

Random subgroups of Thompson's group F^*

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Abstract

We consider random subgroups of Thompson's group F with respect to two natural stratifications of the set of all k generator subgroups. We find that the isomorphism classes of subgroups which occur with positive density are not the same for the two stratifications. We give the first known examples of *persistent* subgroups, whose isomorphism classes occur with positive density within the set of k -generator subgroups, for all sufficiently large k .

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Additionally, Thompson's group provides the first example of a group without a generic isomorphism class of subgroup. Elements of F are represented uniquely by reduced pairs of finite rooted binary trees. We compute the asymptotic growth rate and a generating function for the number of reduced pairs of trees, which we show is D-finite and not algebraic. We then use the asymptotic growth to prove our density results.

1 Introduction

We investigate the likelihood of randomly selecting a particular k -generator subgroup of Thompson's group F , up to isomorphism. This is made precise through a notion of *asymptotic density*. This in turn involves a choice of *stratification* of the set of k -tuples of elements, which we view as generating sets for the subgroups, into spheres of size n . Intuitively, the density of an isomorphism class of subgroup with k generators is the probability that a randomly selected k -generator subgroup is in the class.

A k -generator subgroup H of a group G is called *generic* among all k -generated subgroups if a randomly selected subgroup of G with k generators is isomorphic to H with probability which is asymptotically one. Previous results on asymptotic density of subgroups of particular groups, such as braid or free groups, have always found a generic type of subgroup for all k . We find that Thompson's group F , with respect to each of two natural stratifications on the set of k -generator subgroups, does not possess a generic isomorphism class of subgroup for any k . Additionally, for each stratification there are isomorphism classes of subgroups which are chosen at random with small but positive probability among the set of all k -generated subgroups, for *any* sufficiently large k . We call such subgroups *persistent*. Lastly, we exhibit subgroups with positive density with respect to one stratification but not the other, illustrating that different natural notions of stratification can have dramatic effects on the forms of randomly chosen subgroups.

The likelihood that a particular isomorphism class of subgroup of a given group is selected at random is motivated by questions in group-based cryptography. The analysis of the security of algorithms used in cryptography can depend upon the expected isomorphism type of a random subgroup. Many group-based cryptosystems propose the braid group B_n as a platform; recent work of Miasnikov, Shpilrain and Ushakov [16] shows that experimentally, subgroups of B_n generated by k elements where k is small relative to n , and moreover, those k elements are of small size, are generically isomorphic to B_n . Due to the restrictions on the size of the generators we cannot conclude that a subgroup of B_n with k generators is generically isomorphic to B_n . Regardless, their results explain why current cryptosystems based on B_n are vulnerable to attack.

Our definition of the asymptotic density of a particular subgroup H of a group G follows Borovik, Miasnikov and Shpilrain in [3]. They present a detailed discussion of asymptotic and statistical questions in group theory. We also refer the reader to Kapovich, Miasnikov, Schupp and Shpilrain [15] for background on

generic-case complexity and notions of density.

We let G be an infinite group and X a set of representatives of elements that maps onto G . We can associate to each $x \in X$ an integer *size*. For example, a natural notion of size is word length- we can let X be the set of all words in a finite generating set for G , with size corresponding to word length. There are situations where other notions of size, besides word length, are considered. We let X_k be the set of unordered k -tuples of representatives $x \in X$. Then each member of X_k corresponds to a k -generated subgroup of G , taking the k representatives as the generators. We fix a notion of size on X_k . We can define an integer size for each k -tuple in a variety of ways. For example, the size of a k -tuple could be the sum of the sizes of its components. Alternatively, one could take the size of a k -tuple to be the maximum size of any of its components. Once notions of size are fixed, both for elements and tuples, the set of all tuples of size n in X_k is called the n -sphere, and denoted $\text{Sph}_k(n)$. Such a decomposition of X_k into spheres of increasing radii is known as a *stratification* of X_k . We prefer our spheres of a fixed size to be finite and thus we can regard these spheres of increasing radii as an exhaustion of an infinite set X_k by a collection of finite sets.

To quantify the likelihood of randomly selecting a particular subset of X_k , we take a limit of the counting measure on spheres of increasing radii. Let $|T|$ denotes the size of the set T . The *asymptotic density* of a subset T in X_k is defined to be the limit

$$\lim_{n \rightarrow \infty} \frac{|T \cap \text{Sph}_k(n)|}{|\text{Sph}_k(n)|}$$

if this limit exists. We often omit the word asymptotic and refer to this limit simply as the *density* of T .

To understand density not just of k -tuples, but of isomorphism classes of k -generator subgroups, we let T_H be the set of k -tuples that generate a subgroup of G isomorphic to some particular subgroup H . If the density of T_H is positive we say that H is *visible* in the space of k -generated subgroups of G . We call the set of all visible k -generated subgroups of G the *k -subgroup spectrum*, denoted by $\text{Spec}_k(G)$. If the density of T_H is one, we say that H is *generic* in $\text{Spec}_k(G)$; if this density is zero we say that H is *negligible* in $\text{Spec}_k(G)$.

We make a series of choices within this construction, each of which can greatly influence the densities of different subsets; those choices include: the representation of group elements, the size function defined on X , and the stratification of the set of tuples X_k . Additionally, we are asserting that the likelihood of randomly selecting a k -generator subgroup isomorphic to the given one is captured by the limit as defined. It is certainly possible to construct contrived stratifications which various pathological properties, so we concentrate on stratifications which correspond to “natural” definitions of the sphere of size n in X_k . Despite this, we show that for Thompson’s group F , a small change in the stratification has a great impact on the set of visible subgroups.

Below, we show that Thompson’s group F is the first example of a group which has different asymptotic properties with respect to two different, yet natural,

methods of stratification. To define these stratifications, we represent elements of F using reduced pairs of finite rooted binary trees, which we abbreviate to “reduced tree pairs”. These representatives are in one-to-one correspondence with group elements. Each pair consists of two finite, rooted binary trees with the same number of leaves, or equivalently, with the same number of internal nodes or carets, as defined below, satisfying a reduction condition specified in Section 2. The size of a tree pair will be the number of carets in either tree of the pair.

Using reduced tree pairs to represent elements of F , we define the sphere of radius n in X_k in two natural ways:

1. take $\text{Sph}_k(n)$ to be the set of k -tuples in which the sum of the sizes of the coordinates is n , or
2. take $\text{Sph}_k(n)$ to be the set of k -tuples where the maximum size of a coordinate is n .

We will refer to these as the “sum stratification” and “max stratification” respectively. With respect to the sum stratification, every non-trivial isomorphism class of m -generated subgroup for $m \leq k$ is visible. That is, every possible subgroup isomorphism class has non-zero density. With respect to the max stratification, there are subgroup isomorphism classes with zero density.

Perhaps the most natural stratification to consider on F , or on any finitely generated group, is obtained by taking the size of an element of F to be the word length with respect to a particular set of generators. For F we can consider word length with respect to the standard finite generating set $\{x_0, x_1\}$. This stratifies the group itself into metric spheres. Despite work of José Burillo [6] and Victor Guba [12] in this direction, the sizes of these spheres have not been calculated, and thus it is not yet computationally feasible to consider the possible induced stratifications of X_n with respect to word length as a notion of size.

It is striking in our results below that the k -generator subgroups of Thompson’s group F have no generic isomorphism type with respect to either stratification, for any k . All other groups which have been studied in this way exhibit a generic type of subgroup with respect to natural stratifications. Arzhantseva and Olshanskii [2] and Arzhantseva [1] considered generic properties of subgroups of free groups. With respect to the notions of stratification described here, Jitsukawa [14] proved that k elements of any finite rank free group generically form a free basis for a free group of rank k . Miasnikov and Ushakov [17] proved this is true also for the pure braid groups and right angled Artin groups.

To obtain our results on random subgroups of Thompson’s group F we must be able to count the number r_n of reduced pairs of trees with a given number of carets. Woodruff [21], in his thesis, conjectured that the number r_n is proportional to $(8 + 4\sqrt{3})^n/n^3$. We prove Woodruff’s conjectured growth rate, and additionally show that the generating function for the number of reduced tree pairs is not algebraic, but that it is D-finite, meaning that it satisfies a linear ordinary differential equation with polynomial coefficients.

This paper is organized as follows. In Section 2, we consider the number of pairs of reduced trees of size n , which we call r_n . We prove that r_n has a D-finite generating function which is not algebraic. We prove that r_n approaches $A\mu^n/n^3$ uniformly, where A is a constant and $\mu = 8 + 4\sqrt{3} \approx 14.93$.

In Section 3 we describe particular subgroups of Thompson's group F and elementary observations about F that will be important in later sections.

In Section 4 we study the sum stratification and compute the asymptotic density of isomorphism classes of k -generator subgroups. We prove that if G is a non-trivial m -generator subgroup of F , then its isomorphism class is visible in the space of k -generator subgroups of F for $k \geq m$. This stands in stark contrast to previously known examples, since no subgroup is generic in this stratification.

In Section 5 we turn to the max stratification and compute the asymptotic density of isomorphism classes of k -generator subgroups of F and find very different behavior. In this case, not every isomorphism class of m -generator subgroup is visible in the space of k -generator subgroups of F for $k \geq m$. We prove that \mathbb{Z} is visible in the set of k -generated subgroups only for $k = 1$. Yet there are examples of isomorphism classes of subgroups which are *persistent*; that is, visible in the set of k generator subgroups for all sufficiently large k . For example, we show that the isomorphism class of F itself is visible in the set of k -generated subgroups for all $k \geq 2$.

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2 Combinatorics of reduced tree pairs

A *caret* is a pair of edges that join two vertices to a common parent vertex, which we draw as \wedge . An n -caret *tree pair diagram*, or *tree pair* for short, is an ordered pair consisting of two rooted binary trees, each having n carets. A 5-caret tree pair is shown in Figure 1(a). A *leaf* is a vertex of degree one. A tree with n carets will have $n + 1$ leaves. In the trees we consider, all vertices other than the leaves and the root have degree three. The *left child* of a caret is the caret attached to its left leaf; the *right child* is defined analogously. An *exposed caret* is a caret both of whose children are leaves. A pair of trees with at least two carets in each tree is *unreduced* if, when the leaves are numbered from left to right, each tree contains a caret with leaves numbered k and $k + 1$ for some k . In an unreduced tree pair, the caret with identical leaf numbers is removed from both trees, the leaves are renumbered, and the trees are again inspected for possible reductions. For example, the tree pair in Figure 1(a) is unreduced. Removing the exposed caret with leaves labeled 1, 2 in each tree yields the reduced tree pair in Figure 1(b). A pair of trees which is not unreduced is called reduced. Note that

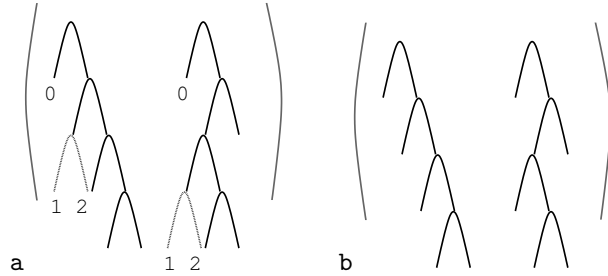


Figure 1: A five caret unreduced tree pair diagram, with its first three leaves numbered, and the corresponding four caret reduced tree pair diagram.

we do not reduce a pair of single carets: we insist that our tree pairs are always nonempty. We denote the number of reduced tree pairs with n -carets by r_n , so we have $r_0 = 0$ and $r_1 = 1$.

Ben Woodruff studied the enumeration of $\{r_n\}$ in his thesis [21] where he derived a formula for r_n (which he denoted N_n), proved an upper bound of $(8 + 4\sqrt{3})^n \approx 14.93^n$ and conjectured an asymptotic growth rate of $(8 + 4\sqrt{3})^n/n^3$. We take a different approach to counting r_n and derive a recursive formula in terms of c_n^2 , where $c_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number. Working in terms of generating functions for r_n and c_n^2 , we obtain a finite-order differential equation which leads to a finite polynomial recurrence for r_n . From this we are able to prove the growth rate conjectured by Woodruff. The key to this section is to show that the generating function for r_n is closely related to that for c_n^2 and many of the properties of the generating function for c_n^2 are inherited by that of r_n .

We let $f(k, m)$ denote the number of ordered k -tuples of possibly empty rooted binary trees using a total of m carets, which we call *forests*. So for example $f(3, 2)$, which is the number of forests of three trees containing a total of two carets, is equal to nine, as shown by Figure 2. A straightforward argument shows that $f(k, n) = \frac{k}{2n+k} \binom{2n+k}{n}$.

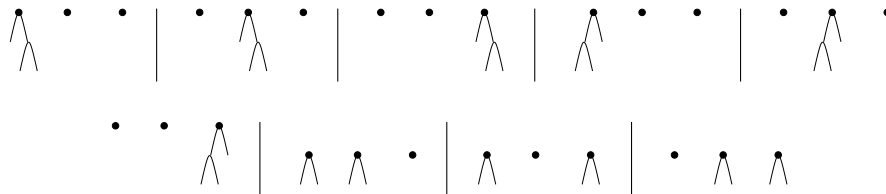


Figure 2: We exhibit that $f(3, 2) = 9$ by enumerating all forests consisting of three trees and a total of two carets.

The n -th Catalan number c_n counts the number of binary trees consisting of n carets, and thus c_n^2 is the number of ordered pairs of rooted binary trees with

n carets in each tree. Some of these pairs will be reduced, and some not. For those that are not reduced, we can cancel corresponding pairs of carets to obtain an underlying reduced tree pair. In a reduced tree pair consisting of i carets, each tree has $i+1$ leaves. We describe a process which is the inverse of reduction, which we call “decoration.” To decorate a reduced tree pair diagram (S, T) with i carets in each tree, we take a forest of $i+1$ trees, some of which may be empty, and $n-i$ carets (for $n \geq i$), duplicate it, then append the trees in the forests to the corresponding leaves of S and T . The first tree in the forest is appended to the first leaf, the second tree in the forest to the second leaf and so on. We can do this in $f(i+1, n-i)$ different ways. This decorating process yields a new unreduced tree pair with n carets, which will reduce to the original reduced tree pair (S, T) with i carets. For example, the reduced 2-caret tree pair drawn in bold in Figure 3 can be decorated in 9 different ways with a forest consisting of three trees A, B and C with a total of three carets between them, to yield unreduced pairs of 5 carets all of which would all reduce to the original tree pair diagram. This leads to the following lemma.

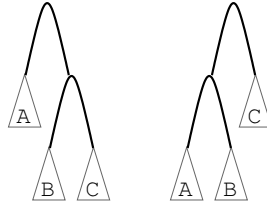


Figure 3: Decorating a reduced tree with a forest of three trees A, B and C .

Lemma 1 (Relating r_n and c_n^2) For $n \geq 1$

$$c_n^2 = r_n + r_{n-1}f(n, 1) + r_{n-2}f(n-1, 2) + \dots + r_1f(2, n-1).$$

Proof: Each n -caret tree pair is either reduced or must reduce to a unique reduced tree pair of i carets for some $i \in [1, n-1]$. Hence the total number of n -caret tree pairs, c_n^2 , is the number of pure reduced pairs of n -carets, r_n , plus the number r_i of reduced i -caret tree pairs multiplied by the number of ways to decorate them with a forest of $n-i$ carets, $f(i+1, n-i)$, for each possible value of i . \square

We can reformulate this recursion in terms of generating functions. We define the generating functions for r_n, c_n and c_n^2 respectively as:

$$\begin{aligned} R(z) &= r_1z + r_2z^2 + r_3z^3 + \dots \\ C(z) &= c_0 + c_1z + c_2z^2 + c_3z^3 + \dots \\ P(z) &= c_1^2z + c_2^2z^2 + c_3^2z^3 \dots \end{aligned}$$

Note that $R(z)$ and $P(z)$ have no constant term while $C(z)$ does. We prove in the following proposition that $R(z)$ can be obtained from $P(z)$ via a simple substitution. Using knowledge of $P(z)$ we can find a closed form expression for

$R(z)$ and asymptotic growth rate for r_n . Note that if $G(z)$ is the generating function for a set of objects, then $G(z)^k$ is the generating function for ordered k -tuples of those objects. In this way we can express the generating function of $f(k, n)$ for fixed k as $C(z)^k$.

Proposition 2 (Relating $R(z)$ and $P(z)$) *The generating functions for r_n and c_n^2 are related by the following equation:*

$$R(z) = (1 - z)P(z(1 - z)),$$

which is equivalent to

$$P(x) = C(x)R(xC(x)).$$

Proof: The generating function for the Catalan numbers is well known and may be written in closed form as

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x};$$

it satisfies the algebraic equation $C(x)(1 - xC(x)) = 1$. See Stanley [19] for example. If we rewrite the equation $R(z) = (1 - z)P(z(1 - z))$ substituting the variable z with $xC(x)$ then we obtain

$$R(xC(x)) = (1 - xC(x))P(xC(x)(1 - xC(x))) = \frac{1}{C(x)}P(x)$$

which rearranges to

$$P(x) = C(x)R(xC(x))$$

This substitution is inverted by $x \mapsto z(1 - z)$, and so proving this equation implies the proposition. By examining the coefficients of x^n we will show that this statement is equivalent to Lemma 1.

The right hand side can be written as

$$C(x)R(xC(x)) = C(x) \sum_{k=0}^{\infty} r_k (xC(x))^k = \sum_{k=0}^{\infty} r_k x^k (C(x))^{k+1}$$

We will use the notation $[x^i]G(x)$ to denote the coefficient of x^i in the expansion of a generating function $G(x)$. Considering the above equation in terms of the coefficient of x^n we have

$$\begin{aligned} c_n^2 &= [x^n]P(x) = [x^n]C(x)R(xC(x)) \\ &= [x^n] \sum_{k=0}^{\infty} r_k x^k (C(x))^{k+1} = \sum_{k=0}^{\infty} [x^n] x^k r_k (C(x))^{k+1} \\ &= \sum_{k=0}^n [x^{n-k}] r_k (C(x))^{k+1} \end{aligned}$$

As noted above, $(C(x))^{k+1}$ is the generating function for the number of ordered $(k+1)$ -tuples of rooted binary trees, which are counted by $f(k+1, n)$. Thus the coefficient of x^{n-k} in $(C(x))^{k+1}$ is precisely $f(k+1, n-k)$, that is, $[x^{n-k}](C(x))^{k+1} = f(k+1, n-k)$. So the above equation becomes

$$c_n^2 = r_n f(n+1, 0) + r_{n-1} f(n, 1) + \dots + r_1 f(2, n-1) + r_0 f(1, n)$$

which is precisely Lemma 1 since $f(n+1, 0) = 1$ and $r_0 = 0$. \square

A function is said to be *D-finite* if it satisfies a homogeneous linear ordinary differential equation with polynomial coefficients, for example, see [19]. The class of D-finite functions strictly contains the class of algebraic (and rational) functions. If one has a differential equation for a generating function it is possible to obtain the asymptotic growth rate of its coefficients by studying the differential equation. Following [19], a generating function is D-finite if and only if its coefficients satisfy a finite polynomial recurrence.

Lemma 3 (*R(z) is D-finite*) *The generating function R(z) satisfies the following linear ordinary differential equation*

$$\begin{aligned} & z^2(1-z)(16z^2 - 16z + 1)(2z - 1)^2 \frac{d^3 R}{dz^3} \\ & - z(2z - 1)(16z^2 - 16z + 1)(8z^2 - 11z + 5) \frac{d^2 R}{dz^2} \\ & - (128z^5 - 320z^4 + 365z^3 - 232z^2 + 76z - 4) \frac{dR}{dz} \\ & + 36z(z - 1)R(z) = 0. \end{aligned}$$

It follows that R(z) is D-finite.

Proof: Starting from a recurrence satisfied by the Catalan numbers we can find a differential equation satisfied by $P(z)$ and then standard tools allow us to transform this equation into one satisfied by $R(z)$.

Since $c_n = \frac{1}{n+1} \binom{2n}{n}$, we have the following recurrence for the Catalan numbers:

$$(n+2)c_{n+1} = 2(2n+1)c_n.$$

Squaring both sides yields

$$(n+2)^2 c_{n+1}^2 = 4(2n+1)^2 c_n^2.$$

Thus we have a finite polynomial recurrence for the coefficients of $P(z)$, which means that we can find a linear differential equation for $P(z)$. We do this using the Maple package GFUN [18] to obtain

$$(z^2 - 16z^3) \frac{d^2 P}{dz^2} + (3z - 32z^2) \frac{dP}{dz} + (1 - 4z)P(z) = 1.$$

The original recurrence can be recovered by extracting the coefficient of z^n in the above equation. We can then make this differential equation homogeneous

$$(16z^3 - z^2) \frac{d^3 P}{dz^3} + (80z^2 - 5z) \frac{d^2 P}{dz^2} + (68z - 4) \frac{dP}{dz} + 4P(z) = 0.$$

Making the substitution $z \mapsto z(1-z)$ using the command `algebraicsubs()` in `GFUN` we find a differential equation satisfied by $P(z(1-z))$. This in turn leads to the homogeneous differential equation for $R(z)$ given above. \square

Following the notation of Flajolet [11], we say that two functions are *asymptotically equivalent* and write $f(n) \sim g(n)$ when

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1.$$

Proposition 4 (Woodruff's conjecture) $r_n \sim A\mu^n/n^3$ where $\mu = 8+4\sqrt{3}$ and $A > 0$ is a constant.

Proof: We begin by establishing a rough bound on the exponential growth of r_n and refine this bound by analyzing a polynomial recurrence satisfied by r_n using techniques from Wimp and Zeilberger [20].

Since reduced tree pairs are a subset of the set of all tree pairs, it follows that $r_n \leq c_n^2$. We obtain a lower bound on r_n by the following construction. For each tree T with n carets, number the leaves from left to right starting with 0. Let S_1 denote the tree consisting of n left carets, each the left child of its parent caret. Let S_2 denote the tree with $n-1$ left carets, and a single interior caret attached to the right leaf of the leftmost caret. This interior caret has leaves numbered 1 and 2. If T does not have an exposed caret with leaves labeled 0 and 1, then the pair (T, S_1) is reduced. If T does have an exposed caret with leaves labeled 0 and 1, then form the reduced tree pair (T, S_2) . Thus for each tree T with n carets, there is at least one distinct reduced tree pair diagram with n carets, and we conclude that $c_n \leq r_n$.

It follows that $c_n^{1/n} \leq r_n^{1/n} \leq c_n^{2/n}$. Since $c_n \sim B4^n n^{-3/2}$ for a constant B (see Flajolet and Sedgewick [11] for example), it follows that $4 \leq \lim_{n \rightarrow \infty} r_n^{1/n} \leq 16$.

The differential equation satisfied by $R(z)$ can be transformed into a linear difference equation satisfied by r_n using the Maple package `GFUN` [18]:

$$\begin{aligned} 0 &= (n+5)(n+6)^2 r_{n+5} - (n+5)(n+4)(21n+101) r_{n+4} \\ &\quad + 2(4n+15)(n+4)(13n+33) r_{n+3} - 4(n+3)(53n^2+208n+195) r_{n+2} \\ &\quad + 32(6n+5)(n+2)(n+1) r_{n+1} - 64n^2(n+1) r_n. \end{aligned}$$

To compute the asymptotic behavior of the solutions of this recurrence we will use the technique described in [20]. This technique has also been automated by the command `Asy()` in the `GuessHo1o2` Maple package. This package is available from Doron Zeilberger's website. We outline the method below.

Theorem 1 of [20] implies that the solutions of linear difference equations

$$\sum_{\ell=0}^{\nu} a(n) f_{n+\ell} = 0,$$

where $a(n)$ are polynomials, have a standard asymptotic form. While this general form is quite complicated (and we do not give it here), we note that in the enumeration of combinatorial objects which grow exponentially rather than super-exponentially one more frequently finds asymptotic expansions of the form

$$f_n \sim \lambda^n n^\theta \sum_{j \geq 0} b_j n^{-j}.$$

By substituting this asymptotic form into the recurrence one can determine the constants λ, θ and b_j . For example, substituting the above form into the recurrence satisfied by r_n , one obtains (after simplifying):

$$\begin{aligned} 0 &= (\lambda - 1)(\lambda^2 - 16\lambda + 16)(\lambda - 2)^2 \\ &+ (\lambda - 2)(5\lambda^4\theta + 17\lambda^4 - 256\lambda^3 - 74\lambda^3\theta + 164\lambda^2\theta + 558\lambda^2 - 352\lambda - 96\lambda\theta + 32)/n \\ &+ O(1/n^2). \end{aligned}$$

In order to cancel the dominant term in this expansion we must have

$$\lambda = 1, 8 - 4\sqrt{3}, 2, 8 + 4\sqrt{3}.$$

Each of these values for λ implies different values of θ so as to cancel the second-dominant term. In particular, if $\lambda = 8 + 4\sqrt{3}$, then $\theta = -3$. Since $4 \leq \lim_{n \rightarrow \infty} r_n^{1/n} \leq 16$, it follows that the value of λ which corresponds to the dominant asymptotic growth of r_n must be $8 + 4\sqrt{3}$.

The application of this process using the full general asymptotic form has been automated by the `GuessHo1o2` Maple package. In particular, we have used the `Asy()` command to compute the asymptotic growth of r_n :

$$\frac{n^3}{A\mu^n} r_n \sim 1 + \frac{33/2 - 11\sqrt{3}}{n},$$

for some constant A . □

Though we do not need the exact value of the constant A in our applications below, we can estimate the constant A as follows. Using Stirling's approximation we know that $c_n^2 \sim \frac{1}{\pi n^3} 16^n$. This dictates the behavior of $P(z)$ around its dominant singularity, which forces the behavior of $R(z)$ around its dominant singularity. Singularity analysis using methods of Flajolet and Sedgewick [11] then yields

$$r_n \sim \frac{6 - 3\sqrt{3}}{\pi n^3} \mu^n \sim \frac{12}{\mu \pi n^3} \mu^n.$$

While this argument is not rigorous as it uses the estimate for A , the above form is in extremely close numerical agreement with r_n for $n \leq 1000$.

Proposition 5 (Not algebraic) *The generating function $R(z)$ is not algebraic.*

Proof: Theorem D of [10] states that if $l(z)$ is an algebraic function which is analytic at the origin then its Taylor coefficients l_n have an asymptotic equivalent of the form

$$l_n \sim A \beta^n n^s$$

where $A \in \mathbb{R}$ and $s \notin \{-1, -2, -3, \dots\}$. Since r_n is not of this form, in particular it has an n^{-3} term, the generating function $R(z)$ cannot be algebraic. \square

The generating function, or “growth series,” for the actual word metric in Thompson’s group F with respect to the $\{x_0, x_1\}$ generating set (see below), is not known to be algebraic or even D-finite. Burillo [6] and Guba [12] have estimates for the growth but there are significant gaps between the upper and lower bounds which prevent effective asymptotic analysis at this time. Since finding differential equations for generating functions can lead to information about the growth rate of the coefficients, more precise understanding of the growth series for F with respect to the standard generating set (or any finite generating set) would be interesting and potentially quite useful.

In the following sections we regularly use following lemma which follows immediately from the asymptotic formula for r_n .

Lemma 6 (Limits of quotients of r_n) *For any $k \in \mathbb{Z}$*

$$\lim_{n \rightarrow \infty} \frac{r_{n-k}}{r_n} = \mu^{-k}.$$

Proof: From Proposition 4 we have

$$r_{n-k} \sim A \mu^{n-k} (n-k)^3 = A \mu^k n^3 \mu^{-k} \left(\frac{n-k}{n}\right)^3 \sim r_n \mu^{-k}.$$

\square

Finally, we give a formula for r_n . Woodruff ([21] Theorem 2.8) gave the following formula for the number of reduced tree pairs on n carets for $n \geq 2$

$$\sum_{k=1}^{\lceil n/2 \rceil} 2^{n-2k+1} \binom{n-1}{n-2k+1} c_{k-1} \sum_{i=0}^k (-1)^i \binom{k}{i} c_{n-i}.$$

One may readily verify (numerically) that Woodruff’s formula and ours (below) agree for $n \geq 2$. We have been able to show (using Maple) that both expressions satisfy the same third-order linear recurrence, which together with the equality of the first few terms is sufficient to prove that the expressions are, in fact, equal. Unfortunately we have not been able to prove this more directly.

Lemma 7 (Formula for r_n) *The number of reduced tree pairs with n carets in each tree is given by the formula*

$$r_n = \sum_{k=1}^n (-1)^{n-k} \binom{k+1}{n-k} c_k^2$$

Proof: From Proposition 2 we have $R(z) = (1-z)P(z(1-z))$ which expands to

$$\begin{aligned} \sum_{n \geq 1} r_n z^n &= (1-z) \sum_{k \geq 1} c_k^2 z^k (1-z)^k \\ &= \sum_{k \geq 1} c_k^2 z^k (1-z)^{k+1} \\ &= \sum_{k \geq 1} c_k^2 z^k \left(\sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} z^j \right) \\ &= \sum_{k \geq 1} c_k^2 \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} z^{k+j} \end{aligned}$$

Now we look at the coefficient of z^n on both sides. For the right side, as k runs from 1 up, we get exactly one term from the second summation, when $j = n - k$. Thus we get

$$r_n = \sum_{k \geq 1} c_k^2 (-1)^{n-k} \binom{k+1}{n-k}$$

which yields the result, since the binomial term becomes 0 for $k > n$. □

3 Thompson's group F

Richard Thompson's group F is a widely studied group which has provided examples of and counterexamples to a variety of conjectures in group theory. We refer the reader to Cannon, Floyd and Parry [7] for additional background information about this group. Briefly, F is defined using the standard infinite presentation

$$\langle x_0, x_1, \dots \mid x_i^{-1} x_j x_i = x_{j+1}, i < j \rangle.$$

It is clear that x_0 and x_1 are sufficient to generate the entire group, and the standard finite presentation for this group is thus

$$\langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle,$$

where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$. Group elements $w \in F$ can be uniquely represented by a reduced tree pairs as defined in the previous section. Equivalently, each element corresponds uniquely to a piecewise-linear map $\phi_w :$

$[0, 1] \rightarrow [0, 1]$ whose slopes are all powers of two, the coordinates of the breakpoints are dyadic rationals and the slope changes at each breakpoint. As described by Cannon, Floyd and Parry [7], each leaf of the reduced tree pair diagram defining $w \in F$ corresponds uniquely to an interval with dyadic endpoints in the domain or range of the map ϕ_w . The tree pair diagrams for x_0 and x_1 are given in Figure 4. F has a diverse range of subgroups, but notably, it has no free subgroups of rank more than 1.



Figure 4: Tree pair diagrams for the elements x_0 and x_1 respectively.

3.1 Recognizing support and commuting elements

Two elements of F can commute for many reasons, but one of the simplest is that they have disjoint supports. The *support* of an element of F regarded as a homeomorphism of $[0, 1]$ is the closure of the set of points $x \in [0, 1]$ such that $f(x) \neq x$; that is, the set of points which are moved by f . Away from the support of f , the map f will coincide with the identity. From the graph representing a group element as a homeomorphism, it is easy to recognize the complement of the full support of an element by inspecting where it coincides with the identity; x_1 , for example, has support $[1/2, 1]$ as it coincides with the identity for the first half of the interval. It is not as easy to recognize the complete support of an element directly from the reduced tree pair diagrams representing it. Nevertheless, it is possible to tell easily if the support extends to the endpoints 0 and 1 of the interval, by inspecting the locations of first and last leaves of the trees S and T representing an element.

If the distances of the leftmost leaves (the leaves numbered 0) in S and T from their respective roots are both k , then the homeomorphism represented by this pair of trees coincides with the identity at least on the interval $[0, \frac{1}{2^k}]$. If there are, in addition to the leaves numbered 0, a sequence of leaves numbered $1, \dots, m$, each of which have the same distances from the root in both trees, then the homeomorphism will coincide with the identity from 0 to the endpoint of the dyadic interval represented by leaf m . Similarly, near the right endpoint 1, if the distances of the rightmost leaves (those numbered n) in S and T from their respective roots are both l , then the homeomorphism represented coincides with the identity at least on the interval $[1 - \frac{1}{2^l}, 1]$. Again, if there are sequences of leaves numbered from $n - m$ up to n which have the same levels in the trees S and T , then the homeomorphism will coincide with the identity on the corresponding dyadic interval, ending at the right endpoint of 1. Elements that have homeomorphisms

that coincide with the identity for intervals of positive length at both the left and right endpoints are of particular interest as those elements lie in the commutator subgroup of F , as described below.

A simple method for generating pairs of commuting elements of F is to construct them to have disjoint supports. An illustrative example is simply the construction of a subgroup of F isomorphic to $F \times F$, where the four generators used are pictured in Figure 5. The first two generators have support lying in the inter-

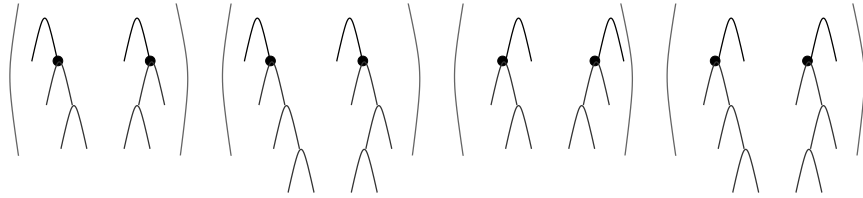


Figure 5: Generators of the standard $F \times F$ subgroup of F .

val $[\frac{1}{2}, 1]$ and generate a copy of F with support in that interval. Similarly, the second two generators have support lying in $[0, \frac{1}{2}]$ and generate a commuting copy of F in that interval. We refer to this example as the *standard* $F \times F$ subgroup of F and will make use of it in later sections.

3.2 More subgroups of F

One important subgroup of F is the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$. Guba and Sapir [13] proved a dichotomy concerning subgroups of F : any subgroup of F is either free abelian or contains a subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. A representative example of a subgroup of F isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ is easily seen to be generated by the elements x_0 and $y = x_1 x_2 x_1^{-2}$. The conjugates of y by x_0 have disjoint support and thus commute.

Other wreath product subgroups of F include $F \wr \mathbb{Z}$ and $H \wr \mathbb{Z}$ for any $H < F$. Generators for $H \wr \mathbb{Z}$ are obtained as follows. Let $\{h_1, \dots, h_k\}$ be a generating set for H where $h_i = (T'_i, S'_i)$. Let T be the tree with two right carets, and leaves numbered 1, 2, 3. Define generators $k_i = (T_i, S_i)$ for $H \wr \mathbb{Z}$ by letting T_i be the tree T with T'_i attached to leaf 2, and S_i be the tree T with S'_i attached to leaf 2. Then $\{k_i\} \cup \{x_0\}$ forms a generating set for $H \wr \mathbb{Z}$.

The group F contains a multitude of subgroups isomorphic to F itself; any two distinct generators from the infinite generating set for F will generate such a subgroup. More generally, Cannon, Floyd and Parry [7] describe a simple arithmetic condition to guarantee that a set of analytic functions of the interval with the appropriate properties generates a subgroup of F which is isomorphic to F . A combinatorial description of their construction of proper subgroups of F isomorphic to F is as follows.

Given a finite string of zeros and ones, we construct a rooted binary tree by attaching to a root caret a left child if the first letter of the string is zero, and a right child otherwise. Continue in this way, adding a child to the left leaf of the previous caret if the next letter in the string is a zero, to the right leaf of the previous caret otherwise. For the final letter in the string, do not add a caret, but mark a distinguished leaf v in the tree in the same manner, that is, mark the left leaf of the last caret added if the final letter is a zero, and the right leaf otherwise. Let T be a tree constructed in this way, and form two tree pair diagrams h_0 and h_1 based on T as follows. Denote $x_0 = (T_{x_0}, S_{x_0})$ and $x_1 = (T_{x_1}, S_{x_1})$. Draw four copies of the tree T , numbered T_1 through T_4 . To the marked vertex v in T_1 attach the tree T_{x_0} and to the marked vertex v in T_2 attach S_{x_0} , forming the tree pair diagram representing h_0 . Do the same thing with T_3, T_4, T_{x_1} and S_{x_1} respectively to form h_1 . Then h_0 and h_1 generate a subgroup of F isomorphic to F , which is called a *clone subgroup* in [9] and consists of elements whose support lies in the dyadic interval determined by the vertex v . Subgroups of this form are easily seen to be quasi-isometrically embedded. This geometric idea is easily extended to construct subgroups of F isomorphic to F^n .

Another family of important subgroups of F are the subgroups isomorphic to \mathbb{Z}^n , which will play a role in the proofs in Sections 4 and 5. We let T be the tree with $n - 1$ right carets, and n leaves, and (A_i, B_i) for $i = 1, 2, \dots, n$ reduced pairs of trees so that for each i , A_i and B_i have the same number of carets. We construct generators $h_i = (C_i, D_i)$ of \mathbb{Z}^n as follows. We let C_i be the tree T with A_i attached to leaf i , and D_i the tree T with B_i attached to leaf i , as shown in Figure 6. We

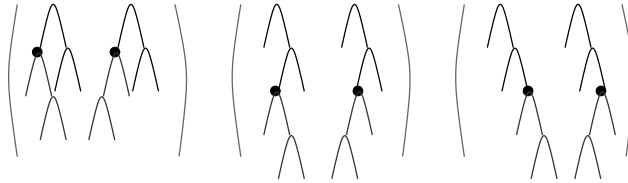


Figure 6: Three tree pairs h_1, h_2, h_3 used to generate \mathbb{Z}^3 . We have used the tree pair diagram for x_0 as each pair (A_i, B_i) . Note that the first pair can be reduced to a tree pair diagram containing only three carets by deleting the rightmost exposed caret.

reduce the pair (C_i, D_i) if necessary. It is easy to check by multiplying the tree pair diagrams that $h_i h_j = h_j h_i$ for $i, j = 1, 2, \dots, n$ and thus these elements generate a subgroup of F isomorphic to \mathbb{Z}^n . Burillo [5] exhibits a different family of subgroups of F isomorphic to \mathbb{Z}^n using the generators $\{x_0 x_1^{-1}, x_2 x_3^{-1}, x_4 x_5^{-1}, \dots, x_{2n-2} x_{2n-1}^{-1}\}$ which he shows are quasi-isometrically embedded. In fact, Burillo proves that any infinite cyclic subgroup of F is undistorted; that is, that the cyclic subgroups are quasi-isometrically embedded.

3.3 The commutator subgroup of F

In the proofs in Sections 4 and 5 below, we use both algebraic and geometric descriptions of the commutator subgroup $[F, F]$. This subgroup of F has two equivalent descriptions:

- The commutator subgroup of F consists of all elements in F which coincide with the identity map (and thus have slope 1) in neighborhoods both of 0 and of 1. This is proven as Theorem 4.1 of [7].
- The commutator subgroup of F is exactly the kernel of the map $\varphi : F \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ given by taking the exponent sum of all instances of x_0 in a word representing $w \in F$ as the first coordinate, and the exponent sum of all instances of x_1 as the second coordinate.

The exponent-sum homomorphism φ is closely tied to another natural homomorphism ϕ from F to $\mathbb{Z} \oplus \mathbb{Z}$. The “slope at the endpoints” homomorphism ϕ for an element $f \in F$ takes the first coordinate of the image to be the logarithm base 2 of the slope of f at the left endpoint 0 of the unit interval and the second coordinate to be the logarithm base 2 of the slope at the right endpoint 1. The images of the generators under the slope-at-the-endpoints homomorphism ϕ are $\phi(x_0) = (1, -1)$ and $\phi(x_1) = (0, -1)$ and ϕ and φ have the same kernel.

It is not hard to see that the first description above has the following geometric interpretation in terms of tree pair diagrams. An element of the commutator subgroup will have slope 1 at the left and right endpoints and coincide with the identity on intervals of the form $[0, b_0]$ and $[b_1, 1]$ where b_0 and b_1 are, respectively, the first and last points of non-differentiability in $[0, 1]$. These points must lie on the line $y = x$, and the element is represented by tree pair diagrams in which the first leaves (numbered 0) in each tree lie at the same *level* or distance from the root, and the same must be true of the last leaf in each of the trees. Thus, elements of the commutator subgroup are exactly those which have a reduced tree pair diagram (S, T) where the leaves numbered zero are at the same level in both S and T and the last leaves are also at the same level in both S and T . For example, if (A, B) is any reduced n -caret tree pair, then the $(n+2)$ -caret tree pair in Figure 7 is also reduced and represents an element in $[F, F]$.

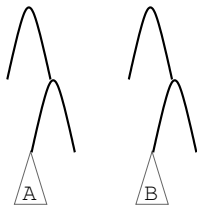


Figure 7: Constructing a tree pair representing a group element which lies in the commutator subgroup $[F, F]$.

We refer the reader to [7] for a proof that the commutator of F is a simple group, and that $F/[F, F] \cong \mathbb{Z} \oplus \mathbb{Z}$.

In our arguments below we will be interested in isomorphism classes of subgroups of F . It will sometimes be necessary to assume that a particular finitely generated subgroup of F is not contained in the commutator subgroup $[F, F]$. We now show that within the isomorphism class of any subgroup H of F , it is always possible to pick a representative not contained in $[F, F]$. The proof of this lemma follows the proof of Lemma 4.4 of [7].

Lemma 8 (Finding subgroups outside the commutator) *Let H be a finitely generated subgroup of F . Then there is a subgroup H' of F which is isomorphic to H and not contained in the commutator subgroup.*

Proof: If H is not contained in the commutator subgroup $[F, F]$, then take $H' = H$. Otherwise, let H be generated by h_1, h_2, \dots, h_k where each $h_i \in [F, F]$. Then each h_i has an associated ordered pair (a_i, b_i) where a_i is x -coordinate of the first point of non-differentiability of h_i as a homeomorphism of $[0, 1]$ (necessarily at a_i the slope will change from 1 to something which is not 1.) Similarly, we let b_i be the x -coordinate of the final point of non-differentiability of h_i . We let $a = \min\{a_i\}$ and $b = \max\{b_i\}$. By the choice of a and b , all $h \in H$ have support in $[a, b]$.

Following the proof of Lemma 4.4 of [7], we let $\phi : [a, b] \rightarrow [0, b - a]$ be defined by $\phi(x) = x - a$. We use ϕ to define a map on $h \in H$ by $h \mapsto \phi h \phi^{-1}$, assuming that $\phi h \phi^{-1}$ acts as the identity for $x \in (b - a, 1]$. It is clear from the definition of ϕ that the breakpoints of $\phi h \phi^{-1}$ are again dyadic rationals, and the slopes are again powers of two. Since ϕ is an isomorphism, we know that $H \cong \langle \phi h_i \phi^{-1} \rangle$. But this subgroup cannot be in the commutator, since at least one element, the one which had its minimal breakpoint at $x = a$, now has slope not equal to 1 at $x = 0$, and thus is not in the commutator subgroup. \square

In the proofs in Sections 4 and 5 below, we often want to make a more specific choice of representative subgroup from an isomorphism class of a particular subgroup of F , as follows.

Let $E_i(w)$ for $i = 0, 1$ denote the exponent sum of all instances of x_i in a word w in x_0 and x_1 .

Lemma 9 *Let $H = \langle h'_1, h'_2, \dots, h'_k \rangle$ be a finitely generated subgroup of F . Then there is a subgroup $H' = \langle h_1, h_2, \dots, h_k \rangle$ isomorphic to H so that $E_0(h_1) \neq 0$ and $E_0(h_j) = 0$ for $j = 2, 3, \dots, k$.*

Proof: By Lemma 8, we assume without loss of generality that H is not contained in the commutator subgroup $[F, F]$. By replacing some generators with their inverses, we may assume that $E_0(h'_i) \geq 0$ for all i , and that $E_0(h'_1)$ is minimal among those $E_0(h'_i)$ which are positive. For these h'_i with $i > 1$, we replace h'_i by $h'_i h_1^{-d_i}$ where d_i is chosen so that $E_0(h'_i h_1^{-d_i})$ is as small as possible while non-negative. Repeating this process yields a generating set for a subgroup isomorphic to H with one element having exponent sum on all instances of x_0 equal to zero.

We can repeat this process with the remaining generators, possibly reindexing at each step, until a generating set with the desired property is obtained. \square

4 Subgroup spectrum with respect to the sum stratification

We now introduce the first of two stratifications of the set of k generator subgroups of Thompson's group F . We view group elements as non-empty reduced tree pairs and denote by X_k the set of unordered k -tuples of non-empty reduced tree pairs $t_i = (T_1^i, T_2^i)$ for $i = 1, \dots, k$. We denote the number of carets in T_1^i by $|t_i|$. We define the sphere of radius n in X_k as the set of k -tuples having a total of n carets in the k tree pair diagrams in the tuple:

$$\text{Sph}_k^{\text{sum}}(n) = \left\{ (t_1, \dots, t_k) \mid \sum_{i=1}^k |t_i| = n \right\}$$

which induces a stratification on X_k that we will call the *sum stratification*. Note that since each tree in a tree-pair has the same number of carets, we only count (without loss of generality) the carets in the left tree. For example, the triple of tree pairs in Figure 6, once h_1 is reduced, lies in $\text{Sph}_3^{\text{sum}}(11)$.

Recall from Section 1 that the density of a set T of k -tuples of reduced tree pairs is given by

$$\lim_{n \rightarrow \infty} \frac{|T \cap \text{Sph}_k^{\text{sum}}(n)|}{|\text{Sph}_k^{\text{sum}}(n)|}$$

with respect to this stratification. Let H is a subgroup of F , and T_H the set of k -tuples whose coordinates generate a subgroup of F that is isomorphic to H . Recall that H is *visible* if T_H has positive density, and the k -*spectrum* $\text{Spec}_k^{\text{sum}}(F)$ is the set of visible subgroups with respect to the sum stratification of X_k . In this section we explicitly compute these subgroup spectra. We find that any isomorphism class of nontrivial subgroup H of F which can be generated by m generators is an element of in $\text{Spec}_k^{\text{sum}}(F)$ for all $k \geq m$ (Theorem 11). We conclude that this stratification does not distinguish any particular subgroups through the subgroup spectrum, in contrast to the results we will describe in Section 5 when the max stratification is used.

We begin by determining upper and lower bounds on the size of the sphere of radius n in this stratification. Since our k -tuples are unordered, we may assume that they are arranged from largest to smallest.

Lemma 10 (Size of $\text{Sph}_k^{\text{sum}}(n)$) *For $k \geq 1$ and $n \geq k$, the size of the sphere of radius n with respect to the sum stratification satisfies the following bounds:*

$$r_{n-k+1} \leq |\text{Sph}_k^{\text{sum}}(n)| \leq r_{n+k-1}.$$

Proof: For the lower bound, $\text{Sph}_k^{\text{sum}}(n)$ contains all k -tuples where the first pair has $n - k + 1$ carets and the remaining $(k - 1)$ pairs consist of two single carets. There are r_{n-k+1} ways to choose this first pair, which yields the lower bound.

For the upper bound, we consider the set of all r_{n+k-1} reduced tree pairs with $n+k-1$ carets in each tree. A (small) subset of these correspond to the k -tuples of $\text{Sph}_k^{\text{sum}}(n)$ as follows. Take the subset of these tree pairs where each tree contains at least $k - 1$ right carets, as in Figure 8, where leaf i for $0 \leq i \leq n - 1$ has a possibly empty left subtree labeled A_i in T_- and B_i in T_+ . Let A_n and B_n respectively denote the right subtrees attached to leaf n in T_- and T_+ . The sum of the number of carets in the A_i must equal n .

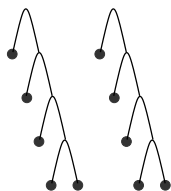


Figure 8: A tree-pair consisting of $k - 1$ right caret pairs (with $k = 5$).

When the number of carets in A_i equals the number of carets in B_i for all i , this pair of trees can be associated to an (ordered) k -tuple of tree pairs with a total of n carets. Amongst these we can find every unordered k -tuple in $\text{Sph}_k^{\text{sum}}(n)$. So this is a gross overcount which suffices to prove the lemma. \square

Theorem 11 (All subgroup types are visible with respect to sum) *Let $H = \langle h_1, h_2, \dots, h_m \rangle$ be a nontrivial subgroup of F . Then $H \in \text{Spec}_k^{\text{sum}}(F)$ for all $k \geq m$.*

We use the notation from Section 3.3 to represent the exponent sum of different generators in a word in x_0 and x_1 . Let $E_i(w)$ for $i = 0, 1$ denote the exponent sum of x_i in a group element given by a word w .

Proof: Applying Lemmas 8 and 9, we may assume that H is a representative of its isomorphism class which is not contained in the commutator subgroup $[F, F]$ and such that $E_0(h_1) \neq 0$ but $E_0(h_i) = 0$ for $i > 1$.

We now construct a set of k generators $l_i = (T_i, S_i)$ for $i = 1, 2, \dots, k$ using a total of n carets which we will show generate a subgroup of F isomorphic to H . We let $h_i = (T'_i, S'_i)$ as a tree pair diagram, and $s = \sum_{i=1}^m |h_i|$. We let (A, B) be a reduced pair of trees with $n - (s + k)$ carets in each tree. We take n to be larger than $s + k$ in order to construct (A, B) in this way. We define l_1 by taking T_1 to be the tree with a root caret whose left subtree is T'_1 and whose right subtree is A . Similarly, we let S_1 be the tree with a root caret whose left subtree is S'_1 and whose right subtree is B .

For $2 \leq i \leq m$, we let T_i be the tree consisting of a root caret whose left subtree is T'_i and whose right subtree is empty. We let S_i be the tree consisting of a root

caret whose left subtree is S'_i and whose right subtree is empty. For $m+1 \leq i \leq k$, we let l_i be the identity represented by a pair of trees each containing a single caret.

We note that by construction, all tree pair diagrams constructed in this way are reduced. We have k root carets (counting one caret per pair), to which we attached s carets for all the (T'_i, S'_i) pairs, $n - (s + k)$ carets for the (A, B) pair. This totals to $k + s + n - (s + k) = n$ ensuring that the k -tuple constructed lies in the desired sphere.

It is clear that $\langle l_1, l_2, \dots, l_k \rangle$ generate a subgroup of $H \times \mathbb{Z}$, where the isomorphic copy of H lies in the first factor of the standard $F \times F$ subgroup of F and where we take (A, B) to be the generator of the \mathbb{Z} factor which lies in the second factor of the standard $F \times F$ subgroup. We now claim that $\langle l_1, l_2, \dots, l_k \rangle \cong H$. We use the coordinates (h, t^a) on $H \times \mathbb{Z}$, where $h \in H$ and $t = (A, B)$. We define a homomorphism from $H \times \mathbb{Z}$ to H by taking the first coordinate of (h, t^a) . When restricted to $\langle l_1, l_2, \dots, l_k \rangle$, this map is onto by construction.

To show this projection map is injective, we suppose that $(1, t^a)$ lies in the kernel, for $a \neq 0$. Thus $\langle l_1, l_2, \dots, l_k \rangle \subset H \times \mathbb{Z}$ has a relator ρ which, when projected to H , yields a relator r of H , and when considered as a word in $\langle l_1, l_2, \dots, l_k \rangle$, has a second coordinate not equal to the identity. But any relator r of H , when each h_1 is written as a word in x_0 and x_1 , satisfies $E_0(r) = 0$. Since the only generator of H with $E_0(h_i) \neq 0$ is h_1 , we see that r must have the same number of h_1 and h_1^{-1} terms in it. Thus ρ must have the same number of $l_1 = (h_1, t)$ and l_1^{-1} terms. Since l_1 is the only generator of $\langle l_1, l_2, \dots, l_k \rangle$ which can change the \mathbb{Z} coordinate of a product, having equal numbers of l_1 and l_1^{-1} terms in our relator ρ implies that when the H coordinate is the identity, the second coordinate must be t^0 . Thus projection to the first factor is an isomorphism when restricted to $\langle l_1, l_2, \dots, l_k \rangle$, and we conclude that this group is isomorphic to H .

We now show that the set of k -tuples of tree pair diagrams constructed in this way is visible in $\text{Spec}_k^{\text{sum}}(F)$. There are $r_{n-(s+k)}$ ways to choose the pair (A, B) , which had $n - (s + k)$ carets, and which determined the l_1 generator in this construction. Thus we see that

$$\lim_{n \rightarrow \infty} \frac{r_{n-(s+k)}}{|\text{Sph}_k^{\text{sum}}(n)|} \geq \lim_{n \rightarrow \infty} \frac{r_{n-(s+k)}}{r_{n+k-1}} = \mu^{-(s-1+2k)} > 0$$

using Lemmas 10 and 6. □

The probabilistic motivation for the definition of a visible subgroup H is that a set of k randomly selected reduced pairs of trees will generate a subgroup isomorphic to H with nonzero probability. In the preceding proof, we were able to show that any given m -generator subgroup is visible in $\text{Spec}_k^{\text{sum}}(F)$ using a k -tuple of pairs of trees consisting of one “large” tree pair diagram, $m - 1$ “small” tree pair diagrams, and finally $k - m$ “tiny” tree pair diagrams representing the identity.

Given a subgroup H of F , the estimate given above on a lower bound for the density of the isomorphism class of H is small but positive. It follows from the proof of Theorem 11 that we obtain larger estimates of this lower bound when the original subgroup H is generated by elements with small tree pair diagrams.

For example, the asymptotic density of the isomorphism class of the subgroup \mathbb{Z} is at least $\mu^{-5} \approx \frac{1}{750000}$ in the set of all 2-generator subgroups, since $k = 2$ and \mathbb{Z} can be generated by x_0 which has size 2. For other nontrivial subgroups, the construction in this proof will require more carets and the lower bounds we obtain will be even smaller, but always positive.

5 Subgroup spectrum with respect to the max stratification

We now begin to compute the subgroup spectrum with respect to a different stratification, the “max” stratification, of the set of all k -generator subgroups of F . We again let X_k be the set of unordered k -tuples of reduced pairs of trees, and define the sphere of size n to be the collection of k -tuples in which the maximum size of any component is n :

$$\text{Sph}_k^{\max}(n) = \left\{ (t_1, \dots, t_k) \mid \max_{i \in \{1, 2, \dots, k\}} \{|t_i|\} = n \right\}$$

For example, the triple of tree pairs in Figure 6 (once h_1 is reduced) lies in $\text{Sph}_3^{\max}(4)$. Defining spheres in this way induces the desired stratification of X_k .

We define the density of a subset $T \subseteq X_n$ with respect to the max stratification by

$$\lim_{n \rightarrow \infty} \frac{|T \cap \text{Sph}_k^{\max}(n)|}{|\text{Sph}_k^{\max}(n)|}$$

and $\text{Spec}_k^{\max}(F)$ to be the set of visible isomorphism classes of subgroups of F with respect to the max stratification. As noted at the end of Section 4, the sum stratification is biased towards k -tuples of tree pair diagrams which contain multiple copies of the identity and other “small” pairs of trees having few carets. Using the maximum number of carets in a tree pair diagram to determine size seems to yield a more natural stratification.

We find strikingly different results when we compute $\text{Spec}_k^{\max}(F)$ as compared to $\text{Spec}_k^{\text{sum}}(F)$. For example, we show that \mathbb{Z} lies in $\text{Spec}_1^{\max}(F)$ but not in $\text{Spec}_k^{\max}(F)$ for larger values of k .

As in Section 4, we must first obtain bounds on the size of the sphere of radius n with respect to the max stratification. We will use these bounds in the proofs below. We begin with a lemma about sums of r_n .

Lemma 12 (Sums of r_n) For $n \geq 2$, $\sum_{i=1}^{n-1} r_i \leq r_n$.

Proof: Since $r_1 = 1 < r_2 = 2$ the statement holds for $n = 2$. We assume for induction the statement is true for $k \geq 2$. Then

$$\sum_{i=1}^k r_i = \sum_{i=1}^{k-1} r_i + r_k \leq 2r_k$$

by inductive assumption. We consider the set of reduced tree pairs with $k + 1$ carets in each tree, where either the right child of each root is empty, or the left child of each root is empty. In each case there are r_k ways to arrange the k carets on the nonempty leaf, and these tree pairs form disjoint subsets of the set of all reduced pairs of trees with $k + 1$ carets. Thus $2r_k \leq r_{k+1}$ which completes the proof. \square

Lemma 13 (Size of $\text{Sph}_k^{\max}(n)$) For $k \geq 1$ and $n \geq k$,

$$\frac{1}{k!}(r_n)^k \leq |\text{Sph}_k^{\max}(n)| \leq k(r_n)^k$$

Proof: For the lower bound, there are $(r_n)^k$ ordered k -tuples of reduced tree pairs where every pair has n carets. Since $\text{Sph}_k^{\max}(n)$ consists of unordered tuples then dividing this by $k!$ gives a lower bound.

For the upper bound, at least one of the k tree pairs must have n carets. For $1 \leq i \leq k$ suppose that i tree pairs have exactly n carets, and the remaining $k - i$ tree pairs have strictly less than n carets. There are at most $(r_n)^i$ ordered i -tuples of n -caret tree pairs, and so at most this many unordered i -tuples, and at most $\left(\sum_{j=1}^{n-1} r_j\right)^{k-i}$ ordered $(k - i)$ -tuples of tree pairs with at most $n - 1$ carets each, and so at most this many unordered $(k - i)$ -tuples.

So for each i the number of unordered k -tuples of tree pairs where i pairs have n carets and $k - i$ pairs have less than n carets is at most

$$(r_n)^i \binom{n-1}{\sum_{j=1}^{k-i} r_j} \leq (r_n)^i (r_n)^{k-i} = (r_n)^k$$

by Lemma 12. Since our k -tuples of tree pairs are unordered, without loss of generality we can list the ones containing n carets first.

Thus for the total number of k -tuples, we have at most

$$\sum_{i=1}^k (r_n)^k = k(r_n)^k.$$

\square

We begin by showing that \mathbb{Z}^k is present in $\text{Spec}_k^{\max}(F)$ for all $k \geq 1$. We prove that $\mathbb{Z} \notin \text{Spec}_k^{\max}(F)$ for $k > 1$, and conjecture that \mathbb{Z}^m is not visible in $\text{Spec}_k^{\max}(F)$ for $k > m$. In the proof below, we construct a particular collection of subgroups of F isomorphic to \mathbb{Z}^k , all of whose generators have a common form, and show that this collection of subgroups is visible. Presumably, the actual density of the isomorphism class of subgroups of F isomorphic to \mathbb{Z}^k is considerably larger.

Lemma 14 ($\text{Spec}_k^{\max}(F)$ is nonempty) $\mathbb{Z}^k \in \text{Spec}_k^{\max}(F)$ for all $k \geq 1$.

Proof: We let T be the tree consisting of a string of $k - 1$ right carets. We construct a set of k pairs of trees which generate a subgroup of F isomorphic to \mathbb{Z}^k as described in Section 3.2.

We let (A_i, B_i) be a reduced pair of trees each with $n - (k - 1)$ carets for $i = 1, 2, \dots, k$. We let h_i be the pair of trees obtained by taking the pair (T, T) and attaching A_i to the i -th leaf of the first copy of T , and B_i to the i -th leaf of the second copy of T . We reduce the tree pair generated in this way (which will be necessary for $i = 1, \dots, k - 2$) to obtain the reduced representative for h_i , which we again denote h_i . We note that h_k will have n carets in each tree in its pair, so this tuple does lie in the proper sphere of the stratification. As discussed above, the set $\{h_1, h_2, \dots, h_k\}$ will generate a subgroup of F isomorphic to \mathbb{Z}^k .

We compute the density of the set of k -tuples of pairs of trees constructed in this way to be at least:

$$\lim_{n \rightarrow \infty} \frac{(r_{n-k+1})^k}{k(r_n)^k} = \frac{1}{k} \mu^{-k^2+k} > 0$$

using Lemma 6 and the upper bound from Lemma 13. Thus \mathbb{Z}^k is visible in $\text{Spec}_k^{\max}(F)$. \square

For example, this shows that the density of \mathbb{Z}^2 in the set of 2-generator subgroups is at least $\frac{1}{2} \mu^{-2} \approx \frac{1}{500}$.

We now show that a subgroup H of F cannot appear in $\text{Spec}_k^{\max}(F)$ for values of k smaller than the rank of the abelianization H_{ab} .

Lemma 15 (Abelianization) *We let H be a subgroup of F , and let n be the rank of the abelianization H_{ab} of H . Then $H \notin \text{Spec}_k^{\max}(F)$ for $k < n$.*

Proof: Since the rank of H_{ab} is n , we know that H cannot be generated with fewer than n elements. Thus H cannot be visible in $\text{Spec}_k^{\max}(F)$ for $k < n$. \square

Aside from straightforward obstructions like the group rank and the rank of the abelianization, it is not clear what determines the presence of an isomorphism class of subgroup in a given spectrum. In general, it is difficult to show that an isomorphism class of subgroup is not present in a particular spectrum. This is because it can be difficult to systematically describe all possible ways of generating a subgroup isomorphic to a given one. However, in the case of \mathbb{Z} , we can show that \mathbb{Z} is not present in the k -spectrum for $k \geq 2$. This highlights a major difference between the composition of $\text{Spec}_k^{\text{sum}}(F)$ and $\text{Spec}_k^{\max}(F)$, since \mathbb{Z} appears in all spectra with respect to the sum stratification. As a subgroup of F with a single generator is either the identity or infinite cyclic, it follows that $\text{Spec}_1^{\max}(F)$ contains only \mathbb{Z} .

5.1 Showing that the sum and max spectra are different

The goal of this section is to prove the following theorem.

Theorem 16 (\mathbb{Z} not visible) *With respect to the max stratification, the spectrum $\text{Spec}_1^{\max}(F) = \{\mathbb{Z}\}$ and for any $k \geq 2$, we have that $\mathbb{Z} \notin \text{Spec}_k^{\max}(F)$.*

The essence of this proof is that if k group elements generate a subgroup isomorphic to \mathbb{Z} , then they must all be powers of a common element. Thus we make precise the notion that counting the number of k -tuples which generate a subgroup isomorphic to \mathbb{Z} is, up to a polynomial factor, the same problem as choosing a single reduced tree pair as the generator of the subgroup.

We begin with some elementary lemmas relating the slope of the first non-identity linear piece of a nontrivial element $f \in F$ and the number of carets in the reduced tree pair diagram representing that element.

Lemma 17 *If $f \in F$ has a breakpoint with coordinates $(\frac{x}{2^m}, \frac{y}{2^r})$ where x, y are odd integers, then the reduced tree pair diagram for f has at least $\max(m, r)$ carets in each tree.*

Proof: In each tree within the tree pair diagram, carets at level k correspond to points in $[0, 1]$ with denominator 2^k . The lemma follows. \square

Lemma 18 *Suppose that the first non-identity linear piece of $f \in F$ has slope 2^r for $r \neq 0$. Then the reduced tree pair diagram for f has at least $|r|$ carets.*

Proof: Suppose that the first non-identity linear piece of f with slope 2^r has endpoints with coordinates $(\frac{a}{2^s}, \frac{a}{2^s})$ and $(\frac{b}{2^x}, \frac{c}{2^y})$ where a is either zero or an odd integer and b and c are odd integers. We easily see that

$$2^r = \frac{\frac{c}{2^y} - \frac{a}{2^s}}{\frac{b}{2^x} - \frac{a}{2^s}}$$

Factoring out the highest power of 2 possible from the denominator and the numerator of this fraction, and letting $m_1 = \min(s, x)$ and $m_2 = \min(s, y)$, we obtain

$$\frac{\frac{1}{2^{m_1}} \left(\frac{c}{2^{y-m_1}} - \frac{a}{2^{s-m_1}} \right)}{\frac{1}{2^{m_2}} \left(\frac{c}{2^{y-m_2}} - \frac{a}{2^{s-m_2}} \right)} = \frac{2^{m_2}}{2^{m_1}} A = 2^r$$

where no additional powers of 2 can be factored out of the A part of this expression. Thus we see that one of m_1, m_2 must be at least $|r|$, and thus it follows from Lemma 17 that the tree pair diagram for f has at least $|r|$ carets. \square

We will use the coordinates of the first breakpoint to vastly overcount the number of pairs of tree pair diagrams that we are considering. However, even this vast overcounting will work for the final argument. We also need the following elementary lemma that follows from Lemma 18.

Lemma 19 *Let $f \in F$ have a reduced tree pair diagram with n carets. Then f does not have an m -th root for $m > n$.*

Proof: Suppose that f has an m -th root h for some $m > n$. If f is the identity on $[0, \epsilon]$, then any root or power of f will be the identity on this interval as well. Let the slope of the first non-identity linear piece of h be 2^r for $r \neq 0$, and have left endpoint $\alpha = (\frac{a}{2^s}, \frac{a}{2^s})$ for $a = 0$ or a odd. Then the slope of h^m near α is 2^{rm} and $|rm| > n$ since $m > n$. Thus it follows from Lemma 18 that the tree pair diagram for $f = h^m$ has more than n carets, a contradiction. \square

The proof of Theorem 16 is divided into the following three lemmas. Note that Lemma 21 is a special case of Lemma 22, but is included to illustrate the ideas involved.

Lemma 20 *With respect to the max stratification, the spectrum $\text{Spec}_1^{\max}(F) = \{\mathbb{Z}\}$.*

Proof: It follows from Lemma 14 that $\mathbb{Z} \in \text{Spec}_1^{\max}(F)$. The only other possible candidate for a subgroup isomorphism class in $\text{Spec}_1^{\max}(F)$ is that of the identity, and the only reduced tree pair diagram representing the identity is of size 1. The number of reduced tree pairs representing the identity is 0 for size $n > 1$, and thus the density of the isomorphism class of the identity subgroup when $k = 1$ is 0. We conclude that $\text{Spec}_1^{\max}(F) = \{\mathbb{Z}\}$. \square

To see that $\mathbb{Z} \notin \text{Spec}_k^{\max}(F)$ for any $k \geq 2$, we begin by overcounting the number of k -tuples of elements which can generate a subgroup isomorphic to \mathbb{Z} .

Lemma 21 *For a fixed $n > 1$, there are at most $(2n + 1)(n + 1)r_n$ distinct unordered pairs of elements $f, g \in F$ so that*

1. *the number of carets in each tree pair diagram is at most n ,*
2. *the number of carets in at least one tree pair diagram is equal to n , and*
3. *$\langle f, g \rangle \cong \mathbb{Z}$.*

Proof: Since $\langle f, g \rangle \cong \mathbb{Z}$ we know that f and g are powers of a common element. Note that this includes the case where this common element is either f or g . By assumption, one of f and g has n carets in its tree pair diagram; without loss of generality we assume that it is f . Thus there are r_n choices for f .

From Lemma 19 we know that f may have m -th roots for $0 \leq m \leq n$. It follows from [4], Theorem 4.15 that if $f \in F$ has an m -th root, then that root is unique. Denote the possible roots of f by q_0, q_1, \dots, q_j for $0 \leq j \leq n$. Note that we are including f itself as the 0-th root. We also know that g must be a power of one of those (at most) $n + 1$ possible roots, so there is an i so that $g = q_i^a$ for some integer a . Since g has at most n carets in its tree pair diagram, it follows from Lemma 18 that this exponent a is at most n in absolute value. To see this, let (x, y) be the first breakpoint of f so that the slope of the linear piece following (x, y) is 2^α for $\alpha \neq 0$. Then it is easy to see that the slope to the right of (x, y) in f^k is $2^{k\alpha}$ and the statement then follows from Lemma 18. Thus there are $2n + 1$

choices for the exponent a so that $h^a = g$ since $|a| \leq n$. In total, the number of ways we can construct a pair of this form is at most $(2n+1)(n+1)r_n$. Again, this count includes many pairs of elements that do not satisfy the requirements of the proposition, but all elements that do satisfy those conditions are counted in this argument. \square

Lemma 22 *For a fixed $n > 1$, there are at most $(2n+1)^{k-1}(n+1)r_n$ distinct unordered k -tuples of elements $f_1, f_2, \dots, f_k \in F$ so that*

1. *the number of carets in each tree pair diagram is at most n ,*
2. *the number of carets in at least one tree pair diagram is equal to n , and*
3. *$\langle f_1, f_2, \dots, f_k \rangle \cong \mathbb{Z}$.*

Proof: The argument follows the proof of Lemma 21. There must be some element h which generates this copy of \mathbb{Z} , that is, all f_i are powers of this element h . Suppose without loss of generality that f_1 has n carets in its tree pair diagram. Then f_1 may have m -th roots for $0 \leq m \leq n$, which we denote q_0, q_1, \dots, q_j for $0 \leq j \leq n$. The same reasoning shows that there is some l with $0 \leq l \leq n$ so that for each i we must have $f_i = q_i^{e_i}$, with $|e_j| \leq n$, where the latter inequality follows from Lemma 18. We then see that the number of such k -tuples satisfying the conditions of the lemma is at most $(2n+1)^{k-1}(n+1)r_n$. \square

We now finish the proof of Theorem 16.

Proof of Theorem 16. For $k \geq 2$ we see that the density of k -tuples of pairs of trees which generate a subgroup isomorphic to \mathbb{Z} is at most

$$\lim_{n \rightarrow \infty} \frac{(2n+1)^{k+1}(n+1)r_n}{\frac{1}{k!}(r_n)^k} = 0$$

using the bound on the size of the n -sphere in the max stratification given in Lemma 13 as well as the upper bounds proven in Lemmas 21 and 22. The first statement in the theorem follows from Lemma 20 and the second from the above limit. \square

We note that this approach does not appear to generalize to show that \mathbb{Z}^m is not visible in $\text{Spec}_k^{\max}(F)$ for $k > m$, as it is difficult to recognize when a collection of tree pair diagrams generates a subgroup isomorphic to \mathbb{Z}^m for $m \geq 2$.

5.2 Further results for the max spectrum

Apart from $\text{Spec}_1^{\max}(F)$, it seems quite difficult to compute the complete list of subgroups which appear in $\text{Spec}_k^{\max}(F)$. Indeed, ignoring any consideration of densities, a complete list of even the 2-generated subgroups of F is not known (see [8] Problem 2.4). For $k = 2$ we can say the following.

Proposition 23 (2-spectrum of F) *Let $H = \langle h_1, h_2 \rangle$ be a subgroup of F . Then either H or $H \times \mathbb{Z}$ lies in $\text{Spec}_2^{\max}(F)$. If $H_{ab} \cong \mathbb{Z} \oplus \mathbb{Z}$, then $H \in \text{Spec}_2^{\max}(F)$, otherwise $H \times \mathbb{Z} \in \text{Spec}_2^{\max}(F)$.*

Proof: We may assume, quoting Lemmas 8 and 9 that if $H = \langle h_1, h_2 \rangle$ that

- $h_1 \notin [F, F]$
- when h_1 is expressed as a word in x_0 and x_1 , the exponent sum of all the instances of x_0 is not equal to 0, and
- when h_2 is expressed as a word in x_0 and x_1 , the exponent sum of all the instances of x_0 is equal to 0.

As tree pair diagrams, we use the notation $h_i = (S_i, T_i)$.

We create a new set of generators $k_1 = (X_1, Y_1)$ and $k_2 = (X_2, Y_2)$ for a two generator subgroup of F as follows. We let T be the tree consisting entirely of two right carets, whose leaves are numbered 1, 2 and 3, and let (A, B) and (C, D) be arbitrary reduced pairs of trees so that (A, B) has $n - N(h_1) - 2$ carets in each tree and (C, D) has $n - N(h_2) - 2$ carets in each tree. We construct X_1 by attaching S_1 to leaf 1 of T and A to leaf 2 of T . We construct Y_1 by attaching T_1 to leaf 1 of T and B to leaf 2 of T . We construct X_2 by attaching S_2 to leaf 1 of T and C to leaf 3 of T . We construct Y_2 by attaching T_2 to leaf 1 of T and D to leaf 3 of T , as in Figure 9. Note that each tree has size n , and we assume without loss of generality that $n > \max\{N(h_i)\} + 4$ so that the trees A, B, C and D each have at least two carets.

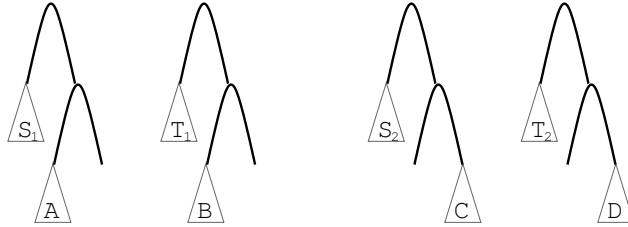


Figure 9: Constructing the tree pairs k_1, k_2 which generate a subgroup of $H \times \mathbb{Z}$.

One may easily verify that k_1 and k_2 generate a subgroup of the standard $F \times F$ subgroup in which the subgroup you obtain on the first factor of F is simply H . Also, $t = (A, B)$ and $s = (C, D)$ each generate a copy of \mathbb{Z} in the second factor of $F \times F$ provided that neither tree pair diagram represents the identity. Let $K \cong \langle k_1, k_2 \rangle$. Then by construction, $K \subset H \times \mathbb{Z}^2$, where the first \mathbb{Z} is generated by $t = (A, B)$ and the second by $s = (C, D)$.

We first show that the set of subgroups K constructed in this way is visible in $\text{Spec}_2^{\max}(F)$, and then we discuss of what isomorphism class of subgroups we

have constructed using these elements. By Lemmas 13 and 6 the density of pairs of tree pair diagrams constructed in this way is at least

$$\lim_{n \rightarrow \infty} \frac{\binom{r_n - N(h_1) - 2}{r_n - N(h_1) - 2} \binom{r_n - N(h_2) - 2}{r_n - N(h_2) - 2}}{2(r_n)^2} = \frac{1}{2} \mu^{-N(h_1) - N(h_2) - 4} > 0.$$

We claim that K is either isomorphic to H or to $H \times \mathbb{Z}$. Use the coordinates (w, t^a, s^b) on $H \times \mathbb{Z}^2$ where $w \in H$. It is easy to see that for every element $h \in H$, there is at least one $k \in K$ represented by the coordinates (h, t^a, s^b) for some $a, b \in \mathbb{Z}$. We first show that for each $h \in H$, there is a unique second coordinate. Suppose that $w_1 = (h, t^a, s^b)$ and $w_2 = (h, t^c, s^d)$ both lie in K , and thus the product $w_1 w_2^{-1} = (Id, t^{a-c}, s^{b-d})$ also lies in K . Thus there is some relation ρ in H expressed in terms of h_1 and h_2 so that when we replace h_i with k_i we obtain the element $(Id, t^{a-c}, s^{b-d}) \in K$. Since the generator t of \mathbb{Z} is linked to h_1 in k_1 , and the t coordinate of (Id, t^{a-c}, s^{b-d}) is not zero, we conclude that in ρ , the exponent sum of all instances of the generator h_1 is not equal to zero.

Recall that h_1 was chosen so that when h_1 is expressed as a word in x_0 and x_1 , the exponent sum of all the instances of x_0 is not equal to 0, but h_2 does not have this property. Any relation in H can be written in terms of x_0 and x_1 to yield a relation of F , and thus any relation in H must have the total exponent sum of all instances of x_0 equal to 0. By our choice of h_1 and h_2 , we see that a relation of H *must* have the exponent sum of all instances of the generator h_1 equal to zero. Thus we must have $a = c$ in our coordinates above.

We have now shown that either $K \cong H \times \mathbb{Z}$ or $K \cong H$. Suppose that $H_{ab} \cong \mathbb{Z} \oplus \mathbb{Z}$. Then $(H \times \mathbb{Z})_{ab} \cong \mathbb{Z}^3$ and it follows from Lemma 15 that $H \times \mathbb{Z} \notin \text{Spec}_2^{\max}(F)$. In this case we must have $K \cong H$.

Suppose that $H_{ab} = \mathbb{Z}$. In this case, either $h_2 \in [H, H]$ or $h_1^c = h_2^d w$ for some non-identity element $w \in [H, H]$ and integers c, d . In either case, there is a relator of H in which the total exponent sum on the instances of h_2 is nonzero. Since the s coordinate of the second \mathbb{Z} factor in $H \times \mathbb{Z}^2$ is linked to the h_2 generator in k_2 , there is a way to realize both (h, t^a, s^b) and (h, t^a, s^d) in K with $d \neq b$. Thus we must have $K \cong H \times \mathbb{Z}$. \square

It follows from Proposition 23 that $\text{Spec}_2^{\max}(F)$ contains \mathbb{Z}^2, F , and $\mathbb{Z} \wr \mathbb{Z}$, and from Theorem 16 that it does not contain \mathbb{Z} or $\{Id\}$.

We have seen above that it can be difficult to ascertain when a particular isomorphism class of subgroup is present in a given spectrum. Furthermore, the example of \mathbb{Z} shows that presence in a given spectrum does not necessarily imply presence in spectra of higher index.

We find that F is a very special two generator subgroup of itself, and exhibits behavior unlike that of \mathbb{Z} . As long as $k \geq 2$, we can show that $F \in \text{Spec}_k^{\max}(F)$. We call this behavior *persistence*; that is, a subgroup H is *persistent* if there is an l so that $H \in \text{Spec}_k^{\max}(F)$ for all $k \geq l$. In the small set of groups whose spectra have been previously studied, no subgroups have shown this persistent behavior. As noted in the introduction, the current known examples of subgroup spectra all find that the free group F_k is generic in the k -spectrum. In Thompson's group F ,

we find a wealth of examples of this persistent behavior. In the previous section, we effectively proved that *every* non-trivial finitely generated subgroup of F is persistent with respect to the sum stratification (Theorem 11). As a corollary of Theorem 24 below and the techniques in Lemma 14 above, it will follow that $F^n \times \mathbb{Z}^m$ and $F^n \wr \mathbb{Z}^m$ are also persistent with respect to the *max* stratification, with $l = 2n + m$.

Theorem 24 (F is persistent) F lies in $\text{Spec}_k^{\max}(F)$ for all $k \geq 2$.

Proof: Since F can be generated by two elements, and $F_{ab} \cong \mathbb{Z} \oplus \mathbb{Z}$, it follows from Proposition 23 that $F \in \text{Spec}_2^{\max}(F)$. We now show that $F \in \text{Spec}_k^{\max}(F)$ for all $k > 2$.

We define k generators h_1, h_2, \dots, h_k which generate a subgroup of F isomorphic to F , in such a way that the set of k -tuples pairs of trees of this form is visible. As reduced tree pair diagrams, we use the notation $h_i = (T_i, S_i)$. We begin by defining h_1 and h_2 . We let $x_0 = (T_{x_0}, S_{x_0})$ and $x_1 = (T_{x_1}, S_{x_1})$ as tree pair diagrams, (C_1, D_1) any reduced pair of trees with $n - 4$ carets in each tree and (C_2, D_2) any reduced pair of trees with $n - 5$ carets in each tree. We let T be the tree with two right carets, and three leaves numbered 1, 2, 3. We construct h_1 and h_2 as follows:

- We let T_1 be the tree T with T_{x_0} attached to leaf 1 and C_1 attached to leaf 2.
- We let S_1 be the tree T with S_{x_0} attached to leaf 1 and D_1 attached to leaf 2.
- We let T_2 be the tree T with T_{x_1} attached to leaf 1 and C_2 attached to leaf 3.
- We let S_2 be the tree T with S_{x_1} attached to leaf 1 and D_2 attached to leaf 3.

This construction is shown in Figure 10.

For fixed n , let (A_i, B_i) be any reduced pair of trees with $n - 3$ carets for $i = 3, 4, \dots, k$. Note that there are r_{n-3} ways to choose each such pair. Construct a reduced $(n - 1)$ -caret tree pair that represents an element of $[F, F]$ by attaching the pair (A_i, B_i) to a 2-caret tree as in Figure 7 in Section 3.3. Call this pair (A'_i, B'_i) . We now define $h_i = (T_i, S_i)$ for $i = 3, 4, \dots, k$ as follows:

- let T_i consist of a root caret with A'_i attached to its left leaf, and
- let S_i consist of a root caret with B'_i attached to its left leaf.

The subgroup generated by the $\{h_i\}$ is clearly a subgroup of $F \times \mathbb{Z}^2$, since the subtrees of the h_i which are the left children of the root carets, when taken as

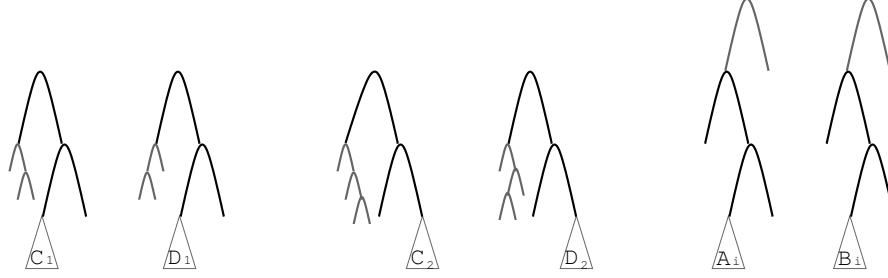


Figure 10: Constructing the tree pairs h_1, h_2 , and h_i generating a subgroup of $F \times \mathbb{Z}^2$.

independent tree pair diagrams, clearly generate a subgroup H which is isomorphic to F , as they contain the tree pair diagrams for x_0 and x_1 .

Any relator which is introduced into H by the inclusion of the commutators (A'_i, B'_i) as generators must hold true in F as well. Since all relators of F are commutators or conjugates of commutators, all relators have exponent sum on all instances of either x_0 and x_1 equal to zero. Additionally, we know that x_0 and x_1 are not commutators themselves. Thus any new relators introduced into H by the inclusion of the commutators (A'_i, B'_i) as generators must also have exponent sum on all instances of either x_0 and x_1 equal to zero. Using the coordinates (w, t^a, s^b) for elements of H , where $w \in F$, $t = (C_1, D_1)$ and $s = (C_2, D_2)$, the argument given in Proposition 23 goes through exactly to show that $w \in F$ has unique second and third coordinates, and thus $H \cong F$.

To see that the set of k -tuples constructed in this way is visible, note that the number of ways to construct them is $r_{n-4}r_{n-5}(r_{n-3})^{k-2}$. The choices are in the C_i, D_i trees which generate \mathbb{Z}^2 , and the A'_i, B'_i trees which are used to construct elements of $[F, F]$. Thus we compute the density of this set of k -tuples to be at least

$$\lim_{n \rightarrow \infty} \frac{r_{n-4}r_{n-5}(r_{n-3})^{k-2}}{k(r_n)^k} = \frac{1}{k} \mu^{-4} \mu^{-5} (\mu^{-3})^{k-2} > 0$$

by Lemmas 13 and 6. □

This proof used two very special properties of the whole group F which are not generally true for subgroups of F . First, there is an explicit way of characterizing tree pair diagrams corresponding to elements in the commutator subgroup $[F, F]$, which allows us to construct commutators containing a large arbitrary tree. Second, the relators of F are all commutators themselves, and thus including additional commutators as generators yields relators with the appropriate exponent sums on x_0 and x_1 . Thus we do not expect this persistent behavior from many other subgroups of F . However, we can adapt the ideas used above to prove that if a subgroup H of F is visible in a particular spectrum, $\text{Spec}_k^{\max}(F)$, then both the product $H \times \mathbb{Z}$ and the wreath product $H \wr \mathbb{Z}$ are visible in $\text{Spec}_{k+1}^{\max}(F)$. As a corollary of this fact and Theorem 24, we find that subgroups which contain F as

a factor are indeed persistent. We first need the following straightforward lemma about densities of visible subgroups.

Lemma 25 *We let $H_k(n)$ denote the set of all k -tuples of tree pair diagrams which generate a subgroup of F isomorphic to H with a maximum of n carets in any pair of trees, such that at least one coordinate realizes this maximum. If a subgroup H is visible in $\text{Spec}_k^{\max}(F)$ then*

$$\lim_{n \rightarrow \infty} \frac{|H_k(n)|}{(r_n)^k} \geq \lambda_k$$

for some $\lambda_k \in (0, 1]$.

Proof:

$$\lim_{n \rightarrow \infty} \frac{|H_k(n)|}{(r_n)^k} \geq \lim_{n \rightarrow \infty} \frac{|H_k(n)|}{k! |\text{Sph}_k^{\max}(n)|}$$

by Lemma 13. Since H is visible this limit equals the density of H with respect to the max stratification, and is positive, which gives the result. \square

Proposition 26 (Closure under products) *If $H \in \text{Spec}_k^{\max}(F)$ then $H \times \mathbb{Z}$ and $H \wr \mathbb{Z}$ lie in $\text{Spec}_{k+1}^{\max}(F)$.*

Proof: We construct the $k+1$ generators necessary to obtain a family of subgroups of F isomorphic to $H \times \mathbb{Z}$ in such a way that the set of $(k+1)$ -tuples of this form is visible. The techniques are similar to those used above.

We let h_1, h_2, \dots, h_k be a set of k generators for H . We will construct a set l_1, l_2, \dots, l_{k+1} of generators for $H \times \mathbb{Z}$. We let $h_i = (T'_i, S'_i)$ as a reduced pair of trees, and we must define $l_i = (T_i, S_i)$. For $i = 1, \dots, k$ we let T_i consist of a root caret with T'_i as its left subtree, and S_i consist of a root caret with S'_i as its left subtree. We let (A, B) be a reduced pair of trees with $n-1$ carets. To define l_{k+1} , let T_{k+1} consist of a root caret with A as its right subtree, and S_{k+1} consist of a root caret with B as its right subtree.

It is clear that the set $\{l_i\}$ generate a subgroup of F isomorphic to $H \times \mathbb{Z}$. We now show that the set of $(k+1)$ -tuples constructed in this way is visible in $\text{Spec}_{k+1}^{\max}(F)$.

To compute the density of the set of $(k+1)$ -tuples constructed in this way which generate a subgroup of F isomorphic to $H \times \mathbb{Z}$, we compute the following limit.

$$\lim_{n \rightarrow \infty} \frac{|H_k(n-1)|r_{n-1}}{|\text{Sph}_{k+1}^{\max}(n)|} \geq \lim_{n \rightarrow \infty} \frac{|H_k(n-1)|r_{n-1}}{(k+1)(r_n)^{k+1}}$$

by Lemma 13

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \frac{1}{k+1} \frac{|H_k(n-1)|}{(r_n)^k} \frac{r_{n-1}}{r_n} \\
&= \lim_{n \rightarrow \infty} \frac{1}{k+1} \frac{|H_k(n-1)|}{(r_{n-1})^k} \frac{(r_{n-1})^k}{(r_n)^k} \frac{r_{n-1}}{r_n} \\
&\geq \frac{\lambda_k \mu^{-k-1}}{k+1} > 0.
\end{aligned}$$

by Lemmas 25 and 6.

To see that $H \wr \mathbb{Z}$ lies in $\text{Spec}_{k+1}^{\max}(F)$ under the same assumption on H , we construct slightly different generators, and make an argument analogous to that in Theorem 24. As above, we let h_1, h_2, \dots, h_k be a set of k generators for H . We will construct a set l_1, l_2, \dots, l_{k+1} of generators which will generate a subgroup of $(H \wr \mathbb{Z}) \times \mathbb{Z}$ which we show to be isomorphic to $H \wr \mathbb{Z}$.

Let $H_k(n-3)$ be the set of all k -tuples which generate a subgroup of F isomorphic to H , where at least one tree pair contains $n-3$ carets. Let $\{h_i = (T'_i, S'_i)\} \in H_k(n-3)$. Since H is visible in $\text{Spec}_k^{\max}(F)$, Lemma 25 implies that

$$\lim_{n \rightarrow \infty} \frac{|H_k(n-3)|}{r_{n-3}^k} > 0.$$

We define $l_i = (T_i, S_i)$ for $i = 1, 2, \dots, k+1$ as follows. We let T be the tree with two left carets, and one interior caret attached to the right leaf of the caret which is not the root. Number the leaves of T by 1, 2, 3, 4. For $i = 1, 2, \dots, k$, let T_i be the tree T with T'_i attached to leaf 2. We let S_i be the tree T with S'_i attached to leaf 2. We let (A, B) be any reduced pair of trees with $n-3$ carets. We let $x_0 = (T_{x_0}, S_{x_0})$. We define l_{k+1} by taking T_{k+1} to be a single root caret with T_{x_0} attached to its left leaf and A attached to its right leaf. We let S_{k+1} be a single root caret with S_{x_0} attached to its left leaf and B attached to its right leaf. See Figure 11.

It is clear by the construction of our generators that any element of $H \wr \mathbb{Z}$ can appear as the pair of left subtrees of the root carets in any element of $\langle h_i \rangle$. However, we must show that $\langle h_i \rangle$ generates a subgroup of F isomorphic to $H \wr \mathbb{Z}$ and not $(H \wr \mathbb{Z}) \times \mathbb{Z}$. To do this, we note that since $H \wr \mathbb{Z}$ is a wreath product, all relators are commutators. Thus the argument in Theorem 24 can be applied to show that $\langle h_i \rangle \cong H \wr \mathbb{Z}$ rather than $(H \wr \mathbb{Z}) \times \mathbb{Z}$.

We must now show that the set of $(k+1)$ -tuples generated in this way is visible in $\text{Spec}_{k+1}^{\max}(F)$. We let $H_k(n)$ be the set of all k -tuples of tree pair diagrams which generate a subgroup of F isomorphic to H with a maximum of n carets in any pair of trees, such that at least one coordinate realizes this maximum. The density of the set of $(k+1)$ -tuples constructed in this way which generate a subgroup of

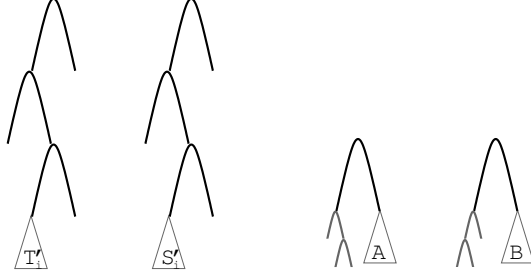


Figure 11: Constructing the pairs l_i and l_{k+1} generating a subgroup of $(H \wr \mathbb{Z}) \times \mathbb{Z}$.

F isomorphic to $H \wr \mathbb{Z}$ is computed as follows. We have r_{n-3} choices for the pair (A, B) , and $|H(n-3)|$ is the number of (T'_i, S'_i) generating sets for H with a maximum of $n-3$ carets in some pair. So together the density is

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{|H(n-3)|r_{n-3}}{|\text{Sph}_{k+1}^{\max}(n)|} &\geq \lim_{n \rightarrow \infty} \frac{1}{k+1} \frac{|H(n-3)|r_{n-3}}{(r_n)^k r_n} \\
&\geq \lim_{n \rightarrow \infty} \frac{1}{k+1} \frac{|H(n-3)|}{(r_{n-3})^k} \frac{(r_{n-3})^k r_{n-3}}{(r_n)^k r_n} \\
&= \frac{1}{k+1} \lambda_k \mu^{-3k-3} > 0
\end{aligned}$$

by Lemmas 25 and 6. □

This proposition combined with Theorem 24 allows us to find many isomorphism classes of subgroups in $\text{Spec}_k^{\max}(F)$ for the appropriate value of k .

- The l -fold iterated wreath product of \mathbb{Z} with itself $\mathbb{Z} \wr \dots \wr \mathbb{Z}$ lies in $\text{Spec}_l^{\max}(F)$.
- If H is a persistent subgroup present in $\text{Spec}_k^{\max}(F)$ for $k \geq l$, then $H \times \mathbb{Z}$ and $H \wr \mathbb{Z}$ are persistent subgroups present in $\text{Spec}_k^{\max}(F)$ for $k \geq l+1$.
- For $n \geq 1, m \geq 0$, and for all $k \geq 2n+m$, we have that $F^n \times \mathbb{Z}^m$ lies in $\text{Spec}_k^{\max}(F)$.

This shows that it is possible to have a subgroup H of F so that both H and $H \times \mathbb{Z}$ are contained in the $\text{Spec}_k^{\max}(F)$ for the same value of k ; we can take $H = F^n \times \mathbb{Z}^m$ and $k > 2m+n$.

- $F^n \wr \mathbb{Z}$ lies in $\text{Spec}_k^{\max}(F)$ for $n \geq 1$ for all $k \geq 2n+1$.

More generally, we can see that persistent subgroups can “absorb” visible subgroups to form new persistent subgroups.

Theorem 27 (Products with persistent subgroups are persistent) *If H is a subgroup which is present in $\text{Spec}_k^{\max}(F)$ and K is a persistent subgroup which is present in $\text{Spec}_l^{\max}(F)$ for $l \geq l_0$, then $H \times K$ is persistent and present in $\text{Spec}_l^{\max}(F)$ for $l \geq l_0 + k$.*

Proof: Let $H_k(n)$ denote the set of all k -tuples of tree pair diagrams which generate a subgroup of F isomorphic to H with a realized maximum of n carets in some coordinate. Since $H \in \text{Spec}_k^{\max}(F)$ we know from Lemma 25 that

$$\lim_{n \rightarrow \infty} \frac{|H_k(n)|}{(r_n)^k} \geq \lambda_k$$

for some $\lambda_k \in (0, 1]$.

Let $K_l(n)$ denote the set of all l -tuples of tree pair diagrams which generate a subgroup of F isomorphic to K with a realized maximum of n carets in some coordinate. Since K is persistent, we know that for any $l \geq l_0$, the limit

$$\lim_{n \rightarrow \infty} \frac{|K_l(n)|}{(r_n)^l} \geq \lambda_l$$

for some $\lambda_l \in (0, 1]$.

Let $m = k + l$ for any $l \geq l_0$. Form a generating set $\{t_1, t_2, \dots, t_m\}$, where $t_i = (T_i, S_i)$, for $H \times K$ as follows. Take any k -tuple $\delta \in H_k(n)$, where $\delta_i \in \delta$ is represented by the pair of trees (T_i^δ, S_i^δ) . Take any l -tuple $\eta \in K_l(n)$, where $\eta_j \in \eta$ is represented by the pair of trees (T_j^η, S_j^η) .

- For $1 \leq i \leq k$, let T_i consist of a root caret with left subtree T_i^δ , and let S_i consist of a root caret with left subtree S_i^δ .
- For $k + 1 \leq i \leq m$, let T_i consist of a root caret with right subtree T_i^η , and let S_i consist of a root caret with right subtree S_i^η .

This set of tree pairs generates a subgroup of $F \times F$ isomorphic to $H \times K$. A lower bound on the density of the isomorphism class of $H \times K$ is given by the following positive valued limit:

$$\lim_{n \rightarrow \infty} \frac{|H_k(n)||K_l(n)|}{r_n^{k+l}} = \lim_{n \rightarrow \infty} \frac{|H_k(n)|}{r_n^k} \frac{|K_l(n)|}{r_n^l} \geq \lambda_k \lambda_l > 0.$$

□

Thus, our analysis shows that the following subgroups are present in the k -spectrum with respect to the max stratification:

- The persistent subgroups $F, F \times F, \dots, F^n$ for $2n \leq k$.
- The persistent subgroups $F^n \times \mathbb{Z}^m$, for $2n + m \leq k, n \geq 1$.
- The persistent subgroups $F^n \wr \mathbb{Z}$ for $2n + 1 \leq k, n \geq 1$.

- The abelian subgroup \mathbb{Z}^k and the k -fold iterated product of \mathbb{Z} with itself.
- The mixed direct and wreath products of \mathbb{Z} with itself with k terms, including for example $\mathbb{Z}^{k-1} \wr \mathbb{Z}$ and $(\mathbb{Z} \wr \mathbb{Z} \wr \mathbb{Z}) \times \mathbb{Z}^{k-3}$.
- Various mixed direct and wreath products with \mathbb{Z} such as $(F^2 \times \mathbb{Z}^3) \wr \mathbb{Z} \times \mathbb{Z}$ which is present in all $k \geq 9$, for example.

While the isomorphism classes of subgroups described above occur with positive densities in $\text{Spec}_k^{\max}(F)$ for appropriate k , the lower bounds on their densities are very small. In fact, the lower bound on the sum of the densities of all of these isomorphism classes of subgroups amounts to much less than 1% of all isomorphism classes of subgroups in $\text{Spec}_k^{\max}(F)$.

We conclude with an open question about the isomorphism type of a random subgroup of the other Thompson's groups T and V . Although these groups contain F as a proper subgroup, unlike F they also contain free subgroups of rank 2 and above. What is the density of the set of free subgroups of a given rank within $\text{Spec}_k(T)$? Within $\text{Spec}_k(V)$? Are these groups like F in that their subgroup spectra contain many isomorphism classes, or does one find a generic isomorphism class of subgroup in $\text{Spec}_k(T)$ and $\text{Spec}_k(V)$?

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