronous falsification by fellow traveler property are equivalent.*

Proof: Suppose (G, X) has the asynchronous falsification by fellow traveler property with constant k . If w is not geodesic, take $t \in \mathbb{N}$ minimal such that $w(t)$ is geodesic and $w(t+1)$ is not geodesic. Let $w = w_1 w_2$ with $w_1 = [w(0), w(t+1)]$. There is a word u that asynchronously k -fellow travels w_1 and $|u| < t + 1$. If u is not geodesic then there is a word v that asynchronously k -fellow travels u and $|v| < |u| < t + 1$. Then v must be geodesic, so we have two geodesics that asynchronously $2k$ -fellow travel, so by the lemma they synchronously $4k$ -fellow travel. Then vw_2 is shorter than w and they synchronously $4k$ -fellow travel, provided $k \geq 1$. The other direction is obvious. \square

An important fact is that if (G, X) has the falsification by fellow traveler property then the language of geodesics is regular [6]. It is not known whether the converse is true.

A finitely generated group is said to be *almost convex*(i) if there is a constant $C(i)$ such that any two elements of the Cayley graph which lie in the metric ball of (arbitrary) radius n in the graph and lie within distance i of each other also lie within distance $C(i)$ of each other in the ball of radius n . Cannon proves that almost convex(i) implies almost convex($i+1$) for $i \geq 2$, so we say a group is *almost convex* if it is almost convex(2).

Proposition 1. *If (G, X) enjoys the falsification by fellow traveler property then (G, X) is almost convex (Figure 3).*

Proof: Suppose $k > 0$ is the falsification by fellow traveler property constant, and that $g, g' \in S(n)$ with $d(g, g') \leq 2$ realized by a path γ , where $S(n)$ is the set of all points of the Cayley graph that lie distance n from the identity. Let w be a geodesic path for g . Now $w\gamma$ is not geodesic for g' , so by the falsification by fellow traveler property there is a path u for g' which k -fellow travels $w\gamma$, and

$$|u| < |w\gamma| \leq |w| + 2 = n + 2.$$

If u is not geodesic then there is a path v for g' which k -fellow travels u and

$$|v| < |u| < n + 2$$

hence v must be geodesic. If u is geodesic put $v = u$.

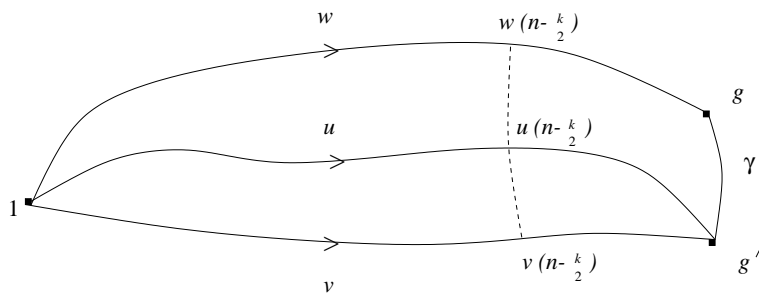


FIGURE 3. The falsification by fellow traveler property implies almost convexity.

Since these paths pairwise k -fellow travel, then it is easily checked that the path from $w(n - \frac{k}{2})$ to $u(n - \frac{k}{2})$ to $v(n - \frac{k}{2})$ is contained inside the ball $B(n)$ of radius n . Thus we have shown that (G, X) is almost convex with constant at most $3k$. \square

An example in [3] shows that this implication is not reversible. We have already noted that both properties are generating set dependent; Thiel [8] gives an example of a group that is almost convex for one generating set but not another, and Neumann and Shapiro [6] give a virtually abelian group which enjoys the falsification by fellow traveler property for one generating set but not another. Cannon proved that almost convex groups are finitely presented and have at most an exponential isoperimetric function [1].

Proposition 2. *If (G, X) enjoys the falsification by fellow traveler property then G is finitely presented and has at most a quadratic isoperimetric function.*

Proof: Suppose $w \in X^*$ is a word evaluating to 1 in G . Then the edge path described by w in the Cayley graph is a loop. Unless w is the

empty word, it is not geodesic, so by the falsification by fellow traveler property there is a shorter path which k -fellow travels it. Iteratively we can find successively shorter paths until we get the empty word. This requires at most $|w|$ iterations. Then the loop w can be filled by at most $|w|^2$ relators of length at most $2k + 2$, and the result follows. \square

3. THE MAIN THEOREM

One concrete way to construct an Eilenberg MacLane space is to start with a presentation 2-complex for a group, which is a space having one vertex, an edge for each generator and a 2-cell for each relator, glued in appropriately. The fundamental group of this 2-complex is the group, and if it has nontrivial second order homotopy we glue in 3-cells to kill it. Inductively we can glue in higher dimensional balls to obtain an Eilenberg MacLane space for G . The universal cover of this construction has the Cayley graph for its 1-skeleton, and the *Cayley complex* (or “filled Cayley graph” [4]) for its 2-skeleton.

The metric on the two complex can be defined by taking the metric on the 1-skeleton as the metric on the Cayley graph, and saying that a 2-cell is “in the n -ball $B(n)$ ” if its boundary is in $B(n)$ of the Cayley graph.

The following observation is required for the main theorem.

Lemma 2. *Let G be any group with isoperimetric function ρ and suppose that the universal cover of an Eilenberg MacLane space for G has 2-cells of perimeter at most $2k + 2$. Consider a loop L of length s with its vertices lying inside $B(r)$ (the ball of radius r about the identity of G) in the 1-skeleton. Then L can be filled by 2-cells so that all interior points lie in $B(r + (2k + 2)\rho(s))$.*

Proof: The loop L can be filled by at most $\rho(s)$ 2-cells, and each cell has at most $2k + 2$ edges. Then there are at most $\rho(s)(2k + 2)$ edges in this filling. A geodesic path from an interior point to a point in L has no more than the number of edges in the interior, hence every point lies in $B(r + \rho(s)(2k + 2))$. \square

Theorem 1. *If (G, X) has the falsification by fellow traveler property then G is of type F_3 .*

Proof: Let K be the Cayley complex for G with respect to the presentation

$$\langle X \mid \{r \in X^* : r =_G 1, |r| \leq 2k + 2\} \rangle.$$

We will show that any combinatorial 2-sphere occurring in K can be filled by 3-cells of a bounded size. This bound will depend only on the falsification by fellow traveler property constant k , the isoperimetric function ρ for G and a constant $\epsilon \in \mathbb{N}$.

Let Θ be a combinatorial 2-sphere of arbitrary size in K . By isometry we can assume the identity is a vertex of Θ . Let $n = \min\{m \in \mathbb{N} : \Theta \subseteq B(m)\}$. For each 2-cell of Θ we will attach a 3-cell so that the boundary of $\Theta \cup \{3\text{-balls}\}$ is Θ and another combinatorial 2-sphere $\Theta' \subseteq B(n - \epsilon + \frac{k}{2})$. Provided ϵ is chosen to be greater than $\frac{k}{2}$ we can inductively fill Θ by 3-cells.

Following Gersten we can think of our 3-cells as “drums”, albeit distorted ones. The top of each drum will be a unique 2-cell of Θ . The sides will be described presently, and they will match up with adjacent drums, that is, drums having tops adjacent in Θ . Each drum will have a base that need not match up, as shown in Figure 4, and we require that the entire base is contained in $B(n - \epsilon + \frac{k}{2})$. The extra $\frac{k}{2}$ here is due to some of the base cells “bulging out” of the base (these will be “type 2” below). After attaching one such 3-cell for each 2-cell of Θ the bases will glue together to form a homotopic copy Θ' of Θ inside $B(n - \epsilon + \frac{k}{2})$. This is a sketch of the argument; now let us fill in the details.

Constructing the sides

Fix a set of geodesics w_g from 1 to each vertex g of Θ . Fix the constant $\epsilon \in \mathbb{N}$. Consider each edge (g, g') of Θ that lies outside of $B(n - \epsilon)$. Retrace the geodesics $w_g, w_{g'}$ back to $w_g(n - \epsilon), w_{g'}(n - \epsilon)$. Recall that since (G, X) has the falsification by fellow traveler property, (G, X) is almost convexity so there is a path from $w_g(n - \epsilon), w_{g'}(n - \epsilon)$ inside $B(n - \epsilon)$ of length at most $C(2\epsilon + 1)$ where $C(i)$ is the almost convexity constant for points distance at most i apart. Now $C(i)$ is a function of k and i , so $C(2\epsilon + 1)$ depends on k and ϵ . A *side cell* is either the edge (g, g') if it lies inside $B(n - \epsilon)$, or a 2-disc of perimeter at most $C(2\epsilon + 1) + 2\epsilon + 1$, as seen in Figure 5. Each side cell has at most $\rho(C(2\epsilon + 1) + 2\epsilon + 1)$ 2-cells of the Cayley complex K . Each drum has a *top* a 2-cell of Θ of perimeter at most $2k + 2$, so has at most $(2k + 2)\rho(C(2\epsilon + 1) + 2\epsilon + 1)$ 2-cells for its sides.

Now we have a loop of length at most $(2k + 2)C(2\epsilon + 1)$ to which we must attach a base. This loop lies in $B(n - \epsilon)$. Effectively we have taken the 1-skeleton of Θ and pushed it down into $B(n - \epsilon)$ by homotoping each edge outside $B(n - \epsilon)$ to a path of length at most $C(2\epsilon + 1)$, the homotopy for each edge realized by a side cell. So we have a homotopic copy of the 1-skeleton of Θ inside $B(n - \epsilon)$. For each 2-cell of Θ we

have a loop in this copy of length at most $(2k + 2)C(2\epsilon + 1)$. We will attach a base of bounded size to each such loop, and ensure that each base lies in $B(n - \epsilon)$.

Constructing the base

Each drum so far has a top, sides, and a loop of length at most $(2k + 2)C(2\epsilon + 1)$ in $B(n - \epsilon)$ to which we must attach a base. Suppose the loop has length l . Fix a vertex on the loop, let u_0 be a geodesic from 1 to it, and write the loop as an edge path $a_1a_2 \dots a_l$. By the falsification by fellow traveler property, if u_0a_1 is not geodesic then there is a word $u_1 =_G u_0a_1$ such that $|u_1| < |u_0| + 1$, so $|u_1| \leq |u_0| \leq n - \epsilon$, and u_0, u_1 k -fellow travel. If u_0a_1 is geodesic then put $u_1 = u_0a_1$ and note that $|u_1| \leq n - \epsilon$ since it is a geodesic for to point on the loop. Recursively we can find u_0, u_1, \dots, u_l such that $|u_i| \leq n - \epsilon$ and u_{i-1}, u_i k -fellow travel for $i \in [1, l]$. Note that u_l, u_0 need not k -fellow travel, but they do kl -fellow travel. We call this the “tear” in the drum.

Type 1 base cells:

Retrace each path u_i back to $u_i(n - \epsilon - M)$ where M is a constant to be determined below. This gives at most l type 1 cells as in Figure 7. The cell at the top has perimeter at most $2k + 2$, and below it each cell has perimeter at most $2k + 2$. So in total we have at most Ml 2-cells of K to make up the type 1 base cells for each drum. For each integer $t \geq \frac{k}{2}$ each pair of points $u_{i-1}(t), u_i(t)$ has a path of length at most k between them, thus each of these cross paths lies in $B(n - \epsilon)$. It follows that these cells lie in $B(n - \epsilon)$.

Type 2 base cells:

We want to fill in the tear with cells inside $B(n - \epsilon)$. Let v_0 be the path from $u_0(n - \epsilon - M)$ to $u_0(n - \epsilon) = u_l(n - \epsilon)$ to $u_l(n - \epsilon - M)$ (some of the points $u_0(n - \epsilon - t)$ could be the identity if $n - \epsilon < M$). This path has length at most $2M$, and we know there is a path of length at most kl for it. So if v_0 is not geodesic then there is a shorter path v_1 from $u_0(n - \epsilon - M)$ to $u_l(n - \epsilon - M)$ which k -fellow travels v_0 . Recursively if v_i is not geodesic we can find a shorter path v_{i+1} which k -fellow travels v_i . This gives at most $2M - kl$ paths, as shown in Figure 8. Now each path v_i has length at most $2M$, so lies in $B(n - \epsilon)$. For each integer t there is a path of length at most k from $v_i(t)$ to $v_{i+1}(t)$, and this path lies in $B(n - \epsilon + \frac{k}{2})$. So we can fill in the tear with at most $(2M - kl)2M$ 2-cells of perimeter at most $2k + 2$, which lie in $B(n - \epsilon + \frac{k}{2})$. Note that we need an extra $\frac{k}{2}$ here.

Type 3 base cells:

After including the above base cells in our drum we are left with a loop of length at most $2kl$ that lies in $B(n - \epsilon - M + \frac{kl}{2})$. By Lemma 2 this loop can be filled by at most $\rho(2kl)$ 2-cells of K so that the interior lies in

$$B(n - \epsilon - M + \frac{kl}{2} + (2k + 2)\rho(2kl)).$$

By choosing

$$\begin{aligned} M &= \frac{k(2k + 2)C(2\epsilon + 1)}{2} + (2k + 2)\rho(2k(2k + 2)C(2\epsilon + 1)) \\ &\geq \frac{kl}{2} + (2k + 2)\rho(2kl) \end{aligned}$$

we ensure the type 3 base cells lie in $B(n - \epsilon)$. Note that ρ is a monotone increasing function, so these inequalities are justified.

In total each drum has a boundary of at most

$$\left\{ \begin{array}{ll} 1 & \text{top} \\ (2k+2)\rho(C(2\epsilon+1)+2\epsilon+1) & \text{sides} \\ (2k+2)M & \text{base type 1} \\ 2M(2M-k(2k+2)C(2\epsilon+1)) & \text{base type 2} \\ \rho(2k(2k+2)C(2\epsilon+1)) & \text{base type 3} \end{array} \right.$$

2-cells of K , thus each drum has a bounded size dependent on the constants k, ϵ and ρ . \square

Corollary to the Proof. *If (G, X) has the falsification by fellow traveler property then G has at most exponential second order isoperimetric function.*

Proof: Fix the 3-complex constructed above. Suppose a combinatorial 2-sphere Θ has area N , that is, it consists of N 2-cells. By isometry we may assume $1 \in \Theta$, and let n be the smallest integer such that $\Theta \subseteq B(n)$. For each 2-cell in Θ we attach one 3-ball. Let

$$b = (2k+2)M + 2M(2M - k(2k+2)C(2\epsilon+1)) + \rho(2k(2k+2)C(2\epsilon+1))$$

which is greater than the number of 2-cells in the base of a 3-ball from the proof of the theorem above. Then after attaching N 3-balls we obtain another combinatorial 2-sphere having area at most Nb and which lies in $B(n - \epsilon + \frac{k}{2})$. If we repeat this procedure $\frac{n}{\epsilon - k/2}$ times we get a 2-sphere inside $B(0)$ so it must be the identity vertex. We will have filled Θ with at most $N + Nb + (Nb)b + \dots + Nb^{\frac{n}{\epsilon - k/2}}$ 3-balls. Now since N is the area of Θ , and $1 \in \Theta$, then n can be at most $N(2k+2)$ which is the maximum number of edges in Θ . Therefore the number of 3-balls required to fill a combinatorial 2-sphere of area N is

$$\sum_{i=0}^{(2k+2) \cdot N} Nb^i = Nc^N < d^N$$

for c, d constants. \square

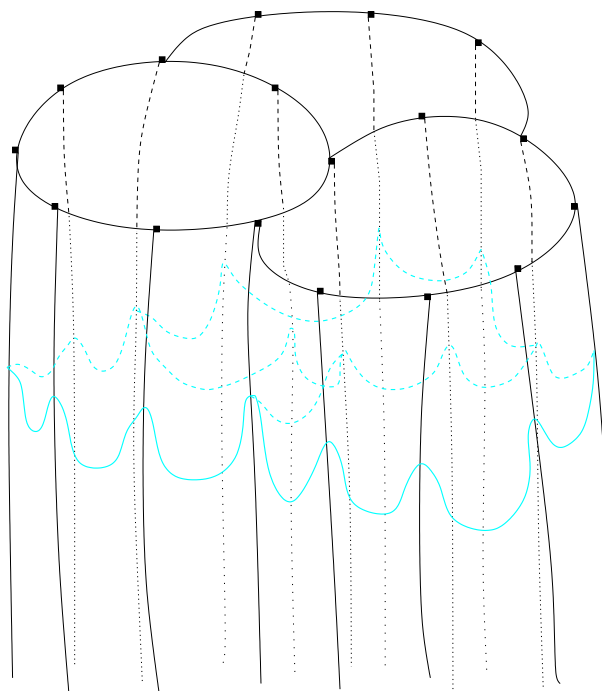
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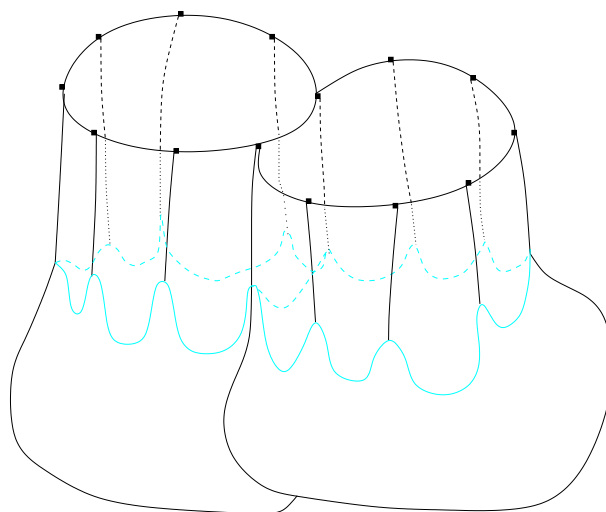
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(a) Tops and sides matching up



(b) Gluing on bases

FIGURE 4. The drum construction

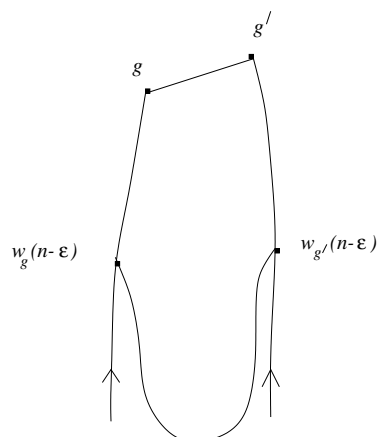


FIGURE 5. A side cell

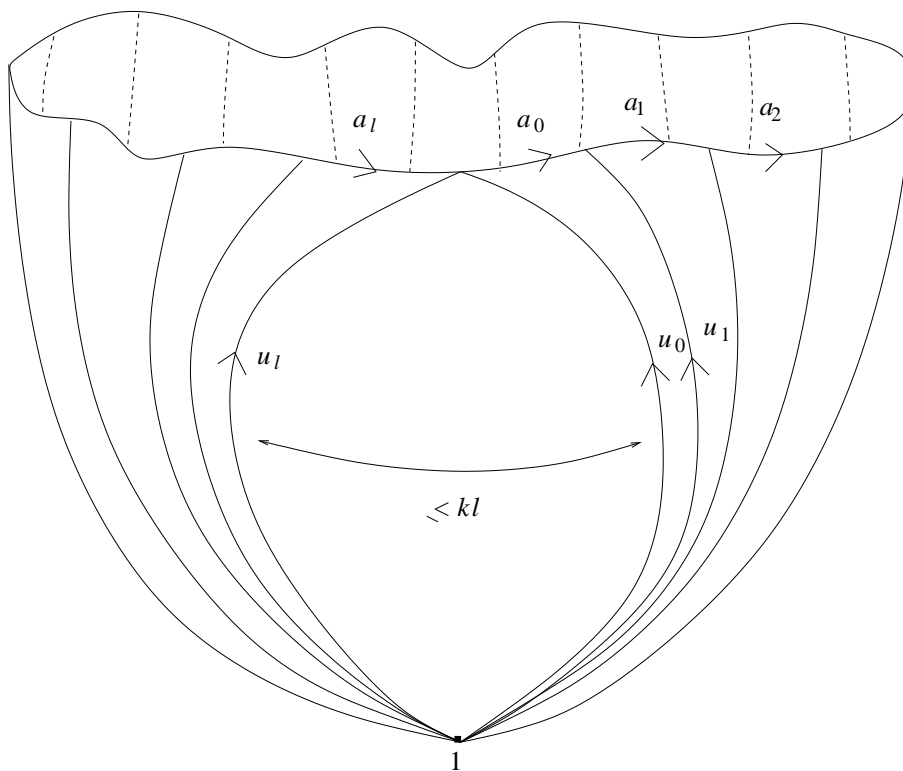


FIGURE 6. The “tear” in the base

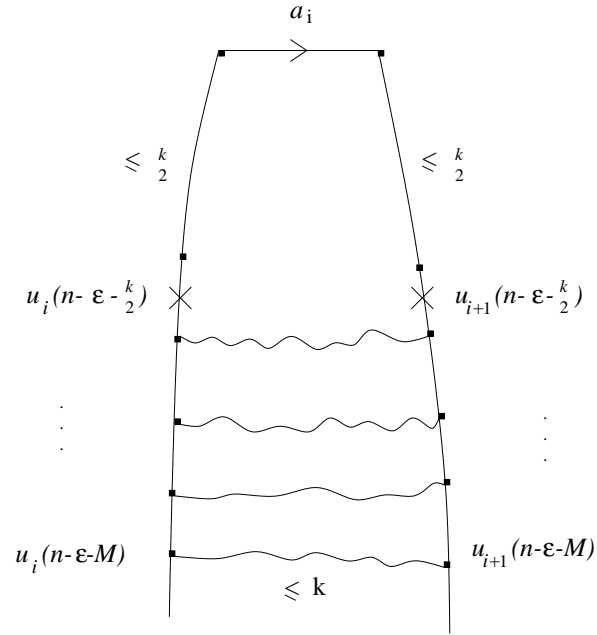


FIGURE 7. Base type 1

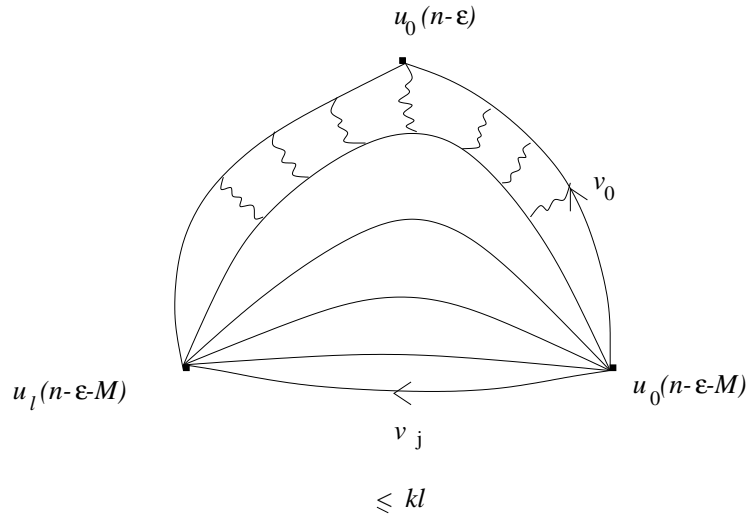


FIGURE 8. Base type 2