On the Wilf-Stanley limit of 4231-avoiding permutations and a conjecture of Arratia.

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Abstract

We construct a sequence of finite automata that accept subclasses of the class of 4231-avoiding permutations. We thereby show that the Wilf-Stanley limit for the class of 4231-avoiding permutations is bounded below by 9.35. This bound shows that this class has the largest such limit among all classes of permutations avoiding a single permutation of length 4 and refutes the conjecture that the Wilf-Stanley limit of a class of permutations avoiding a single permutation of length k cannot exceed $(k-1)^2$.

Key words: permutation classes, automata
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1 Introduction

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ and $\pi = \pi_1\pi_2\cdots\pi_n$ be permutations of $\{1, 2, 3, \ldots, k\}$ and $\{1, 2, 3, \ldots, n\}$ respectively, written as their sequences of values. Then σ *occurs as a pattern in* π if for some subsequence τ of π of the same length as σ all the values in τ occur in the same relative order as the corresponding values in σ. If σ does not occur as a pattern in π we say that π *avoids* σ. A *pattern class of permutations*, or simply class, is any set of permutations of the form:

$$\mathrm{Av}(X) = \{\pi \,:\, \forall\, \sigma \in X,\ \pi \text{ avoids } \sigma\}$$

where X is any set of permutations. We usually write $\mathrm{Av}(\sigma)$ rather than $\mathrm{Av}(\{\sigma\})$. Pattern classes are the lower ideals of the set of all finite permutations with respect to the partial order "occurs as a pattern in" and so are closed under arbitrary intersections and unions.

Much of the study of pattern classes has concentrated on enumerating classes $\mathrm{Av}(X)$ when X is a relatively small set of relatively short permutations. We write $s_n(X)$ for $|\mathrm{Av}(X) \cap \mathcal{S}_n|$. Results in this area led to the proposal of the Wilf-Stanley conjecture. A somewhat simplified version of this conjecture is:

Conjecture 1 (Wilf-Stanley) *Let X be any non-empty set of permutations. Then there exists a real number c_X such that $s_n(X) \leq c_X^n$.*

The resolution of the Wilf-Stanley conjecture by Marcus and Tardos [1], together with a result of Arratia's on the classes defined by avoiding a single permutation [2] implies that for each permutation π there exists a positive real number $L(\pi)$ called the *Wilf-Stanley limit* of the class $\mathrm{Av}(\pi)$ such that:

$$\lim_{n\to\infty} s_n(\pi)^{1/n} = L(\pi).$$

The values of $L(\pi)$ are known exactly for all permutations of length 3, and for all permutations of length 4 except 4231 and 1324 (which have the same Wilf-Stanley limit, by the obvious isomorphism between the corresponding classes). Using a result of Regev [3], Bóna [4,5,6] provided bounds:

$$9 \leq L(4231) \leq 288.$$

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Further results of Bóna [7] show that $L(\pi) \geq (k-1)^2$ for all layered permutations π of length k (a permutation π is layered if $\pi_{j+1} < \pi_j$ implies $\pi_{j+1} = \pi_j - 1$). Arratia [2] conjectured that for all permutations π of length k, $L(\pi) \leq (k-1)^2$. Regev's result shows that the value $(k-1)^2$ is attained for the permutation $\pi = 123 \cdots k$. In this paper we refute this conjecture by proving that:

$$L(4231) \geq 9.35.$$

Our proof of this result makes use of the insertion encoding for permutations. We establish that there is a class of permutations, strictly contained in Av(4231) whose elements are in one to one correspondence with the words of a language accepted by a certain finite automaton. Using the standard transfer matrix approach we are able to determine the growth rate of this language, which thus provides the lower bound cited above.

In order to make this paper self-contained we provide a brief introduction to the insertion encoding in the next section. Then we will describe the automaton (actually a sequence of automata) referred to above, and prove the required correspondence. We include a brief discussion of the computational methodology and then a summary and conclusions.

2 The Insertion Encoding

The insertion encoding is a general method for describing permutations. It shares some similarity with the generating tree approach of West [8,9] and also the enumeration schemes of Zeilberger [10] two approaches which have been used in these papers and elsewhere [11,12,13,14,15,16] to enumerate or determine structural information about a number of permutation classes.

A permutation π is viewed as "evolving" by the successive insertion of new maximal elements. Thus, the stages in the evolution of 264153 are: ϵ (the empty word), 1, 21, 213, 2413, 24153 and 264153. Each step of the evolution is described by a code letter of the form $\mathbf{f}_i$, $\mathbf{l}_i$, $\mathbf{r}_i$ or $\mathbf{m}_i$ where i is a positive integer. The intent of the symbols will become more clear if in the evolution of π we also include placeholders, called *slots* in positions where an element will eventually be inserted. We denote a slot by the symbol $\diamond$. Now the evolution of 264153 can be written as:

$$\diamond \to \diamond 1 \diamond \to 2 \diamond 1 \diamond \to 2 \diamond 1 \diamond 3 \to 24 \diamond 1 \diamond 3 \to 24 \diamond 153 \to 246153.$$

It can be seen that each event in the evolution is of one of four types: filling a slot (the last two events), insertion on the left hand end of a slot (the addition of 4), on the right hand end of a slot (the addition of 3), or in the middle of a slot splitting it in two (the addition of 1). The code letters then describe the type of insertion to carry out, and the subscript denotes the slot in which to perform the insertion (counted from left to right). Thus the insertion encoding of 246153 is $\mathbf{m}_1\mathbf{l}_1\mathbf{r}_2\mathbf{l}_1\mathbf{f}_2\mathbf{f}_1$.

In considering Av(4231) it turns out that a small modification of this encoding provides a more natural description of the resulting language. In this modification, the rightmost slot is distinguished by not allowing either $\mathbf{r}$ or $\mathbf{f}$ code letters in that slot. This ensures that there is always a slot present at the right hand end – an evolution may be complete when this is the only remaining slot. With respect to this convention, the evolution of 246153 becomes:

$$\diamond \to \diamond 1 \diamond \to 2 \diamond 1 \diamond \to 2 \diamond 1 \diamond 3 \diamond \to 24 \diamond 1 \diamond 3 \diamond \to 24 \diamond 153 \diamond \to 246153 \diamond.$$

The corresponding encoding is $\mathbf{m}_1\mathbf{l}_1\mathbf{m}_2\mathbf{l}_1\mathbf{f}_2\mathbf{f}_1$. For the remainder of this paper, it is this variation of the insertion encoding which we refer to as *the* insertion encoding.

We mention without proof the following result which will appear in [17]. It is not actually used in the next section, but provides the motivation for it.

Theorem 1 *Let k be a fixed positive integer. The collection of permutations whose evolution requires at most k slots at any point forms a pattern class $\mathcal{B}_k$. The insertion encodings of $\mathcal{B}_k$ form a regular language, as do the insertion encodings of any pattern class $\mathcal{B}_k \cap \mathrm{Av}(X)$ where X is a finite set of permutations.*

The theoretical methods of [17] provide, in principle, an effective method for determining the regular languages representing the insertion encodings of $\mathcal{B}_k \cap \mathrm{Av}(4231)$. In practice, these methods require various operations on automata which are of exponential complexity and hence are impractical for most values of k.

Instead, in the next section, we describe a direct construction of the automata which recognize words belonging to the insertion encodings of elements of $\mathcal{B}_k \cap \mathrm{Av}(4231)$.

3 The automata

Consider a configuration of elements and slots which might arise in the evolution of a 4231 avoiding permutation. In this configuration there will be some instances of patterns of the form $\cdots \diamond \cdots b \cdots \diamond \cdots a \cdots$ where $b > a$. Wherever such an instance occurs the first slot must be filled before the second slot can be. Otherwise we would obtain four elements $\cdots d \cdots b \cdots c \cdots a \cdots$ with $a < b < c < d$ in the resulting permutation, that is, an instance of 4231. Conversely, the only way we could ever create a 4231 pattern would be by insertion into such a slot. Borrowing terminology from [18] we say that in this configuration the second slot is *locked* until such time as the first slot (and any other slots participating in such patterns with it) are filled.

We now turn to the question of how locks are created, and how they interact. Suppose that we have a configuration of t slots:

$$\alpha_1 \diamond \alpha_2 \diamond \cdots \diamond \alpha_j \diamond \alpha_{j+1} \diamond \cdots \alpha_t \diamond$$

where α_1 through α_t are certain sequences of elements, α_1 might be empty, but the remaining α's are not. Suppose that the jth slot is not locked and we insert a new maximum element b into it, on the left for the sake of argument. The new configuration is:

$$\alpha_1 \diamond \alpha_2 \diamond \cdots \diamond \alpha_j \, b \diamond \alpha_{j+1} \diamond \cdots \alpha_t \diamond$$

Taking any slot from the first through the $(j-1)$st, b, any slot from the jth through the $(t-1)$st and any element from α_t yields a $\diamond b \diamond a$ pattern. Thus all the slots from the jth through the $(t-1)$st are now locked until all the slots from the first through the $(j-1)$st have been filled.

We can record this in the new configuration by subscripting the jth slot with the value $t-j$ – which is to be read as "the $t-j$ consecutive slots beginning from this one are locked, until the slots before it have been filled". Alternatively, a more attractive visual representation would be to place a bar over this block of slots. Any slot under a bar cannot be filled, but bars are removed when there are no slots to the left of them.

Other insertions into the jth slot create similar locks or bars. If the intersection of two locks is non-empty then one must be contained in the other, since a lock when created always begins at the remaining slot just to the right of the current insertion and ends at the penultimate slot.

It is possible for locks to be extended – in the example above the construction might proceed by adding a few more slots on the right hand end (using middle insertions in the final slot), and then an insertion on the right of the $(j-1)$st slot. Since this new lock properly contains the old one, we can at this point discard the old lock or simply extend its bar in the visual representation.

Observation 2 *If we know all the locking information about a configuration, then we can determine which insertions are allowed. Furthermore, we can determine the locking information of the configuration resulting from any allowed insertion.*

The first part of the observation is trivial since, by definition, insertions are allowed in the unlocked slots. The second follows from the notes above, since the lock formed by any insertion does not depend on the actual values present, only on the slots. Locks are removed precisely when their left hand endpoint becomes the leftmost slot.

By giving slots that are not at the left hand end of a lock a subscript of 0 and then reading a configuration only as a sequence of subscripts we see that the configurations that can arise in the construction of a 4231-avoiding permutation are in one to one correspondence with sequences $s_1 s_2 \cdots s_m$ (for $m \geq 1$) of non-negative integers satisfying $s_1 = s_m = 0$, and if $s_k > 0$ then for all $j < k + s_k$, $j + s_j \leq k + s_k$. The first condition expresses the fact that the first and last slots are always unlocked, and the second that if the jth slot lies within the lock on the kth slot, then its lock cannot extend beyond the end of that one. Sequences satisfying these conditions will be called *lock sequences*. It can easily be established inductively (but is not actually required for the following constructions) that every lock sequence can arise in the evolution of some 4231-avoiding permutation.

If we ignore the first and last slots (which can never be locked) and think of the locks as subintervals of $\{1, 2, 3, \ldots, m\}$ we see that they form a family of subintervals no two of which have the same left endpoint, and with the property that if two intersect, then one is a subinterval of the other. Of course this can be thought of as a recursive description of how such arrangements of locks can be created and it follows directly that the number of configurations of locks on these m elements is exactly the mth large Schröder number (sequence A006318 of [19]). The large Schröder numbers count paths in the nonnegative half plane from $(0, 0)$ to $(2n, 0)$ using steps $\mathbf{u} = (1, 1)$, $\mathbf{d} = (1, -1)$ and $\mathbf{h} = (2, 0)$. The correspondence is most easily seen from the set of such paths to arrangements of locks. Associate the numbers 1 through n with the $\mathbf{u}$'s and $\mathbf{h}$'s of such a sequence in order. The locks are precisely the subintervals of numbers that occur between some $\mathbf{u}$ and its matching $\mathbf{d}$. So, for example the sequence **uhudhuhddh** corresponds to the subintervals $[1, 6]$, $[3, 3]$ and $[5, 6]$ of the interval $[1, 7]$.

If we consider only locking sequences of length at most k (for some fixed positive integer k) and the symbols of the insertion encoding which are allowed to operate on them, then Observation 2 and the discussion in the first paragraph of this section immediately imply the following result.

Theorem 3 *Let k be a fixed positive integer. There is a finite automation $\textsc{Aut}_k$ whose accepted language consists of the insertion encodings of the permutations in $\mathcal{B}_k \cap \mathrm{Av}(4231)$. The states of $\textsc{Aut}_k$ can be taken to be the lock sequences of length at most k and the transitions of $\textsc{Aut}_k$ from a given sequence s are labelled by the codes of the allowed insertions in the slot configuration corresponding to s, and are from s to the lock sequence labelling the result of the corresponding insertion.*

The automata above are simply the restrictions of an automaton $\textsc{Aut}$ (with infinitely many states and an infinite language) that produces the insertion encoding of all and only the elements of $\mathrm{Av}(4231)$. Its states are arbitrary lock sequences and its transitions are precisely the allowed insertions within a lock sequence.

For illustrative purposes, consider $\textsc{Aut}_4$. This automaton has 10 states represented by the lock sequences 0, 00, 000, 010, 0000, 0010, 0100, 0110, 0200 and

0210. A representative slot configuration for each of these states is: ◇, ◇1◇, ◇1◇2◇, ◇2◇1◇, ◇1◇2◇3◇, ◇1◇3◇2◇, ◇2◇1◇3◇, ◇2◇14◇3◇, ◇3◇1◇2◇ and ◇3◇2◇1◇. A complete transition table for this automaton is shown below. Each row illustrates the transitions available from the state specified at the left hand end of the row and double subscripts such as $\mathbf{f}_{12}$ indicate that both $\mathbf{f}_1$ and $\mathbf{f}_2$ induce the same transition.

	0	00	000	010	0000	0010	0100	0110	0200	0210
0	$\mathbf{l}_1$	$\mathbf{m}_1$								
00	$\mathbf{f}_1$	$\mathbf{l}_{12}\mathbf{r}_1$	$\mathbf{m}_2$	$\mathbf{m}_1$						
000		$\mathbf{f}_{12}$	$\mathbf{l}_{13}\mathbf{r}_2$	$\mathbf{r}_1\mathbf{l}_2$	$\mathbf{m}_3$	$\mathbf{m}_2$			$\mathbf{m}_1$	
010		$\mathbf{f}_1$		$\mathbf{l}_{13}\mathbf{r}_1$			$\mathbf{m}_3$			$\mathbf{m}_1$
0000			$\mathbf{f}_{13}$	$\mathbf{f}_2$	$\mathbf{l}_{14}\mathbf{r}_3$	$\mathbf{r}_2\mathbf{l}_3$			$\mathbf{r}_1\mathbf{l}_2$	
0010				$\mathbf{f}_{12}$		$\mathbf{l}_{14}\mathbf{r}_2$				$\mathbf{r}_1\mathbf{l}_2$
0100			$\mathbf{f}_1$	$\mathbf{f}_3$			$\mathbf{l}_{14}\mathbf{r}_3$	$\mathbf{l}_3$	$\mathbf{r}_1$	
0110				$\mathbf{f}_1$				$\mathbf{l}_{14}$		$\mathbf{r}_1$
0200			$\mathbf{f}_1$						$\mathbf{l}_{14}\mathbf{r}_1$	
0210				$\mathbf{f}_1$						$\mathbf{l}_{14}\mathbf{r}_1$

4 Computational Methodology

Let $\mathcal{L}$ be the set of all finite lock sequences. We order this set first by length, and then lexicographically within each length. This assigns an index (the position in this ordering) to each possible lock sequence. Armed with a table of Schröder numbers, the recursive description of $\mathcal{L}$ makes it relatively easy to compute these indices directly. Let

$$\mathsf{ind} : \mathcal{L} \to \mathbb{N}$$

be the function which computes the index of a lock sequence.

Using ind and its inverse the states of AUT can be indexed by the natural numbers, and the transitions of AUT can be determined. As our goal is primarily to determine the growth rate of the language accepted by AUT_k we can use these to construct the matrix A_k whose entry in row i and column j is the number of transitions between the state $\mathsf{ind}^{-1}(i)$ and $\mathsf{ind}^{-1}(j)$ (here i and j are any pair of integers in the image of the lock sequences of length at most k under ind).

The matrix A_k is irreducible because the underlying directed multigraph is strongly connected. Furthermore it is primitive as all the diagonal entries are non-zero (each state has a loop labelled $\mathbf{l}_1$). Thus we can apply the Perron Frobenius theorem and conclude that A_k has a unique dominant eigenvalue λ_k which lies on the positive real axis and that the corresponding eigenvector is positive. Hence the limit

$$\lim_{n\to\infty}(e_1^T A_k^n e_1)^{1/n} = \lambda_k.$$

Moreover, the generating function for the language accepted by $\textsc{Aut}_k$ is simply:

$$\sum_{n=0}^{\infty} e_1^T A_k^n e_1 t^n$$

In other words, λ_k is the growth rate of the language accepted by $\textsc{Aut}_k$ and hence the Wilf-Stanley limit of the class $\mathcal{B}_k \cap \mathrm{Av}(4231)$.

The matrix A_k is relatively sparse, so the eigenvalue λ_k can be computed without great difficulty even for moderately large values of k. For instance, if $k = 13$, A_k is a square matrix with 6589728 rows. There are at most 46 transitions from any state in the automaton (this is achieved in the state with 12 slots having no locks – many states have significantly fewer transitions). However, no row has quite this many non zero entries as there are always several transitions to the same state.

Theorem 4 *The Wilf-Stanley limit,* $L(4231)$ *is at least* 9.35*.*

PROOF. Let $A = A_{13}$. Because A is irreducible, primitive, and non-negative an iterative scheme to compute its dominant eigenvalue is guaranteed to converge. That is, we may define a sequence of vectors $\vec{v}_k$ where $\vec{v}_1 = e_1$ and $\vec{v}_{k+1}$ is a scalar multiple of $A\vec{v}_k$ having some fixed norm. This method, implemented in Java, produced a dominant eigenvalue of 9.3508 for the matrix A together with an approximate eigenvector $\vec{v}$. Direct computation then showed that: $A\vec{v} \geq (9.35)\vec{v}$. Since the entries of A are all non-negative and the diagonal entries are all positive it follows that $A^n\vec{v} \geq (9.35)^n\vec{v}$ for all positive integers n. Since the first coordinate of $\vec{v}$ is non-zero, it also follows that:

$$\lim_{n\to\infty}(e_1^T A_k^n e_1)^{1/n} \geq 9.35$$

which, as noted above, establishes the claim of the theorem. $\square$

The values of $s_n(4231)$ are reported for $n \leq 20$ as sequence A061552 in [19]. The recursive method used to compute these numbers is described in [20] and its exact complexity has not been analysed. A permutation requiring more than k slots to produce in the insertion encoding must have length at least $2k$ so $s_n(1324)$ is the $(1, 1)$ entry of A_k^n for any $k > n/2$. Choosing $k = 13$

allows us to report that the values of the sequence $s_n(4231)$ for n between 21 and 25 are: 1535346218316422, 12015325816028313, 94944352095728825, 757046484552152932 and 6087537591051072864.

As A_k has asymptotically $O(k(1+\sqrt{2})^{2k})$ non-zero entries, the complexity of the computation of $s_n(4231)$ by this method is not more than $O(n^2(1+\sqrt{2})^n)$ (and the constants are not large).

5 Conclusions

The lower bounds presented here leave the question of the true growth rate of Av(4231) intriguingly open. Since the sequence λ_k is monotone increasing and bounded above by $L(4231)$ it has a limit $\lambda_\infty \leq L(4231)$. Although the generating functions for the language accepted by $\textsc{Aut}_k$ and Av(4231) agree through at the first $2k$ terms, this does not necessarily guarantee that $\lambda_\infty = L(4231)$. So this raises:

Question 1 *Is* $\lim_{k\to\infty} \lambda_k = L(4231)$*?*

The growth rates of the automata languages for different values of k are presented below.

k	λ_k
1	1.0000
2	3.4142
3	5.1120
4	6.2262
5	7.0014
6	7.5693
7	8.0029
8	8.3450
9	8.6220
10	8.8511
11	9.0439
12	9.2085
13	9.3508

We leave it to the reader to decide how to extrapolate this sequence. However, the value obtained will depend on how one models the behaviour of the difference $\lambda_\infty - \lambda_k$ as a function of k. Our best guess, based on an empirical observation that the plot of $1/\sqrt{k}$ against λ_k is roughly linear is that the limiting value lies between 11 and 12. Computing λ_k for larger values of k is possible – though we note that there are over 6.5×10^6 states in $\textsc{Aut}_{13}$ and the number of states goes up by a factor of roughly 5.8 for each additional slot so significant further progress in this direction is limited by the obvious combinatorial explosion. However, the natural structure of the states of $\textsc{Aut}$ leaves open the possibility of a closed form or asymptotic analysis of the limiting case.

We suspect that the answer to Question 1 is yes, but the evidence is not entirely convincing. It consists of the observation that for Av(312) we can carry out a similar analysis (and of course we know that $L(312) = 4$) and because of the simple form of the corresponding automaton which only has one state for each number of slots, we can prove that the maximal eigenvalues do converge to 4. On the other hand for the class Av(4321) with $L(4321) = 9$ it is also the case that the underlying automata are relatively simple, the one for k slots having only $O(k^2)$ states. The corresponding dominant eigenvalues do appear to converge to 9 but the rate of convergence is quite slow.

The results above show that the class of 4231 avoiders has strictly larger growth rate than any other class avoiding a single permutation of length 4. This throws open once again the question of what makes one pattern harder to avoid than another. That is:

Question 2 *Among the classes* Av(π) *where π is a single permutation of length k, which have the largest growth rates? What is this largest growth rate? More generally, given two permutations π and τ are there general methods for deciding whether or not $L(\pi) \geq L(\tau)$?*

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