AMERICAN CALL OPTIONS UNDER JUMP-DIFFUSION PROCESSES - A FOURIER TRANSFORM APPROACH

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Abstract. We consider the American option pricing problem in the case where the underlying asset follows a jump-diffusion process. We apply the method of Jamshidian (1992) to transform the problem of solving a homogeneous integro-partial differential equation (IPDE) on a region restricted by the early exercise (free) boundary to that of solving an inhomogeneous IPDE on an unrestricted region. We apply the Fourier transform technique to this inhomogeneous IPDE in the case of a call option on a dividend paying underlying to obtain the solution in the form of a pair of linked integral equations for the free boundary and the option price. We also derive new results concerning the limit for the free boundary at expiry. Finally we present a numerical algorithm for the solution of the linked integral equation system for the American call price, its delta and the early exercise boundary. We use the numerical results to quantify the impact of jumps on American call prices and the early exercise boundary.

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1. Introduction

The American option pricing problem has been explored in great depth in the option pricing literature. A recent survey by Barone-Adesi (2005) provides an overview of this research for the case of the American put under the classical Brownian motion process for asset returns. In practice, many assets, in particular foreign exchange rates, are found to have return distributions that feature "fatter" tails and higher peaks that cannot be accurately modelled using Brownian motion. In order to capture this leptokurtosis in the data, one must consider alternative dynamics for the underlying asset. One such alternative is the jump-diffusion process originally proposed by Merton (1976). It has been shown that jump-diffusion processes can provide a much better fit for asset returns when leptokurtosis is present. Examples of these findings are provided by Jarrow & Rosenfeld (1984), Ball & Torous (1985), Jorion (1988), Ahn & Thompson (1992), and Bates (1996). Merton (1976) provides a framework for pricing European options under jump-diffusion processes, and in this paper we explore the extension of this model to the pricing of American call options. We derive the linked system of integral equations for the price and early exercise boundary of an American call under Merton's jump-diffusion dynamics, focusing in particular on the use of integral transform techniques to solve the associated integro-partial differential equation (IPDE) for the American call price. We derive the limit of the early exercise boundary at maturity, and provide a numerical algorithm for solving the linked integral equation system based on an extension to the jump-diffusion situation of the quadrature integration technique of Kallast & Kivinukk (2003). The algorithm used readily generates values for the price, delta and early exercise boundary.

When deriving the integral equations for the price and early exercise boundary of American options, there are at least four approaches that can be used. The probabilistic method that is demonstrated by Karatzas (1988) and Jacka (1991) for the pure diffusion case, and has been generalised to the jump-diffusion situation by Pham (1997). More recently Jamshidian (2006) has generalised the probabilistic approach to deal with American options under jump-diffusion by making use of the Itô-Meyer formula. The discrete time approach using compound option theory is demonstrated by Geske & Johnson (1984), and Kim (1990) shows how to take the limit to provide the continuous time solution. Gukhal (2001) extends this solution technique to include Merton's jump-diffusion dynamics. The issue of the existence and uniqueness of the solution to the non-linear integral equation that arises in the solution of the free boundary value problem was raised by Myneni (1992) and finally settled by Peskir (2005) in the case of pure diffusion dynamics.

The remaining two approaches focus on deriving solutions to the partial differential equation (PDE) for the American call price. A natural solution technique here is that of integral transforms, which has been very successfully employed in a wide range of PDE problems in the natural sciences; see for example Debnath (1995). One of the difficulties with dealing with American options using this technique arises from the fact that one has to solve the PDE on a region restricted by the early exercise, or free, boundary. McKean (1965), who seems to have been the first to consider the American option pricing problem, solves the homogeneous PDE in a restricted domain by using an incomplete Fourier transform. An alternative approach was developed by Jamshidian (1992) who replaces the homogeneous PDE with an equivalent inhomoegenous PDE on an unrestricted domain. The solution to this alternative formulation can then be derived by using the standard Fourier transform, or through an application of Duhamel's principle.

Integral transforms are a very useful method for solving option pricing problems, as they can be applied to a range of underlying dynamics and payoff types. There are numerous examples where integral transforms have been used in the option pricing literature. Scott (1997) uses Fourier transforms to value European options under jump-diffusion with both stochastic volatility and stochastic interest rates. Carr & Madan (1999) also use Fourier transforms to motivate numerical algorithms using the fast Fourier transform, with a particular focus on the variance gamma model. This approach is further generalised by Lee (2004), and bounds are derived to help improve computational accuracy. Laplace transforms have been applied to more complex payoff types, including the double barrier option explored by Pelsser (2000), and the American straddle analysed by Alobaidi & Mallier (2002). The American strangle is further analysed by Chiarella & Ziogas (2005) using a Fourier transform approach. In all three cases, the underlying dynamics are the classic geometric Brownian motion.

The main advantage of the Fourier transform method is that the solution may be expressed in terms of a general initial value or payoff function, so that a wider variety of American options such as puts, calls, butterflies, spread options and max-options can all be handled systematically. While integral transforms are clearly capable of handling a wide range of asset dynamics, the extension of these solution methods to the jump-diffusion case for American option pricing has not been covered in the existing literature, and the first contribution of this paper is to provide this extension for Jamshidian's formulation.[1] In this paper we demonstrate how to apply Fourier transform techniques to solve the IPDE for the American call option price and free boundary.

Despite the amount of existing literature on American options with jumps, it seems that there has been little work on the implementation of the integral equations for the price and free boundary of American options under jump-diffusion. While some authors such as Pham (1997) and Gukhal (2001) derive these integral equations, they do not discuss how they can be solved numerically. Here we solve the linked integral equation system that arises for the American call and its free boundary in the case of jump-diffusion by extending the approach of Kallast & Kivinukk (2003) who apply a quadrature scheme to the corresponding pure diffusion case. In developing a numerical scheme for the linked integral equation system we obtain a simplification of the integral terms over the jump-size distribution that reduces the computational burden by reducing the dimension of the multiple integration involved. We also provide results on the behaviour of the early exercise boundary at expiry that are needed to start the numerical procedure. While the focus of this paper is not on finding optimal numerical methods for American option prices with jumps, we do demonstrate that the proposed numerical integration scheme is able to accurately find the price, delta and early exercise boundary of American calls with log-normal jump sizes. Furthermore we find that the method is often more efficient

[1]The extension of McKean's approach to the jump-diffusion case is provided by Chiarella & Ziogas (2006).

than a simple two-pass Crank-Nicolson finite difference scheme and competitive with the method of lines of Meyer (1998).

The remainder of this paper is structured as follows. Section 2 outlines the free boundary problem that arises from pricing an American call option under Merton's jump-diffusion model. Section 3 applies Jamshidian's method to derive an inhomogeneous IPDE for the American call price, which is then solved using Fourier transforms. We thus obtain the linked integral equation system for the free boundary and option price. We derive the limit of the free boundary at expiry in Section 4, and find that this limit is different to that found for pure diffusion models. Section 5 analyses the integral equations in the case where the jump sizes follow a log-normal distribution, as suggested by Merton (1976). Section 6 outlines the numerical integration method used to solve the linked integral equation system for both the free boundary, price and delta of the American call. Since the integral equation for the call value and the integral equation for the free boundary are interdependent, we provide a suitable means by which we can manage the interdependence in order to use the two-pass sequential procedure that works well in the non-jump case. Numerical results detailing the efficiency of this algorithm are provided in Section 7, with additional results exploring the impact of jumps on the American call option and its early exercise boundary provided in Section 8. Concluding remarks are presented in Section 9. Most of the lengthy mathematical derivations are given in appendices.

2. Problem Statement - Merton's Model

Let $C(S, \tau)$ be the price of an American option written on the underlying asset S at time to expiry $\tau = T - t$, where T is the option maturity, t is the current time, and the strike price is K. We assume that S pays a continuous dividend yield of rate q. Let $a(\tau)$ denote the early exercise boundary at time to expiry τ, and assume S follows the jump-diffusion process

$$dS = (\mu - \lambda k)Sdt + \sigma SdW + (Y - 1)SdN, \tag{1}$$

where μ is the instantaneous return per unit time, σ is the instantaneous volatility per unit time, W is a standard Wiener process and N is a Poisson process whose increments satisfy

$$dN = \begin{cases} 1, & \text{with probability } \lambda dt, \\ 0, & \text{with probability } (1-\lambda dt). \end{cases}$$

Let the jump size, Y, be a random variable whose probability measure we denote by $\mathbb{Q}$, with W and N independent processes, and the jump sizes are not correlated with these. We use $G(Y)$ to denote the corresponding probability density function for Y. Thus the expected jump size, k, is given by

$$k = \mathbb{E}_{\mathbb{Q}}[Y-1] = \int_0^\infty (Y-1)G(Y)dY. \tag{2}$$

Following Merton's (1976) argument, it can be shown that C satisfies the integro-partial differential equation (henceforth IPDE)

$$\begin{aligned} \frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \left(r - q - \lambda \int_0^\infty (Y-1)(1-l(Y))G(Y)dY \right) S\frac{\partial C}{\partial S} - rC \\ + \lambda \int_0^\infty [C(SY,\tau) - C(S,\tau)](1-l(Y))G(Y)dY, \end{aligned} \tag{3}$$

in the region $0 \le \tau \le T$ and $0 \le S \le a(\tau)$, where r is the risk-free rate, and $l(Y)$ is the market price of jump risk associated with a jump in the underlying from S to SY. Given a form for $l(Y)$, we can define a new intensity, λ^*, and jump-size distribution, $G^*(Y)$, which fully incorporate the term $l(Y)$, such that (3) can be written as

$$\begin{aligned} \frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \left(r - q - \lambda^* \int_0^\infty (Y-1)G^*(Y)dY \right) S\frac{\partial C}{\partial S} - rC \\ + \lambda^* \int_0^\infty [C(SY,\tau) - C(S,\tau)]G^*(Y)dY. \end{aligned} \tag{4}$$

The market incompleteness of the jump-diffusion option pricing problem is reflected in the fact that the choice of λ^* and $G^*(Y)$ is at the discretion of the model builder, so that the associated risk-neutral density is non-unique. We take the view here that the model builder would calibrate λ^* and $G^*(Y)$ directly to market data without needing to explicitly determine $l(Y)$ and so determine a risk-neutral density for the pricing problem at hand. Henceforth, for ease of notation, we shall simply refer to the jump risk-adjusted intensity and jump-size density in (4) as λ and $G(Y)$ respectively.

By use of (2), the IPDE (4) can be written as[2]

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - q - \lambda k)S\frac{\partial C}{\partial S} - rC + \lambda \int_0^\infty [C(SY,\tau) - C(S,\tau)]G(Y)dY, \quad (5)$$

In the case of an American call option, the IPDE (5) is subject to the initial and boundary conditions

$$C(S,0) = \max(S-K,0), \quad 0 \leq S < \infty \quad (6)$$

$$C(0,\tau) = 0, \quad \tau \geq 0, \quad (7)$$

$$C(a(\tau),\tau) = a(\tau) - K, \quad \tau \geq 0, \quad (8)$$

$$\lim_{S\to a(\tau)} \frac{\partial C}{\partial S} = 1, \quad \tau \geq 0. \quad (9)$$

Condition (6) is the payoff function for the call at expiry, and condition (7) ensures that the option is worthless if S falls to zero. The value-matching condition (8) forces the value of the call option to be equal to its payoff on the early exercise boundary, and the smooth-pasting condition (9) sets the delta of the American call to be continuous at the free boundary to guarantee arbitrage-free prices. For the call under consideration, we note that the standard arbitrage arguments that justify condition (9) are not readily applied under Merton's jump-diffusion model, since this depends upon the price process for S being continuous. The corresponding boundary conditions were proven by Pham (1997) for the American put case, and the arguments he uses, based on the maximum principle, are readily applied to the case of the American call problem with a continuous dividend yield for S. Figure 1 illustrates the payoff, price profile and early exercise boundary for the American call under consideration.

****Insert Figure 1 here****

[2]Note that one can also directly arrive at (5) from (3) by assuming that jump-risk is fully diversifiable, and hence $l(Y) = 0$, as is done by Merton (1976). The extension of the traditional hedging and change of measure approaches to properly incorporate the $l(Y)$ term is given in Cheang, Chiarella & Ziogas (2006). We should point out that Pham (1997) seems to have been the first to report the IPDE (3) with the inclusion of the $l(Y)$ term.

It is possible to solve the free boundary value problem (5)-(9) directly using Fourier transforms. Following McKean (1965), we can make the change of variable $S \equiv Ke^x$, and set $C(S,\tau) \equiv KV(x,\tau)$ and $b(\tau) \equiv a(\tau)/K$. We are then able to introduce an incomplete version of the Fourier transform, defined by

$$\mathcal{F}^b\{V(x,\tau)\} \equiv \int_{-\infty}^{\ln b(\tau)} e^{i\eta x} V(x,\tau)dx. \tag{10}$$

Inversion of the transform is readily carried out, and the end result is a linked system of integral equations for $C(S,\tau)$ and $a(\tau)$.

While this is certainly a valid solution method, it comes with three significant drawbacks. The first is that this approach yields a representation for the American call option the economic interpretation of which if not at all obvious. The second difficulty is that the resulting system of integral equations depends upon the derivative $da(\tau)/d\tau$, which introduces difficulties when trying to solve the integral equations numerically. Furthermore, while it is possible to manipulate McKean's form of the solution into an early exercise premium decomposition, such as is given by Kim (1990) for the pure diffusion case, and Gukhal (2001) for the jump-diffusion setting, this task involves a significant amount of additional manipulation, particularly in the jump-diffusion case. Full details on McKean's method applied to American calls under jump-diffusion are given byChiarella & Ziogas (2006). With these shortcomings in mind, we instead seek a more efficient alternative for solving (5)-(9) using the standard Fourier transform.

3. Jamshidian's Representation

One of the major difficulties in solving the IPDE (5) subject to the boundary conditions (6)-(9) is that the solution is sought on a restricted domain for the stock price S, and furthermore the boundary of this domain is itself unknown a priori and needs to be determined as part of the solution process. In the pure diffusion case, Jamshidian (1992) demonstrates that by evaluating the PDE for the American call price when $S > a(\tau)$, one can reformulate the free boundary problem in the restricted domain $0 \leq S \leq a(\tau)$ as an inhomogeneous PDE in the unrestricted domain $0 \leq S < \infty$. This inhomogeneous PDE can then be more readily solved by traditional solution techniques such as Fourier transforms. We note that the free boundary value problem given by (5)-(9) involves a homogeneous IPDE to be solved in the restricted asset price domain $0 \leq S \leq a(\tau)$. Here we show how to apply Jamshidian's approach to reformulate the IPDE (5) and associated boundary conditions as an inhomogeneous IPDE on an unrestricted domain. We highlight the fact that $C(S,\tau)$ and $\partial C/\partial S$ are continuous for $0 \leq S < \infty$ and $\tau > 0$, as given by the value-matching condition (8) and smooth-pasting condition (9). Jamshidian's approach can only be applied with confidence when such continuity holds. We also point out that in the approach we adopt this is the only point at which the smooth pasting condition is used. We now state the main result that converts the homogeneous IPDE on a restricted domain to an inhomogeneous IPDE on an unrestricted domain.

Proposition 3.1. *The solution to the homogeneous IPDE (5) for $C(S,\tau)$ in the domain $0 \leq S \leq a(\tau)$ subject to the initial and boundary conditions (6) - (9) is equivalent to the solution to the inhomogeneous IPDE*

$$\begin{aligned} \frac{\partial C}{\partial \tau} &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - q - \lambda k) S \frac{\partial C}{\partial S} - rC + \lambda \int_0^\infty [C(SY,\tau) - C(S,\tau)] G(Y) dY \\ &\quad + H(S - a(\tau)) \left\{ qS - rK - \lambda \int_0^{a(\tau)/S} [C(SY,\tau) - (SY - K)] G(Y) dY \right\}, \end{aligned} \tag{11}$$

in the region $0 < \tau \leq T$, $0 \leq S < \infty$, subject to the initial condition (6), where $H(x)$ is the Heaviside step function defined as

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{12}$$

Proof: Refer to Appendix 1.

□

There is a clear economic interpretation for the inhomogeneous term that arises in equation (11), which has also been provided by Gukhal (2001). The $(qS - rK)$ term represents the net cash flows received from holding the portfolio $(S - K)$ whenever S is in the stopping region. This is already familiar from the pure diffusion case (see for example

Kim (1990)). The integral term arises entirely because of the introduction of jumps in the price process for S. Note that if no jumps are present (λ=0) then this term will be zero, and the inhomogeneous term becomes the same one presented by Jamshidian (1992). This additional term captures the rebalancing costs incurred by the option holder whenever the price of the underlying jumps down[3] from the stopping region back into the continuation region. Figure 2 illustrates this effect in detail. Consider the case where, during the life of the option contract, the underlying asset price is at $S_- > a(\tau)$. Since the value of S is in the stopping region, the holder of the option will currently possess the portfolio $(S - K)$. If a jump of size Y occurs while $\tau > 0$ such that $S_+ = YS_- < a(\tau)$, then the portfolio held by the investor will now be worth less than the unexercised American call. This difference is the cost being captured by the integral in the inhomogeneous term in (11).

****Insert Figure 2 here****

It should be pointed out that the inhomogeneous term in (11) is not the standard exogenous forcing term that one encounters in standard PDE applications. The inhomogeneous term here involves the unknown C and a functions. Nevertheless this reformulation turns out to be very useful in obtaining the solution to (11) via the Fourier transform technique.

Having derived the inhomogeneous IPDE for $C(S, \tau)$ we now demonstrate how we can use Fourier transforms to find the solution. Our first step is to transform the IPDE to an equation with constant coefficients and a "standardised" strike of 1. Let $S \equiv Ke^x$ and $C(S, t) \equiv KV(x, \tau)$, with $b(\tau) \equiv a(\tau)/K$. The transformed IPDE for V is then[4]

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + \phi \frac{\partial V}{\partial x} - (r+\lambda)V + \lambda \int_0^\infty V(x + \ln Y, \tau) G(Y) dY \\ &\quad + H(x - \ln b(\tau)) \left\{ \lambda \int_0^{b(\tau)e^{-x}} [V(x + \ln Y, \tau) - (Ye^x - 1)] G(Y) dY \right\}, \end{aligned} \tag{13}$$

[3]Since $S \geq a(\tau)$, we know that $a(\tau)/S \leq 1$.
[4]It should be noted that $C(SY, \tau) = KV(\ln(SY/K), \tau) = KV(x + \ln Y, \tau)$.

where $\phi \equiv r - q - \lambda k - \frac{\sigma^2}{2}$. Equation (13) is to be solved in the time domain $0 \le \tau \le T$, and the unrestricted region $-\infty \le x \le \infty$, subject to the initial and boundary conditions

$$\begin{aligned} V(x,0) &= \max(e^x - 1, 0), \quad -\infty < x < \infty, & (14) \\ \lim_{x \to -\infty} V(x,\tau) &= 0, \quad \tau \ge 0, & (15) \\ V(\ln b(\tau), \tau) &= b(\tau) - 1, \quad \tau \ge 0. & (16) \end{aligned}$$

It is worth noting that the smooth-pasting condition still holds, although we do not explicitly require it when solving (13) for $V(x,\tau)$ since it is incorporated in the inhomogeneous term; see again the remark just prior to Proposition 3.1.

Since the x-domain is now the unrestricted region $-\infty < x < \infty$, the Fourier transform of the inhomogeneous IPDE (13) can be found. Define the Fourier transform of V, $\mathcal{F}\{V(x,\tau)\}$, as

$$\mathcal{F}\{V(x,\tau)\} \equiv \hat{V}(\eta,\tau) = \int_{-\infty}^{\infty} e^{i\eta x} V(x,\tau)dx, \tag{17}$$

with the corresponding inversion formula

$$\mathcal{F}^{-1}\{\hat{V}(\eta,\tau)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\eta x}\hat{V}(\eta,\tau)d\eta, \tag{18}$$

where $i = \sqrt{-1}$. Applying this Fourier transform to (13), we can reduce the inhomogeneous IPDE for V to an inhomogeneous ordinary differential equation (ODE) for $\hat{V}$, whose solution is readily found.

Proposition 3.2. *Using the initial and boundary conditions (14)-(15), the Fourier transform of the IPDE (13) with respect to x satisfies the ODE*

$$\frac{\partial \hat{V}}{\partial \tau} + \left[\frac{\sigma^2\eta^2}{2} + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right]\hat{V} = \hat{F}_J(\eta,\tau) \tag{19}$$

where

$$\hat{F}_J(\eta,\tau) \equiv \mathcal{F}\{F_J(x,\tau)\},$$

with

$$F_J(x,\tau) = H(x-\ln b(\tau))\Big\{(qe^x - r) - \lambda\int_0^{b(\tau)e^{-x}} [V(x+\ln Y,\tau)-(Ye^x-1)]G(Y)dY\Big\} \tag{20}$$

and

$$A(\eta) \equiv \int_0^\infty e^{-i\eta\ln Y} G(Y)dY. \tag{21}$$

Furthermore, the solution to the ODE (19) is given by

$$\begin{aligned}\hat{V}(\eta,\tau) &= \hat{V}(\eta,0)\exp\left\{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)\tau\right\} \\ &\quad + \int_0^\tau \exp\left\{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)(\tau-\xi)\right\}\hat{F}_J(\eta,\xi)d\xi,\end{aligned} \tag{22}$$

where $\hat{V}(\eta,0)=\mathcal{F}\{V(x,0)\}$.

Proof: Refer to Appendix 2.

□

We note that the first term on the right-hand side in (22) is the Fourier transform of the IPDE (13) in the case of a European call option under jump-diffusion dynamics. In this case the last term of the right-hand side of (13) does not appear, and it is the latter term involving the free boundary that gives rise to the second term in equation (22).

Now that $\hat{V}(\eta,\tau)$ has been found, we may use the inversion formula (18) to recover $V(x,\tau)$, the American call price in the x-τ plane. By taking the inverse Fourier transform of (22), we have

$$\begin{aligned}V(x,\tau) &= \mathcal{F}^{-1}\Bigg\{\hat{V}(\eta,0)\exp\left\{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)\tau\right\} \\ &\quad + \int_0^\tau \exp\left\{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)(\tau-\xi)\right\}\hat{F}_J(\eta,\xi)d\xi\Bigg\} \\ &\equiv V_E(x,\tau)+V_P(x,\tau) \\ &\equiv \frac{1}{K}\left[C_E(S,\tau)+C_P(S,\tau)\right] = \frac{1}{K}C(S,\tau)\end{aligned} \tag{23}$$

where $C_E(S,\tau) = KV_E(x,\tau)$ is the value of the corresponding European call written on S and $C_P(S,\tau) = KV_P(x,\tau)$ is the early exercise premium for $C(S,\tau)$. By performing the inversions, we can determine the analytic forms of C_E and C_P and these are given in the following propositions.

Proposition 3.3. *The price of the European call option, $C_E(S,\tau)$, in equation (23) is given by*

$$C_E(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)}\{C_{BS}[SX_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2]\}, \tag{24}$$

where

$$\begin{aligned} C_{BS}[S,K,\beta,r,q,\tau,\sigma^2] &= Se^{-q\tau}\mathcal{N}[d_1(S,\beta,r,q,\tau,\sigma^2)] \\ &\quad -Ke^{-r\tau}\mathcal{N}[d_2(S,\beta,r,q,\tau,\sigma^2)], \qquad (25) \\ d_1(S,\beta,r,q,\tau,\sigma^2) &= \frac{\ln\frac{S}{\beta} + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \\ d_2(S,\beta,r,q,\tau,\sigma^2) &= d_1(S,\beta,r,q,\tau,\sigma^2) - \sigma\sqrt{\tau}, \end{aligned}$$

$\mathcal{N}[\cdot]$ is the cumulative normal density function, and we define $X_n \equiv Y_1Y_2...Y_n$ and $X_0 \equiv 1$, along with[5]

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}^{(n)}\{f(X_n)\} &\equiv \int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} f(X_n)G(Y_1)G(Y_2)...G(Y_n)dY_1dY_2...dY_n \\ &= \int_0^{\infty} f(X_n)G(X_n)dX_n. \end{aligned}$$

The $Y_1, Y_2, ..., Y_n$ are independent draws from the jump size distribution $G(Y)$.

Proof: Refer to Appendix A3.1.

□

We note that equation (24) is of course Merton's (1976) solution for a European call option under jump-diffusion, with general jump size density $G(Y)$. Next we shall determine the early exercise premium $C_P(S,\tau)$.

[5] We assume that the density function G is of the form that facilitates the reduction of the n-dimensional integral to a single integral. This is certainly true of the log-normal density function to be used later in the paper.

Proposition 3.4. *The early exercise premium, $C_P(S,\tau)$, in equation (23) is given by*

$$C_P(S,\tau) = \sum_{n=0}^{\infty}\mathbb{E}_{\mathbb{Q}}^{(n)}\Bigg\{\int_0^\tau \frac{e^{-\lambda(\tau-\xi)}(\tau-\xi)^n}{n!} \times\Bigg[C_P^{(D)}[SX_ne^{-\lambda k(\tau-\xi)},K,a(\xi),r,r,q,\tau-\xi,\sigma^2] -\lambda C_P^{(J)}[SX_ne^{-\lambda k(\tau-\xi)},K,a(\xi),r,q,\tau-\xi,\sigma^2;C(\cdot,\xi)]\Bigg]d\xi\Bigg\}, \quad (26)$$

where

$$C_P^{(D)}[S,K,a(\xi),R,r,q,\tau,\sigma^2] = qSe^{-q\tau}\mathcal{N}\left[d_1\left(S,a(\xi),r,q,\tau,\sigma^2\right)\right] - RKe^{-r\tau}\mathcal{N}\left[d_2\left(S,a(\xi),r,q,\tau,\sigma^2\right)\right], \quad (27)$$

$$C_P^{(J)}[S,K,a(\xi),r,q,\tau,\sigma^2;C(\cdot,\xi)] = e^{-r\tau}\int_0^1 G(Y)\int_{a(\xi)}^{a(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)]\kappa(S,\omega,r,q,\tau,\sigma^2)d\omega dY, \quad (28)$$

and

$$\kappa(S,\omega,r,q,\tau,\sigma^2) \equiv \frac{1}{\omega\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_2^2(S,\omega,r,q,\tau,\sigma^2)\right\}. \quad (29)$$

The operator $\mathbb{E}_{\mathbb{Q}}^{(n)}$ and functions $\mathcal{N}$, d_1 and d_2 have been defined in Proposition 3.3.

Proof: Refer to Appendix A3.2.

□

Note that on the left hand side in (28) we introduce the notation $C(\cdot,\xi)$ to indicate that the dependence on the option price is that of a functional (rather than a function), in fact of the form

$$\int_0^1\int_{a(\xi)}^{a(\xi)/Y} C(\omega Y,\xi)G(Y)g(\omega)d\omega dY, \quad (30)$$

for an appropriate function $g(\omega)$. This notation will recur in many of the subsequent formulae.

Each of the linear terms in (26) represent discounted expected cash-flows incurred by the option holder when $S > a(\tau)$, as discussed previously for the interpretation of the inhomogeneous term in (11); the $C_P^{(D)}$ term essentially being the expected value at time to maturity τ of the $qS - rK$ component in (11), and the $C_P^{(J)}$ term being the expected value of the integral term in (11). Combining C_E and C_P, we can now write down the integral equation for the American call option price, $C(S,\tau)$.

Proposition 3.5. *Substituting (24) and (26) into equation (23), the American call price, $C(S,\tau)$ is given by*

$$\begin{aligned} C(S,\tau) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \{ C_{BS}[SX_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2] \}, \qquad (31) \\ & \quad + \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} \frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!} \\ & \qquad \times \Bigg[C_P^{(D)}[SX_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, r, q, \tau-\xi, \sigma^2] \\ & \qquad -\lambda C_P^{(J)}[SX_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, q, \tau-\xi, \sigma^2; C(\cdot,\xi)] \Bigg] d\xi \Bigg\}, \end{aligned}$$

where C_{BS} is given by equation (25), and the functions $C_P^{(D)}$, and $C_P^{(J)}$ are given by equations (27) and (28) respectively.

Proof: Direct substitution of (24) and (26) into (23) yields equation (31).

□

Note that equation (31) is indeed an integral equation since the unknown option price also appears under the time integral on the right-hand side, in particular through the $C_P^{(J)}$ term. This is in contrast to the pure diffusion case where the equation corresponding to (31) (when $\lambda = 0$ and so that the $C_P^{(J)}$ term drops out) is simply an integral expression for the option price that can be evaluated once the free boundary has been determined.

The solution (31) is readily compared with that of Gukhal (2001), who derives (31) by generalising the compound option approach of Kim (1990) to the jump-diffusion case. The three additive components of the call value in equation (31) each have a clear economic interpretation, as outlined by Gukhal (2001). The first term, C_{BS}, represents the European component of the American call option's value, while the remaining two terms combine to form the total early exercise premium. The middle term is a natural extension of the early exercise premium that arises in the pure diffusion case. More specifically, this term calculates the dividend received when holding the underlying, less the interest payable on a loan of size K. Thus $C_P^{(D)}$ captures the potential income to the option holder should the option be exercised to buy the underlying by borrowing K at the risk-free rate. The third term, $C_P^{(J)}$, arises entirely due to the introduction of jumps in the price process for S, and captures the rebalancing costs incurred by the option holder whenever the price of the underlying jumps down from the stopping region into the continuation region (see Figure 2).

In equation (31), the value of the American call option is expressed as a function of the underlying asset price S, and time to maturity τ. As we have already noted, equation (31) also depends upon the unknown early exercise boundary, $a(\tau)$. By requiring the expression for $C(S,\tau)$ to satisfy the early exercise boundary condition (8), we can derive a similar integral equation for the value of $a(\tau)$. This integral equation is given by

$$\begin{aligned} a(\tau) - K &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)}\{C_{BS}[a(\tau)X_n e^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2]\}, \qquad (32) \\ &\quad + \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}}^{(n)}\Bigg\{ \int_0^{\tau} \frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!} \\ &\qquad \times \Bigg[C_P^{(D)}[a(\tau)X_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, r, q, \tau-\xi, \sigma^2] \\ &\qquad - \lambda C_P^{(J)}[a(\tau)X_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, q, \tau-\xi, \sigma^2; C(\cdot,\xi)] \Bigg] d\xi \Bigg\}, \end{aligned}$$

It is particularly crucial to note that the integral equation (32) depends upon the unknown call value $C(S,\tau)$, and this dependence arises entirely from the integral terms that have been introduced by the presence of jumps in the dynamics for S.

The structure of the integral equation system consisting of (31) and (32) can be made more transparent by writing it succinctly as

$$C(S,\tau) = \Omega_C(S,\tau) + \int_0^{\tau} \Psi_C[a(\xi),\xi,\tau,S;C(\cdot,\xi)]d\xi, \qquad (33)$$

$$a(\tau) = \Omega_a(a(\tau),\tau) + \int_0^{\tau} \Psi_a[a(\xi),\xi,\tau,a(\tau);C(\cdot,\xi)]d\xi, \qquad (34)$$

where the definitions of the functions (Ω_C, Ψ_C) and (Ω_a, Ψ_a) can be inferred from the right hand sides of equations (31) and (32) respectively. The interdependence of (33) and (34) is obvious, and it is this interdependence that makes numerical implementation more involved than for the corresponding no-jump problem[6]. In fact equations (31)-(32) form a linked system of nonlinear Volterra integral equations of the second kind. Thus in order to implement these integral equations for the free boundary and call price, we need to develop numerical techniques to solve the linked integral equation system (31)-(32).

Before concluding this section, we present an alternative form for the double integral involving the function κ in equation (28).

Proposition 3.6. *By changing the order of integration, C_P^J in equation (28) can be rewritten as*

$$\begin{aligned} &C_P^{(J)}[S, K, a(\xi), r, q, \tau, \sigma^2; C(S, \xi)] \\ &\quad = e^{-r\tau} \int_0^1 [C(a(\xi)z, \xi) - (a(\xi)z - K)] \int_0^z G(Y)\kappa(S/a(\xi), z, r, q, \tau, \sigma^2) dY dz. \end{aligned} \tag{35}$$

Proof: Refer to Appendix A3.3.

□

While the modified representation in (35) is less intuitive than the original, from an economic point of view, we will show that it offers significant advantages when attempting to solve (32) numerically for specific forms of $G(Y)$. In particular, we will demonstrate in Section 5 that when $G(Y)$ is the log-normal density function given by Merton (1976), the innermost integral in (35) can be evaluated analytically. In this way we are able to reduce (35) to a one-dimensional integral, which makes the task of numerically evaluating (35) much simpler. We remind the reader that the $C_P^{(J)}$ term in turn must be integrated over time-to-maturity so that altogether the jump term would in this case involve the evaluation of a double integral.

4. Limit of the Early Exercise Boundary at Expiry

In order to implement numerical schemes we need to know the value of the free boundary just prior to expiry, at $\tau = 0^+$. Existing literature (e.g. Amin (1993) and Carr & Hirsa

[6]There the dependence is sequential, that is first one solves for the free boundary which then feeds into an integral expression for the option price

(2003)) simply assumes that this limit is identical to the the corresponding pure diffusion case. Here however we show that this limit is in fact more subtle. We derive the limit by analysing the inhomogeneous term in the IPDE (11), and find that the presence of jumps does in fact have an impact on the early exercise boundary at expiry. This difference can be expressed analytically, as stated in Proposition 4.1. Here we provide a means of deriving the limit of $a(\tau)$ as $\tau \to 0^+$. This derivation is based on the analysis of Wilmott, Dewynne & Howison (1993) who, for the pure diffusion American call, demonstrate how to determine the limit of the early exercise boundary by performing a local analysis of the PDE for small time to maturity. The simple, intuitive method used here is taken from Chiarella, Kucera & Ziogas (2004) who demonstrate that the approach of Wilmott et al. (1993) is equivalent to setting the inhomogeneous term in Jamshidian's (1992) form for the PDE to zero, setting $\tau = 0$, $S = a(0^+)$, and solving for the free boundary.

Proposition 4.1. *The limit of the early exercise boundary, $a(\tau)$, as $\tau \to 0^+$ is given by*

$$a(0^+) = K \max\left(1, \frac{r + \lambda \int_0^{K/a(0^+)} G(Y)dY}{q + \lambda \int_0^{K/a(0^+)} YG(Y)dY}\right). \tag{36}$$

Proof:

Referring to the inhomogeneous IPDE (11), the inhomogeneous term of interest is[7]

$$H(S - a(\tau)) \left\{ qS - rK - \lambda \int_0^\infty [C(SY, \tau) - (SY - K)]G(Y)dY \right\}. \tag{37}$$

Setting the term in braces in (37) equal to zero and evaluating at $\tau = 0$ with $S = a(0^+)$ we have

$$qa(0^+) - rK - \lambda \int_0^\infty [C(a(0^+)Y, 0) - (a(0^+)Y - K)]G(Y)dY = 0. \tag{38}$$

Given that $C(S, 0) = \max(S - K, 0)$, equation (38) becomes

$$qa(0^+) - rK - \lambda \int_0^\infty [\max(a(0^+)Y - K, 0) - (a(0^+)Y - K)]G(Y)dY = 0. \tag{39}$$

[7]Note that since $C(SY, \tau) = SY - K$ for $Y \geq a(\tau)/S$, we have

$$\int_0^{a(\tau)/S} [C(SY, \tau) - (SY - K)]dY = \int_0^\infty [C(SY, \tau) - (SY - K)]dY.$$

Since the integral term is zero for $Y \geq K/a(0^+)$, we have

$$qa(0^+) - rK + \lambda \int_0^{K/a(0^+)} (a(0^+)Y - K)G(Y)dY = 0, \tag{40}$$

which we can rearrange to give

$$a(0^+) = K \frac{r + \lambda \int_0^{K/a(0^+)} G(Y)dY}{q + \lambda \int_0^{K/a(0^+)} YG(Y)dY}. \tag{41}$$

Finally, by noting that $a(\tau) \geq K$ must hold[8] for all $\tau \geq 0$, we arrive at the result.

□

An alternative approach to the derivation of the limit of $a(\tau)$ is given by Kim (1990), in which he takes the limit of the integral equation for the free boundary as $\tau \to 0^+$. This approach is more involved than the one presented here, but we have verified that this approach leads to the same result. We refer the reader to Chiarella & Ziogas (2006) for further details.

It is worthwhile to observe that when $\lambda = 0$ equation (36) simplifies to the limit derived by Kim (1990) for the pure diffusion American call free boundary. Note that (36) is an implicit expression for $a(0^+)$, but it can be solved quickly and accurately using standard root-finding techniques. Furthermore, as $q \to 0$ the solution to the implicit part of equation (36) increases without bound. Thus when $q = 0$, $a(0^+)$ becomes infinite, and we observe the well-known property that it is never optimal to exercise an American call option early in the absence of dividends.

Before concluding this section, we shall take a closer look at the properties of equation (36), specifically with a view to better understanding the solution to

$$a(0^+) = f(a(0^+)), \tag{42}$$

where

$$f(a(0^+)) \equiv K \frac{r + \lambda \int_0^{K/a(0^+)} G(Y)dY}{q + \lambda \int_0^{K/a(0^+)} YG(Y)dY}.$$

[8] This is true because it is never optimal to exercise a call option if $S < K$.

Once (42) is solved, then the max( ) operator can be applied. Since the value of the underlying is always non-negative, we must consider the domain $a(0^+) \geq 0$ when finding the solution to (42). It is not possible to provide a simple, explicit summary of the behaviour of (42) for various values of $a(0^+)$, because the integral terms[9] depend upon $a(0^+)$, and the function $f(a(0^+))$ involves the parameters r, q and λ, as well as the jump-size density $G(Y)$.

Firstly, we see that is is simple to evaluate $f(a(0^+))$ at the limits of the domain. Specifically, we can show that

$$f(0) = K\frac{r+\lambda}{q+\lambda(k+1)} \geq 0, \tag{43}$$

and

$$\lim_{a(0^+)\to\infty} f(a(0^+)) \equiv f(\infty) = K\frac{r}{q}. \tag{44}$$

Thus for $f(a(0^+))$ to be finite at each extremity of the domain, it is sufficient that we have $q > 0$. In this case, it is clear that $f(a(0^+))$ is continuous, and (42) will have at least one solution. Since $a(0^+)$ appears only in the limits of the integral terms over the density $G(Y)$ within $f(a(0^+))$, we can safely claim that the behaviour of $f(a(0^+))$ with respect to $a(0^+)$ will be bounded by the behaviour of $G(Y)$. Further exploration appears difficult without specifying the form of $G(Y)$, and as such we provide a more detailed analysis in Section 5.

5. American Call with Log-Normal Jumps

Before we begin exploring a numerical solution method for the integral equation system (31)-(32), we shall consider a specific example for the jump-size density, $G(Y)$. Here we consider a log-normal distribution for the jump sizes, Y, in accordance with the original model of Merton (1976). The probability density function for Y is given by

$$G(Y) = \frac{1}{Y\delta\sqrt{2\pi}} \exp\left\{-\frac{(\ln Y - (\gamma - \delta^2/2))^2}{2\delta^2}\right\}, \tag{45}$$

where we set $\gamma \equiv \ln(1+k)$, and δ^2 is the variance of $\ln Y$. Furthermore we note that for this choice of $G(Y)$ we have $\mathbb{E}_{\mathbb{Q}}[Y] = e^{\gamma}$.

[9]While we note that these integral terms are expectations over the jump-size density $G(Y)$, this does not aid us when trying to provide a general analysis of $f(a(0^+))$.

Proposition 5.1. *In the case where $G(Y)$ is given by equation (45), the integral equation for $C(S,\tau)$ in (31) becomes*

$$
\begin{aligned}
C(S,\tau) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} C_{BS}[S,K,K,r_n(\tau),q,\tau,v_n^2(\tau)], \\
&\quad + \sum_{n=0}^{\infty} \Bigg\{ \int_0^\tau \frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!} \\
&\qquad \times \Bigg[C_P^{(D)}[S,K,a(\xi),r,r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi)] \\
&\qquad -\lambda C_P^{(J)}[S,K,a(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi);C(\cdot,\xi)] \Bigg] d\xi \Bigg\},
\end{aligned} \tag{46}
$$

where $\lambda' = \lambda(1+k)$, $r_n(\tau) = r - \lambda k + n\gamma/\tau$ and $v_n^2(\tau) = \sigma^2 + n\delta^2/\tau$.

Proof: Refer to Appendix 4.

□

While equation (46) has incorporated the distribution for Y, the last term, which involves a double-integral, may be further simplified before attempting to implement (46) numerically.

Proposition 5.2. *By use of Proposition 3.6, the term $C_P^{(J)}$ in Proposition 5.1 can be expressed as*

$$
\begin{aligned}
&C_P^{(J)}[S,K,a(\xi),r_n(\tau),q,\tau,v_n^2(\tau);C(\cdot,\xi)] \\
&= e^{-r_n(\tau)\tau} \int_0^1 [C(a(\xi)z,\xi) - (a(\xi)z-K)]\kappa(S/a(\xi),z,r_{n+1}(\tau),q,\tau,v_{n+1}^2(\tau)) \\
&\qquad \times \mathcal{N}[D(S/a(\xi),z,r_n(\tau),q,v_n(\tau),v_{n+1}(\tau),\tau,\gamma,\delta)]dz,
\end{aligned} \tag{47}
$$

where

$$
\begin{aligned}
&D(S/a(\xi),z,r_n(\tau),q,v_n(\tau),v_{n+1}(\tau),\tau,\gamma,\delta) \\
&\qquad \equiv \frac{\delta^2 \ln \frac{S}{a(\xi)z} + \left[(\ln z)\, v_{n+1}^2(\tau) + \delta^2[r_n(\tau)-q] - \gamma v_n^2(\tau)\right]\tau}{v_n(\tau)v_{n+1}(\tau)\delta\tau}.
\end{aligned} \tag{48}
$$

Proof: Refer to Appendix 5.

□

We draw the reader's attention to the fact that in the form (47) the $C_P^{(J)}$ term in (46) now only involves a single integral, which will result in a considerable saving in computational effort.

By use of (32), the integral equation for the early exercise boundary, $a(\tau)$, in the case of log-normal jump sizes, is given by

$$\begin{aligned} a(\tau) - K \quad = \quad & \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} C_{BS}[a(\tau), K, K, r_n(\tau), q, \tau, v_n^2(\tau)], \\ & + \sum_{n=0}^{\infty} \Bigg\{ \int_0^\tau \frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!} \\ & \times \Bigg[C_P^{(D)}[a(\tau), K, a(\xi), r, r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi)] \\ & -\lambda C_P^{(J)}[a(\tau), K, a(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi); C(\cdot,\xi)] \Bigg] d\xi \Bigg\}, \end{aligned} \tag{49}$$

where $C_P^{(D)}$ and $C_P^{(J)}$ are given by (27) and (47) respectively.

5.1. **Delta for the American Call.** We now provide one further result regarding the delta of the American call option, $\Delta_C(S,\tau)$. This quantity is obviously important for hedging purposes, but it is also required by the numerical algorithm we consider in Section 6.

By differentiating (46) with respect to S, we find that

$$\begin{aligned} \Delta_C(S,\tau) \quad = \quad & \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} \Delta_{BS}[S, K, K, r_n(\tau), q, \tau, v_n^2(\tau)], \\ & + \sum_{n=0}^{\infty} \Bigg\{ \int_0^\tau \frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!} \\ & \times \Bigg[\Delta_P^{(D)}[S, K, a(\xi), r, r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi)] \\ & -\lambda \Delta_P^{(J)}[S, K, a(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi); C(\cdot,\xi)] \Bigg] d\xi \Bigg\}, \end{aligned} \tag{50}$$

where

$$\Delta_P^{(D)}[S,K,a(\xi),r,r_n(\tau),q,\tau,v_n^2(\tau)]$$
$$= e^{-q\tau}\Bigg\{\frac{1}{v_n(\tau)\sqrt{\tau}}\mathcal{N}'\left[d_1\left(S,a(\xi),r_n(\tau),q,\tau,v_n^2(\tau)\right)\right](q-r)$$
$$+ q\mathcal{N}\left[d_1\left(S,a(\xi),r_n(\tau),q,\tau,v_n^2(\tau)\right)\right]\Bigg\}, \quad (51)$$

$$\Delta_P^{(J)}[S,K,a(\xi),r_n(\tau),q,\tau,v_n^2(\tau);C(\cdot,\xi)]$$
$$= e^{-r_n(\tau)\tau}\int_0^1 \frac{[C(a(\xi)z,\xi)-(a(\xi)z-K)]}{Sv_n(\tau)\sqrt{\tau}}\kappa(S/a(\xi),z,r_{n+1}(\tau),q,\tau,v_{n+1}^2(\tau))$$
$$\times\Bigg[\frac{\delta}{v_{n+1}(\tau)\sqrt{\tau}}\mathcal{N}'[D(S/a(\xi),z,r_n(\tau),q,v_n(\tau),v_{n+1}(\tau),\tau,\gamma,\delta)]$$
$$- d_2(S/a(\xi),z,r_{n+1}(\tau),q,\tau,v_{n+1}^2(\tau))\Bigg]dz, \quad (52)$$

and we note that $\mathcal{N}'[x] = \exp(-x^2/2)/\sqrt{2\pi}$. Once we have found the price and free boundary for the American call option, we can readily evaluate (50) numerically to find delta.

5.2. **Properties of the Free Boundary at Expiry.** Since we are now considering a specific form for $G(Y)$, we return to the topic of analysing the behaviour of the early exercise boundary, $a(\tau)$, as $\tau \to 0^+$. Firstly we evaluate (36) for the log-normal density $G(Y)$.

Proposition 5.3. *When $G(Y)$ is given by the log-normal density (45), the limit of the early exercise boundary $a(\tau)$ as $\tau \to 0^+$ becomes*

$$a(0^+) = K\max\left(1, \frac{r+\lambda\mathcal{N}[\{\ln K/a(0^+)-(\gamma-\frac{\delta^2}{2})\}/\delta]}{q+\lambda'\mathcal{N}[\{\ln K/a(0^+)-(\gamma+\frac{\delta^2}{2})\}/\delta]}\right). \quad (53)$$

Proof: Follows by evaluating the integral terms in (36) using $G(Y)$ from (45).

□

To develop an understanding of the case where $a(0^+) > K$, we shall undertake some numerical explorations of the equation

$$b(0^+) = f(b(0^+)), \tag{54}$$

where

$$f(b(0^+)) \equiv \frac{r + \lambda \mathcal{N}[\{-\ln b(0^+) - (\gamma - \frac{\delta^2}{2})\}/\delta]}{q + \lambda' \mathcal{N}[\{-\ln b(0^+) - (\gamma + \frac{\delta^2}{2})\})/\delta]},$$

and we recall that $b(\tau) = a(\tau)/K$. It is not possible to provide a simple, explicit summary of the behaviour of (54) for various values of $b(0^+)$ because the cumulative normal density functions depend upon $b(0^+)$, and the function $f(b(0^+))$ involves the parameters r, q, λ, γ and δ, all of which have a significant impact on the value of $f(b(0^+))$. Nevertheless, we can use numerical examples to offer some additional insight into the nature of (54).

For log-normal jump-sizes we can show that

$$f(0) = \frac{r + \lambda}{q + \lambda e^{\gamma}} \geq 0, \tag{55}$$

and

$$\lim_{b(0^+) \to \infty} f(b(0^+)) \equiv f(\infty) = \frac{r}{q}. \tag{56}$$

When $q > 0$ it is clear that $f(b(0^+))$ is continuous, and (54) will have at least one solution. We can demonstrate by example that $f(b(0^+))$ is not monotonic, nor is it strictly bounded by the end values (55)-(56). This makes it difficult to prove that for $q > 0$ equation (54) has at most one solution. Since $b(0^+)$ appears only inside cumulative normal functions within $f(b(0^+))$, the behaviour of $f(b(0^+))$ with respect to $b(0^+)$ will be bounded by the behaviour of $\mathcal{N}(\ln x)$. In particular, we recall that $0 \leq \mathcal{N}(\ln x) \leq 1$, and that $\mathcal{N}(\ln x)$ is a smooth, continuous function of x, where $x \geq 0$. From this we postulate that the function $f(b(0^+))$ will not display any oscillating features within the domain under consideration.

****Insert Figure 3 here****

To provide numerical evidence in support of our claims regarding equation (54), we now present a plot of (55)-(56), which is typical of what we have obtained for a range of empirically relevant parameter values. Setting $\lambda = 1$, $\gamma = 0$ and $\delta = 0.2$, we plot the functions $y = b(0^+)$ and $y = f(b(0^+))$ for various values of r and q, as shown in Figure 3. When $r = 0.05$ and $q = 0.03$, we can see that $f(0) < f(\infty)$. On the other hand, when $r = 0.03$ and $q = 0.05$, we now have $f(0) > f(\infty)$. In both cases it is clear that $f(b(0^+))$ is not bounded by these endpoint values, and we can see that the relative values of r and q directly influence the values of $f(0)$ and $f(\infty)$.

Changing the values of the jump-parameters λ, γ will vary the value of $f(0)$. In addition changes to λ, γ and δ will resize the curve $f(b(0)^+)$, although the basic shape remains unchanged. We make this claim based on further numerical examples, which can be found in Chiarella & Ziogas (2006).

****Insert Figure 4 here****

The last and most important scenario we consider here is when $q = 0$. In this case, $f(\infty)$ is no longer finite, instead increasing without bound as $b(0^+) \to \infty$. Figure 4 demonstrates the behaviour of $f(b(0^+))$ with $q = 0$ for a different selection of parameter values. It is clear from the plot that there is no solution for $b(0^+) = f(b(0^+))$. Furthermore, the only way that equation (54) will be satisfied when $q = 0$ is by taking the limit as $b(0^+) \to \infty$, in which case both sides of (54) will increase without bound. This reflects in yet another way the fact that when $q = 0$, the free boundary at $\tau = 0^+$ becomes infinite, and it is never optimal to exercise an American call early in the absence of dividends.

6. Numerical Implementation

We now provide a numerical scheme with which to evaluate the linked integral equation system for the option price and free boundary formed by (47) and (49). The proposed method is an extension of the quadrature scheme used by Kallast & Kivinukk (2003) for the pure diffusion case. Here we focus on the adjustments that are needed to deal with the introduction of jumps in the dynamics for S. We firstly discretise the time to maturity variable, τ, into N equally spaced intervals of length h. Thus $\tau = ih$ for $i = 0, 1, 2, ..., N$, and $h = T/N$. We denote the call price profile at time step i by $C(S, ih) = C_i(S)$, and similarly the free boundary at time step i by $a(ih) = a_i$. Using a standard numerical technique that is applied to Volterra integral equations, we can solve the system (47)-(49) for increasing values of i, until eventually the entire free boundary and price profile are found. When calculating the infinite summations, we continue adding terms until the size of the Poisson coefficient for a given value of n is less than 10^{-20}. For the parameter values considered here, this typically results in the use of around 30 terms for the summations. In order to start the algorithm we require the initial value of $C_0(S)$, which is simply the payoff function, and also a_0, where $a_0 \equiv a(0^+)$, which is given by equation (53).

Since the integral term in (47) depends upon $C(S, \tau)$, an approximation will be needed for $C_i(S)$. At each time step we found that a suitable approximation is given by $C_{i-1}(S)$, which is simply the American call price at the previous time step. The price at the $(i-1)th$ time step is calculated for a suitably large number of evenly-spaced S values. Here we use 25 points in the range $0 \leq S \leq 250$. All necessary interpolation is conducted using cubic splines fitted locally through 7 values of $C_{i-1}(S)$. We then use Newton's method to solve for the early exercise boundary, as in Kallast & Kivinukk (2003), with two necessary additions. The first addition addresses the evaluation of the inner integral (47) over the interval $[0, 1]$. This is computed using Gaussian integration of moments, with parameter $\alpha = -0.5$. Full details for this Gauss-quadrature scheme can be found in Abramowitz & Stegun (1970) (chapter 25, p.921). The second addition relates to finding the derivative of (49) with respect to $a(\tau)$ for use in Newton's method. This is given by (50) for $\Delta_C(S, \tau)$, evaluated at $S = a(\tau)$. Since it is difficult to determine the limit of the integrands in (50) as $\xi \to \tau$, we resolve this by taking the limits as $\xi \to \tau$ for the option price integrands in (46) and differentiating these with respect to $a(\tau)$, as is done by Kallast & Kivinukk (2003). These limits are all finite, including the new limit required for the jump-related integral term, and the required derivatives are easily determined. Since we need to evaluate the expression (50) for $\Delta_C(S, \tau)$ for use in Newton's method, there is no significant additional computation time required to evaluate the American call delta once the free boundary has been estimated.

Having determined the discretised forms for the price and delta of $C_i(S)$, we then use Newton's method to solve for a_i. Before proceeding to the next time step, we use a_i to calculate a new approximation for $C_i(S)$, which is required when evaluating the double integral term at all subsequent time steps. This update for C_i is essential to ensure that the estimated free boundary remains monotonic. Note that as the value of i increases, the computational burden will also increase at a "faster than linear" rate, since the integration at step i depends on all values of a_j and $C_j(S)$ for $j = 0, 1, 2, ...i - 1$.

It should be noted that the proposed numerical scheme does not involve any iterations with respect to the approximation of the integral term. It is possible to improve the accuracy of the algorithm by updating the approximation for the integral term at the ith time step using the most recently computed estimates for $C_i(S)$ and a_i. In practice we have found that such an iteration does not add significantly to the accuracy of the results (up to the order of accuracy under consideration in Section 7), and that computation time is at least doubled by the introduction of the iteration process. Thus for the purposes of these experiments, we have chosen not to iterate with respect to the integral term approximation.

To explore the efficiency of the proposed numerical integration method, we compare it with two alternative numerical methods. This first method involves a finite difference solution for the IPDE (5). We apply the Crank-Nicolson scheme to all terms except for the integral. We initially estimate the integral term by approximating $C_i(S)$ with the explicit approximation $C_{i-1}(S)$, as in Carr & Hirsa (2003). We then evaluate the integral using the Hermite Gauss-quadrature scheme (which can be found in Abramowitz & Stegun (1970)). The resulting tridiagonal matrix is inverted using LU-decomposition, and the early exercise condition is then applied to the solution at each time step. An evenly spaced grid is used, and the free boundary is estimated at each time step using cubic spline interpolation of the price profile, combined with the bisection method.

To improve the accuracy of the Crank-Nicolson solution, we use a two-step procedure at each time step. After determining an initial solution at time step i, denoted here as $C_i^{(1)}(S)$, using the estimate of $C_{i-1}(S)$ in the integral term, we then find an updated estimate by repeating the solution process, now using the $C_i^{(1)}(S)$ values in the integral estimate. This provides a second approximation for the option price, which we denote by$C_i^{(2)}(S)$. In practice we find that $C_i^{(1)}$ typically converges from below, whilst $C_i^{(2)}$ converges from above. Thus we take $C_i(S) = C_i^{(1)}(S)/2 + C_i^{(2)}(S)/2$ for the final Crank-Nicolson solution. This appears to greatly improve the convergence rate for the Crank-Nicolson scheme, although we do not report details of the convergence of $C^{(1)}$ and $C^{(2)}$ here[10]. In all cases we set the S domain to be $0 \leq S \leq 250$. We also calculate the American call delta by taking a central difference approximation using the price estimates.

The second method we consider is the method of lines approach as presented by Meyer (1998). Meyer only considers discrete jump-size distributions. Here we extend this method to allow for a continuous jump-size density, $G(Y)$, again evaluating the integral term using a Hermite Gauss-quadrature scheme, combined with local cubic spline interpolation of the price profile at each iteration. The method of lines readily provides estimates for both the price and delta for the American call option, without the need for additional computation. Unlike the Crank-Nicolson scheme, the method of lines solution is allowed to iterate until the largest observed change in the option price profile is less than 1×10^{-7}. The scheme we use is second-order accurate in time, and a first-order scheme is applied for the first three time steps.

7. Numerical Results

To analyse the efficiency of the numerical integration method, we compute the price and delta of an American call option with 6-months to maturity, and a strike price of

[10]Briani, Chioma & Natalini (2004) note that it is unclear how to select the stopping criteria when using iterative finite difference solutions for (5). Since we observe greater accuracy by using the average of the first and second iteration results than using the second iteration alone, the averaging scheme we use here seems more efficient than using a stopping criteria that involves three or more iterations.

$K = 100$. The global variance[11] of returns, s^2, is set equal to 5.93%. The jump intensity is set to $\lambda = 1$, and the jump variance is $\delta^2 = 0.04$. We then consider six different parameter sets, specifically $\mathbb{E}_{\mathbb{Q}}[Y] = e^{\gamma}$ taking values of 0.95, 1.00 and 1.04, along with the combinations $r = 3\%$, $q = 5\%$, and $r = 5\%$, $q = 3\%$. Note that $\gamma > 0$ implies upward jumps on average, and $\gamma < 0$ implies downward jumps are expected. When $\gamma = 0$, the expected price change from a jump is zero. The diffusion coefficient, σ^2, is chosen such that the global variance was preserved for varying values of γ. Table 1 summarises the values of σ^2 used to ensure that the global variance was the same for each combination of γ and λ.

****Insert Table 1 here****

We compute the root mean square error (RMSE) using option prices and deltas with $S = 80, 90, 100, 110$ and 120. This is repeated for each of the six parameter sets, from which the average runtime and RMSE is then calculated. Note that in all cases the runtimes include the time required to compute the free boundary, price and delta for the American call. For the integration method we use 20 integration points for the Gauss-quadrature scheme, and consider a sequence of 10 different time step values, with $N = 10, 20, \ldots, 90, 100$.

For the Crank-Nicolson method the integral term is approximated using 50 integration points, and we again use 10 time step values, with $N = 50, 100, \ldots, 450, 500$. We set the number of space steps equal to double the number of time steps. Similarly, we use 50 integration points to estimate the integral term for the method of lines. Again we use 10 time step values, with $N = 50, 100, \ldots, 450, 500$, and the number of space steps is set to 5 times the number of time steps. The code for all methods is implemented using LAHEY$^{\text{TM}}$FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system.

In assessing the efficiency of the numerical integration method, we use a Crank-Nicolson solution with 10,000 time steps and 5,000 space steps for the true solution. Since the

[11]By the global variance we mean $\mathbb{E}[(dS/S - (\mu - \lambda k)dt)^2]$ calculated from equation (1) to be $\sigma^2 + \lambda[e^{2\gamma+\delta^2} - 2e^{\gamma} + 1]$ in the case of a log-normal jump density.

numerical integration scheme requires evaluation of the option delta as part of the solution, it is also of value to consider the efficiency with which delta is calculated. The true delta is estimated from the Crank-Nicolson solution using the central difference approximation for $(C_{i+1} - C_{i-1})/(S_{i+1} - S_{i-1})$. To demonstrate that this is a valid choice for the true solution, we also compute the prices and deltas using the method of lines approach with 1,000 time steps and 5,000 space steps. As mentioned previously, the method of lines computes the price, delta and early exercise boundary simultaneously. The prices are provided in Table 2, with the deltas given in Table 3. We also provide the root mean square differences (RMSD) between the two methods for each set of parameters, along with an average RMSD value. In both cases we find that for the values of S under consideration these methods are consistent to at least 4 decimal places, and thus we conclude that the Crank-Nicolson method using 10,000 time steps and 5,000 steps in the space variable provides a satisfactory estimate for the true price and delta.

****Insert Tables 2 and 3 here****

The relative efficiency for each method is shown in Figure 5 by comparing RMSE as a function of runtime, with Figure 5(a) showing the average RMSE error for the American call price, and Figure 5(b) displaying the same information for the delta. Note that the average runtimes for each discretisation level are the same in figures 5(a) and 5(b) since the price and delta were found using a single algorithm. Firstly, we find that for the parameters and discretisations considered, the numerical integration method consistently displays greater efficiency than the Crank-Nicolson scheme for the American call price, and furthermore, numerical integration is more efficient than Crank-Nicolson when computing delta for computation times of up to 50 seconds.

We find that the method of lines consistently outperforms the Crank-Nicolson scheme, and in the majority of cases, the delta values computed by the method of lines are more accurate than those obtained using numerical integration. When computing prices for runtimes of between 1 and 20 seconds, however, numerical integration is able to outperform the method of lines. We also note that the numerical integration algorithm has a slower rate of convergence for the delta relative to the other two methods. This could be attributed to cumulative numerical integration error as the number of time steps, and hence integration points, is increased. The rate of convergence for the numerical integration method is clearly better when computing the option price.

Thus from Figure 5 we conclude that of the three methods under consideration, the method of lines is consistently more accurate for runtimes beyond 20 seconds. We also find that the numerical integration method consistently outperforms the Crank-Nicolson scheme when computing the option price, and is competitive with Crank-Nicolson when computing the delta for runtimes of less than 50 seconds. Thus we can see that a simple extension of the numerical integration scheme presented by Kallast & Kivinukk (2003) to include log-normal jumps produces a numerical method that is comparable with both the method of lines and the Crank-Nicolson scheme, however there is room for improvement in the computation of delta, in particular when the number of time steps is increased beyond 40. We also note that since the magnitude of the RMSE is much larger for the option price, the best way to select the optimal numerical method is to maximise the pricing efficiency whilst being aware of the RMSE for the deltas. For example, for a runtime of 20 seconds the method of lines is more efficient than numerical integration, since both methods have similar pricing accuracy, but the method of lines offers a better delta estimate in this case. Alternatively, for a runtime of 5 seconds, numerical integration is the most efficient, since it provides better price estimates than the method of lines for the same accuracy in delta.

****Insert Figure 5 here****

8. Impact of Jumps on the Free Boundary and Option Prices

In this section we discuss the impact of jumps on the free boundary and price profile. First we present sample free boundary profiles for the American call option. In Figure 6 we consider the case where $r < q$, and in Figure 7 we set $r > q$. We again consider three different values of e^{γ} (the same values used to generate Figure 5), and compare the resulting boundaries with the pure diffusion case of $\lambda = 0$. The diffusion volatility σ was again adjusted in each case as detailed in Table 1. The most obvious feature of these results is the dramatic effect the presence of jumps has on the profile for the free boundary. Close to expiry, the free boundary with jumps is significantly larger than in the pure diffusion case. This follows from the increased probability of large price movements near expiry, made possible by the presence of jumps within the return dynamics. Thus the holder of the call is less likely to exercise near expiry under the jump-diffusion model to best minimise the potential costs from downward jumps.

As time to expiry increases, we see that the pure diffusion boundary increases more rapidly compared with the jump-diffusion examples, since the jump component becomes less dominant within the underlying dynamics for large time intervals. While jumps are more likely to be observed over longer time intervals, they become less influential overall, since there are sufficient opportunities for the jumps to be reversed, either by jumps in the opposite direction or through the diffusion term. Therefore when far from maturity the holder of the call is more likely to exercise early under jump-diffusion than in the pure diffusion case. These findings coincide with those of Amin (1993), who also notes that for a sufficiently large time to expiry, the probability density for the underlying converges under both models, such that there is no clear distinction between pure diffusion and jump-diffusion.

We also point out that Amin does not provide any formal evidence relating to the limit of the free boundary at expiry, although his numerical results are consistent with the limiting value given here by equation (53). In particular, our Figure 6 is closely related to Figure 6 in Amin (1993). We find that, for the parameter values used by Amin, the limit (53) correctly identifies the value of the free boundary at $\tau = 0^+$, and thus our limit result for $a(\tau)$ is in keeping with the numerical results of Amin.

One further observation we can make from Figure 6 is the impact of the value of γ on the free boundary. As γ increases, the value of the early exercise boundary decreases. This is attributable to the potential for the option holder to incur a rebalancing cost when the price jumps from the stopping region back down into the continuation region. Recall that $\gamma > 0$ implies upward jumps on average, thus making the expected cost of downward jumps quite small. When $\gamma < 0$, we expect downward jumps on average, and the holder will therefore require that S be even larger before exercising the call early.

****Insert Figures 6 and 7 here****

Finally, we demonstrate the impact of jumps on the American call price, relative to the pure diffusion case. In figures 8 and 9 we plot the price differences between the pure diffusion and jump-diffusion American call price for the same three values of e^{γ}. All other parameter values are the same as those used in generating the free boundaries in figures 6 and 7. Positive (negative) differences indicate that the jump-diffusion price is greater than (less than) the pure diffusion price. Figure 8 shows the results for $r < q$ and 9 uses $r > q$.

While the shapes of the plots vary somewhat depending on the relative values of r and q, this mostly occurs deep in-the-money, and is related to the impact that r and q have on the value of the free boundary. In general we observe that when the call is at-the-money ($K = 100$) or close to at-the-money, the jump-diffusion price is consistently less than the pure diffusion price. Furthermore, when the call is deep out-of-the-money, the jump-diffusion price is generally larger than for pure diffusion.

****Insert Figures 8 and 9 here****

For deep in-the-money American calls, there are a number of factors that affect the price differences. First we note that the early exercise feature will always reduce this difference to zero for large values of S. When $\gamma < 0$, the difference is mostly positive for S values just below the free boundary, while the opposite is true when $\gamma = 0$ and $\gamma > 0$. For the European call we would expect to see greater prices under jump-diffusion for large values of S, but for American options this depends upon the value of γ, at least in part. Since $\gamma < 0$ indicates downward jumps are expected on average, this will increase the likelihood of the option holder incurring rebalancing costs, and could provide some of the reason for the increased call value relative to the pure diffusion case. Otherwise, the early exercise feature dominates the price profile for large values of S, and thus we do not observe the same behaviour as we would for European calls. Nevertheless, the leptokurtic features introduced into the return dynamics for S are clearly represented by the increased call prices out-of-the-money, and the reduced prices in a region around the strike. This implies that the jump-diffusion model is able to reflect the basic volatility smile structure observed in market option prices. We have elected not to demonstrate this result using Black-Scholes implied volatilities, as this procedure only makes theoretical sense in the case of European options. It is clear, however, from the relative price differences that the jump-diffusion dynamics have the potential to capture volatility smile behaviour.

9. Conclusion

This paper explores the pricing of American call options in the case where the underlying asset follows a jump-diffusion process, as originally proposed by Merton (1976). We use the approach of Jamshidian (1992) to find an inhomogeneous integro-partial differential equation (IPDE) for the American call price in an unrestricted domain, which we then solve using Fourier transforms, extending this very useful solution methodology to the jump-diffusion setting. Furthermore the approach advocated here has the advantage of being readily extended to a variety of underlying asset price dynamics, such as stochastic volatility and stochastic interest rates, as well as handling a variety of different payoff structures, such as puts, calls, butterflies, spread options and max-options.

This paper has also made two significant contributions regarding the integral equation system for the American call price and free boundary. Firstly, we derive the limit of the free boundary as the time to expiry tends to zero. In particular, we show that the limit is clearly dependent upon the jump intensity and jump-size distribution, a fact not reported in existing literature on American option pricing with jumps. This limit is needed when solving numerically for the free boundary, since it provides the exact starting point for the time-stepping algorithm. The second contribution is to express the integral term for the expected costs incurred from downward jumps in a form that is more tractable for numerical integration purposes. In particular, in the case where the jump sizes are log-normally distributed, we are able to reduce the term from a triple integral to a double integral involving the cumulative normal density, resulting in a task far easier to implement with high levels of accuracy.

The other main result of this paper concerns the use of numerical integration to solve for the free boundary, price and delta of the American call with jumps. We propose a quadrature integration scheme based on the one used for the pure diffusion case by Kallast & Kivinukk (2003). We address the difficulty of dealing with the double integral term, and provide a fast, accurate means of evaluating this, along with a means to overcome the implicit dependency of the integral equation on the unknown option price. We compare the numerical integration solution with a suitable Crank-Nicolson scheme, and find that the proposed numerical integration is often more efficient than the finite difference approach, for computing the option price and delta together to the same level of accuracy. The improved efficiency is consistently apparent for the option price, and most prominent for the option delta when large time step sizes are used. We also compare the integration scheme with the method of lines approach by Meyer (1998). We find that the integration scheme can outperform the method of lines for large time steps, although the increased efficiency is far less prevalent in this case.

We use the integration scheme to demonstrate the impact of jumps on the free boundary of the American call, relative to the pure diffusion case with equivalent global volatility. The results presented here correspond with those obtained by Amin (1993) for the American put using tree methods. In particular, option holders are less likely to exercise early close to expiry, and more likely to exercise further from expiry when jumps are introduced. The relative values of time to expiry where these differences occur depends upon the jump parameter value, and in particular we show how different values for the mean jump-size impact on the free boundary. We observe, as does Amin, that the slope of the free boundary at maturity is not infinite, unlike in the pure diffusion case when it has infinite slope. We leave to future research the task of exploring more fully the behaviour of the free boundary close to maturity, the most likely approach being the small time expansions of the type used by Wilmott et al. (1993), Kuske & Keller (1998) and Chen & Chadam (2007). We also demonstrate the price differences between jump-diffusion and pure diffusion American calls, and as expected, find that the call premium is smaller in a region around the strike price when jumps are present, but larger when the option is deep out-of-the-money. For deep in-the-money options, the early exercise feature causes the American call price to rapidly tend towards the payoff function.

While the numerical results presented consider only log-normal jump sizes, the numerical integration approach is readily applicable to a range of jump size distributions, such as that proposed by Kou (2002). One avenue for future research is to explore these alternatives, and in particular observe what difficulties are encountered when trying to simplify and evaluate the triple integral term for other jump size densities. We have only considered the call option here, but the Fourier transform approach allows a broader range of payoff functions to be explored, in particular problems with more complex stopping and continuation regions, such as those that arise with American option portfolios consisting of strangles and butterflies. The numerical algorithm presented has only been compared with the Crank-Nicolson scheme and the method of lines. There are numerous other numerical methods that have not been considered, such as the tree methods of Amin (1993) and Broadie & Yamamoto (2003), and various finite difference scheme implementations, including Andersen & Andreasen (2000) and d'Halluin, Forsyth & Labahn (2004). A detailed analysis of the relative efficiency of these various numerical methods is planned as a future research project.

Appendix 1. Proof of Proposition 3.1 – Deriving the Inhomogeneous IPDE

To derive the required inhomogeneous term, we evaluate (5) in the region $S \geq a(\tau)$ when $C(S,\tau) = S - K$. Thus we consider

$$\begin{aligned}\Psi(S,\tau) \quad &\equiv \quad H(S-a(\tau))\left\{\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q-\lambda k)S\frac{\partial C}{\partial S} - rC - \frac{\partial C}{\partial \tau}\right. \\ &\qquad\qquad \left. +\lambda \int_0^\infty [C(SY,\tau) - C(S,\tau)]G(Y)dY\right\} \\ &= \quad H(S-a(\tau))\left\{K(r+\lambda) - S(q+\lambda[k+1]) + \lambda\int_0^\infty C(SY,\tau)G(Y)dY\right\}.\end{aligned}$$

Recalling that $C(SY,\tau) = SY - K$ when $SY \geq a(\tau)$, and $k = \mathbb{E}_{\mathbb{Q}}[Y-1]$, the expression for $\Psi(S,\tau)$ becomes

$$\Psi(S,\tau) = H(S-a(\tau))\left\{rK - qS + \lambda\int_0^{a(\tau)/S}[C(SY,\tau) - (SY-K)]G(Y)dY\right\}. \quad (57)$$

Since (57) is the value of the right-hand side of IPDE (5) when $S \geq a(\tau)$, the IPDE can be rewritten as given in equation (11) of Proposition 3.1.

Appendix 2. Proof of Proposition 3.2

When taking the Fourier transform of (13), we note that $V(x,\tau)$ and $\partial(x,\tau)/\partial x$ do not approach zero as $x \to \infty$. One means of dealing with this difficulty, suggested by Carr & Madan (1999) and Lee (2004), involves introducing a damping function of the form $e^{-\alpha x}$ for some positive constant α, and instead apply the transform to the dampened option price $U(x,\tau) = e^{-\alpha x}V(x,\tau)$, which does tend to zero, along with $\partial U(x,\tau)/\partial x$, as $x \to \infty$. The desired function $V(x,\tau)$ can be readily recovered after the solution in transform space has been inverted. Another approach is given by Lewis (2000), who proves that the Fourier transform is still valid when solving for $V(x,\tau)$ in this case, although one must instead take the complex Fourier transform in a strip of the complex plane. Regardless of which approach is used to make the Fourier transform applicable, it turns out that both methods are equivalent to simply assuming that $V(x,\tau)$ and $\partial V(x,\tau)/\partial x$ tend to zero as $x \to \infty$, and applying the standard transform accordingly. Thus in order to simplify the technical discussion, we shall simply apply this assumption and suppress the finer details involved.

For the inhomogeneous term, we have $\hat{F}_J(\eta,\tau) \equiv \mathcal{F}\left\{F_J(x,\tau)\right\}$, and the only term that needs to be evaluated is the one involving the integral, namely

$$\mathcal{F}\left\{\int_0^\infty V(x+\ln Y,\tau)G(Y)dY\right\} = \int_{-\infty}^\infty e^{i\eta x}\int_0^\infty V(x+\ln Y,\tau)G(Y)dY\,dx. \tag{58}$$

Using the change of variable $z = x + \ln Y$, equation (58) becomes

$$\mathcal{F}\left\{\int_0^\infty V(x+\ln Y,\tau)G(Y)d(Y)\right\} = A(\eta)\hat{V}(\eta,\tau),$$

where $A(\eta)$ is defined in (21).

Hence, our IPDE is transformed into the ODE

$$\frac{\partial \hat{V}}{\partial \tau} + \left[\frac{\sigma^2\eta^2}{2} + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right]\hat{V} = \hat{F}_J(\eta,\tau),$$

the solution of which is given by (22).

Appendix 3. Derivation of the American Call Integral Equations

A3.1. **Proof of Proposition 3.3.** Consider the function $V_E(x,\tau)$, given by

$$V_E(x,\tau) = \mathcal{F}^{-1}\left\{\hat{V}(\eta,0)e^{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)\tau}\right\}. \tag{59}$$

To evaluate this inversion, recall the convolution theorem for Fourier transforms given by

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f((x-u),\tau)g(u,\tau)du\right\} = \hat{F}(\eta,\tau)\hat{G}(\eta,\tau), \tag{60}$$

where $\hat{F}$ and $\hat{G}$ are the Fourier transforms, with respect to x, of $f(x,\tau)$ and $g(x,\tau)$ respectively. If we let $\hat{F}(\eta,\tau) = \exp(-\left(\frac{1}{2}\sigma^2\eta^2 + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right)\tau)$, then $f(x,\tau)$ is given by

$$f(x,\tau) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda\tau A(\eta)}e^{-\left[\frac{1}{2}\sigma^2\eta^2\tau+i[\phi\tau+x]\eta+(r+\lambda)\tau\right]}d\eta.$$

Furthermore, let $\hat{G}(\eta,\tau) = \hat{V}(\eta,0)$, then $g(x,\tau)$ will simply be the payoff function, $g(x,0) = \max(e^x - 1, 0)$.

Using a Taylor series expansion for $e^{\lambda\tau A(\eta)}$, the expression for $f(x,\tau)$ becomes

$$f(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n A(\eta)^n}{n!}e^{-\left[\frac{1}{2}\sigma^2\eta^2\tau+i[\phi\tau+x]\eta+(r+\lambda)\tau\right]}d\eta. \tag{61}$$

Note that by definition

$$\begin{aligned}
A(\eta)^n &= \left\{\int_0^{\infty} e^{-i\eta\ln Y}G(Y)dY\right\}^n \\
&= \int_0^{\infty} e^{-i\eta\ln Y_1}G(Y_1)dY_1 ... \int_0^{\infty} e^{-i\eta\ln Y_n}G(Y_n)dY_n \\
&= \int_0^{\infty}\int_0^{\infty} ... \int_0^{\infty} G(Y_1)G(Y_2)...G(Y_n)e^{-i\eta\ln(Y_1Y_2...Y_n)}dY_1dY_2...dY_n, \\
&= \mathbb{E}_{\mathbb{Q}}^{(n)}\left\{e^{-i\eta\ln X_n}\right\},
\end{aligned}$$

where

$$\mathbb{E}_{\mathbb{Q}}^{(n)}\{(\cdot)\} = \int_0^{\infty}\int_0^{\infty} ... \int_0^{\infty} (\cdot)G(Y_1)G(Y_2)...G(Y_n)dY_1dY_2...dY_n,$$

with $\mathbb{E}_{\mathbb{Q}}^{(0)}\{(\cdot)\} \equiv (\cdot)$, and $X_n \equiv Y_1 Y_2 ... Y_n$, with $X_0 \equiv 1$.

Substituting for $A(\eta)^n$ from (21), $f(x,\tau)$ becomes

$$f(x,\tau) = \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(r+\lambda)\tau} e^{-\left(\frac{1}{2}\sigma^2\eta^2\tau + i[\phi\tau + x + \ln X_n]\eta\right)} d\eta \right\}.$$

Recalling the result that

$$\int_{-\infty}^{\infty} e^{-p\xi^2 - q\xi} d\xi = \sqrt{\frac{\pi}{p}} e^{q^2/4p}, \tag{62}$$

we finally have the result that

$$\mathcal{F}^{-1}\{\hat{F}(\eta,\tau)\} = f(x,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \exp\left\{ -\frac{[x + \ln X_n + \phi\tau]^2}{2\sigma^2\tau} \right\} \right\}. \tag{63}$$

Thus, by use of the convolution theorem (60) we have

$$V_E(x,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} (e^u - 1) \exp\left\{ -\frac{[x - u + \ln X_n + \phi\tau]^2}{2\sigma^2\tau} \right\} du \right\},$$

which, in terms of S is

$$C_E(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ I_1(S,\tau) - I_2(S,\tau) \right\}, \tag{64}$$

where we set

$$I_1(S,\tau) \equiv \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} K e^u \exp\left\{ -\frac{[\ln(SX_n/K) - u + \phi\tau]^2}{2\sigma^2\tau} \right\} du, \tag{65}$$

and

$$I_2(S,\tau) \equiv \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} K \exp\left\{ -\frac{[\ln(SX_n/K) - u + \phi\tau]^2}{2\sigma^2\tau} \right\} du. \tag{66}$$

Beginning with I_1, we have

$$I_1(S,\tau) = \frac{Ke^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\beta^2/(2\sigma^2\tau)} \int_0^{\infty} \exp\left\{ -\frac{u^2 - 2(\beta + \sigma^2\tau)u}{2\sigma^2\tau} \right\} du,$$

where $\beta \equiv \ln(SX_n/K) + \phi\tau$. Completing the square with respect to u and changing the integration variable, we find that (recall that $\phi = r - q - \lambda k - \sigma^2/2$)

$$I_1(S,\tau) = SX_n e^{-\lambda k\tau} e^{-q\tau} \mathcal{N}[d_1(SX_n e^{-\lambda k\tau}, K, r, q, \tau, \sigma^2)]. \tag{67}$$

For I_2, a suitable change of integration variable gives

$$I_2(S,\tau) = Ke^{-r\tau}\mathcal{N}[d_2(SX_ne^{-\lambda k\tau}, K, r, q, \tau, \sigma^2)], \tag{68}$$

Finally, substituting I_1 and I_2 into (64), we find that

$$C_E(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{C_{BS}[SX_ne^{-\lambda k\tau}, K, K, r, q, \tau, \sigma^2]\right\}, \tag{69}$$

where C_{BS} is the solution the Black-Scholes-Merton solution for a European call option.

A3.2. **Proof of Proposition 3.4.** Consider the function

$$V_P(x,\tau) = \int_0^{\tau} \mathcal{F}^{-1}\left\{\hat{F}_J(\eta,\xi)e^{-(\frac{\sigma^2\eta^2}{2}+i\phi\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)}\right\}d\xi.$$

Using equation (20) we recall that

$$\mathcal{F}^{-1}\left\{\hat{F}_J(\eta,\xi)\right\} = F_J(x,\xi),$$

where F_J is defined by (62).

We use again the result (63) with τ replaced by $(\tau-\xi)$ to see that

$$\begin{aligned}
&\mathcal{F}^{-1}\left\{e^{-(\frac{\sigma^2\eta^2}{2}+i\phi\eta+(r+\lambda)-\lambda A(\eta))(\tau-\xi)}\right\} \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\exp\left\{-\frac{[x+\ln X_n+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\}\right\}.
\end{aligned}$$

Thus by use of the convolution theorem (60) and (62) we obtain

$$\begin{aligned}
V_P(x,\tau) \quad = \quad & \int_0^{\tau}\sum_{n=0}^{\infty}\left[\frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!}\int_{-\infty}^{\infty} H(u-\ln b(\xi))\right. \\
& \times\left(qe^u - r - \lambda\int_0^{b(\xi)e^{-x}}[V(u+\ln Y,\xi)-(Ye^u-1)]G(Y)dY\right) \\
& \left.\times\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\exp\left\{-\frac{[x-u+\ln X_n+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\}\right\}dud\xi\right],
\end{aligned}$$

which in terms of $S(=Ke^x)$ becomes (recall that $V_P(x,\tau) = C_P(S,\tau)/K$)

$$C_P(S,\tau) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\int_0^{\tau}(\tau-\xi)^n e^{-\lambda(\tau-\xi)}\left[I_3(S,\tau)-I_4(S,\tau)-I_5(S,\tau)\right]d\xi, \tag{70}$$

where

$$I_3(S,\tau) \equiv \frac{qe^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\int_{\ln b(\xi)}^{\infty} Ke^u \exp\left\{-\frac{[\ln(SX_n/K)-u+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\} du \quad (71)$$

$$I_4(S,\tau) \equiv \frac{re^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\int_{\ln b(\xi)}^{\infty} K \exp\left\{-\frac{[\ln(SX_n/K)-u+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\} du, \quad (72)$$

and

$$I_5(S,\tau) \equiv \lambda\frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\int_{\ln b(\xi)}^{\infty} \exp\left\{-\frac{[\ln(SX_n/K)-u+\phi(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\} \times\int_0^{b(\xi)e^{-u}}[C(KYe^u,\xi)-(KYe^u-K)]G(Y)dYdu. \quad (73)$$

To simplify I_3 and I_4, we make use of the results for I_1 and I_2 in Appendix A3.1. Firstly, we note that (71) is simply (65) with τ replaced by $(\tau-\xi)$. Thus from (67) we have

$$I_3(S,\tau) = qSX_n e^{-\lambda k(\tau-\xi)}e^{-q(\tau-\xi)}\mathcal{N}[d_1(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2)]. \quad (74)$$

Similarly for I_2, we can use (68) to show that (72) is

$$I_4(S,\tau) = rKe^{-r(\tau-\xi)}\mathcal{N}[d_2(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2)]. \quad (75)$$

For I_5, we change the order of integration using Fubini's theorem, and make the change of integration variable $\omega = Ke^u$, which gives

$$I_5(S,\tau) = \lambda e^{-r(\tau-\xi)}\int_0^1 G(Y)\int_{a(\xi)}^{a(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)] \times\kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2)d\omega dY,$$

where κ is defined by (29). Finally, substituting I_3, I_4 and I_5 into (70) gives equation (26) from Proposition 3.4.

A3.3. **Alternative Representation for $C_P^{(J)}$.** The representation for $C_P^{(J)}$ in (28) cannot be further simplified without explicit knowledge of the density $G(Y)$. In cases where the density is known, however, it may be possible to complete the integration with respect to Y analytically. Here we change the order of integration to develop a form for the double integral that will be easier to evaluate using numerical integration methods. Recall from (28) that

$$C_P^{(J)}[S,K,a(\xi),r,q,\tau,\sigma^2;C(\cdot,\xi)]$$
$$= e^{-r\tau}\int_0^1 G(Y)\int_{a(\xi)}^{a(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)]\kappa(S,\omega,r,q,\tau,\sigma^2)d\omega dY.$$

Making the change of integration variable $z = \omega Y/a(\xi)$ we obtain

$$C_P^{(J)}[S,K,a(\xi),r,q,\tau,\sigma^2;C(\cdot,\xi)]$$
$$= e^{-r\tau}\int_0^1\int_Y^1 G(Y)[C(a(\xi)z,\xi)-(a(\xi)z-K)]\kappa(S/a(\xi),z,r,q,\tau,\sigma^2)dzdY.$$

Finally, changing the order of integration using Fubini's theorem, we obtain

$$C_P^{(J)}[S,K,a(\xi),r,q,\tau,\sigma^2;C(S,\xi)]$$
$$= e^{-r\tau}\int_0^1[C(a(\xi)z,\xi)-(a(\xi)z-K)]\int_0^z G(Y)\kappa(S/a(\xi),z,r,q,\tau,\sigma^2)dYdz.$$

Appendix 4. American Call Evaluation for Log-Normal Jump Sizes

The form (45) for $G(Y)$ implies that

$$\mathbb{E}_{\mathbb{Q}}^{(n)}\{f(X_n)\} = \int_0^\infty f(X_n)\frac{1}{X_n\delta\sqrt{2\pi n}}\exp\left\{-\frac{1}{2}\left(\frac{\ln X_n - n(\gamma-\frac{\delta^2}{2})}{\delta\sqrt{n}}\right)^2\right\}dX_n. \quad (76)$$

We shall use this to evaluate all of the $\mathbb{E}_{\mathbb{Q}}^{(n)}$ operators in equation (31).

A4.1. **European Component.** Using the results from Merton (1976), the European component becomes

$$\sum_{n=0}^\infty \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\left\{C_{BS}[SX_ne^{-\lambda k\tau},K,K,r,q,\tau,\sigma^2]\right\}$$
$$= \sum_{n=0}^\infty \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}C_{BS}[S,K,K,r_n(\tau),q,\tau,v_n^2(\tau)],$$

where $\lambda' = \lambda(1+k)$, $r_n(\tau) = r - \lambda k + n\gamma/\tau$, and $v_n^2(\tau) = \sigma^2 + n\delta^2/\tau$, with C_{BS} as defined in Proposition 3.3.

A4.2. **Early Exercise Premium - First Term.** Consider the first part of the early exercise premium in (31), given by

$$\begin{aligned} C_P^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} \bigg\{ \int_0^{\tau} \frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!} \\ &\qquad \times \mathbb{E}_{\mathbb{Q}}^{(n)}\{C_P^{(D)}[SX_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, r, q, \tau-\xi, \sigma^2]\} d\xi \bigg\}. \end{aligned}$$

Using equation (27) for the definition of $C_P^{(D)}$ and (76), we can show that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}^{(n)}\{X_n \mathcal{N}[d_1(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2)]\} \\ &\quad = e^{n\gamma}\mathcal{N}[d_1(S, a(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi))], \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}^{(n)}\{\mathcal{N}[d_2(SX_n e^{-\lambda k(\tau-\xi)}, a(\xi), r, q, \tau-\xi, \sigma^2)]\} \\ &\quad = \mathcal{N}[d_2(S, a(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi))]. \end{aligned}$$

Noting that $e^{n\gamma} = (k+1)^n$, $C_P^{(1)}$ becomes

$$\begin{aligned} C_P^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} \bigg\{ \int_0^{\tau} \frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!} \\ &\qquad \times C_P^{(D)}[S, K, a(\xi), r, r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi)] d\xi \bigg\}. \end{aligned}$$

A4.3. **Cost Term from Downward Jumps.** The final term to consider is the cost incurred when S jumps from the stopping region into the continuation region. From (45) this term is given by

$$\begin{aligned} C_P^{(2)}(S,\tau) &= \lambda \sum_{n=0}^{\infty} \bigg\{ \int_0^{\tau} \frac{e^{-\lambda(\tau-\xi)}[\lambda(\tau-\xi)]^n}{n!} \\ &\qquad \times \mathbb{E}_{\mathbb{Q}}^{(n)}\{C_P^{(J)}[SX_n e^{-\lambda k(\tau-\xi)}, K, a(\xi), r, q, \tau-\xi, \sigma^2; C(\cdot,\xi)]\} d\xi \bigg\}. \end{aligned}$$

Referring to (28) and (29) for the definitions of $C_P^{(J)}$ and κ respectively, we find that in order to evaluate the the $\mathbb{E}_{\mathbb{Q}}^{(n)}$ operator, we must consider

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}^{(n)}\{\kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2)\} \\
&\qquad = \int_0^\infty \frac{1}{X_n \delta\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{[\ln X_n - n(\gamma - \frac{\delta^2}{2})]^2}{\delta^2 n}\right\} \frac{1}{\omega\sigma\sqrt{2\pi(\tau-\xi)}} \\
&\qquad\quad \times \exp\left\{\frac{-[(r-q-\lambda k - \frac{\sigma^2}{2})(\tau-\xi) + \ln\frac{SX_n}{\omega}]^2}{2\sigma^2(\tau-\xi)}\right\} dX_n.
\end{aligned}$$

Making the change of variable $x_n = \ln X_n$, this expectation can be evaluated as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}^{(n)}\{\kappa(SX_n e^{-\lambda k(\tau-\xi)}, \omega, r, q, \tau-\xi, \sigma^2)\} \\
&\quad = \frac{1}{\omega\sqrt{2\pi(\tau-\xi)v_n^2(\tau-\xi)}} \\
&\qquad\quad \times \exp\left\{-\frac{[\ln\frac{S}{\omega} + (r_n(\tau-\xi) - q - \frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\}.
\end{aligned}$$

Finally, using the definitions for λ' and $r_n(\tau)$ we can rewrite $C_P^{(2)}$ as

$$\begin{aligned}
C_P^{(2)}(S,\tau) &= \lambda\sum_{n=0}^{\infty}\left\{\int_0^\tau \frac{e^{-\lambda'(\tau-\xi)}[\lambda'(\tau-\xi)]^n}{n!}\right. \\
&\qquad \left.\times C_P^{(J)}[S, K, a(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi); C(\cdot,\xi)]\Big] d\xi\right\}.
\end{aligned}$$

A4.4. **Proposition 5.1.** Combining the results from sections A4.1-A4.3, we find that the integral equation for $C(S,\tau)$ in the case of log-normal jumps is given by equation (46) in Proposition 5.1.

Appendix 5. The Simplified Cost Term for Log-Normal Jump Sizes

Referring to the result in Proposition 3.6, the integral term in equation (31) that we seek to evaluate is

$$\begin{aligned}
I(S,z,\tau,\xi) &\equiv \mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\int_0^z G(Y)\kappa(SYX_n e^{-\lambda k(\tau-\xi)}/a(\xi), z, r, q, \tau-\xi, \sigma^2)\right\} dY \\
&= \frac{1}{\delta\sqrt{2\pi}}\int_0^z \frac{1}{Y}\exp\left\{-\frac{1}{2}\left[\frac{\ln Y - (\gamma - \frac{\delta^2}{2})}{\delta}\right]^2\right\} J(S,z,\tau,\xi,Y)dY,
\end{aligned}$$

where

$$J(S,z,\tau,\xi,Y) \equiv \frac{1}{z\sigma\sqrt{2\pi(\tau-\xi)}}\frac{1}{\delta\sqrt{2\pi n}}\int_0^\infty \frac{1}{X_n}\exp\left\{-\frac{1}{2}\left[\frac{\ln X_n - n(\gamma-\frac{\delta^2}{2})}{\delta\sqrt{n}}\right]^2\right\} \times \exp\left\{-\frac{1}{2}\left[\frac{\ln\frac{SYX_n}{a(\xi)z}+(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi)}{\sigma\sqrt{\tau-\xi}}\right]^2\right\} dX_n.$$

To evaluate $I(S,z,\tau,\xi)$ we need to make use of the following integration result. Let α_1, α_2, β_1, β_2 and z be real-valued functions independent of the integration variable ω. Then by completing the square in the exponent, it can be shown that

$$\int_0^z \frac{1}{\omega}\exp\left\{-\frac{[\ln\omega+\beta_1]^2}{\alpha_1}-\frac{[\ln\omega+\beta_2]^2}{\alpha_2}\right\}d\omega = \sqrt{\frac{\alpha_1\alpha_2\pi}{\alpha_1+\alpha_2}}\exp\left\{-\frac{(\beta_1-\beta_2)^2}{\alpha_1+\alpha_2}\right\}\mathcal{N}[f(z)], \tag{77}$$

where $f(z) = \sqrt{2}[(\alpha_1+\alpha_2)\ln z + \alpha_1\beta_2 + \alpha_2\beta_2]/\sqrt{\alpha_1\alpha_2(\alpha_1+\alpha_2)}$. Applying (77) to $J(S,z,\tau,\xi)$ we find that

$$J(S,z,\tau,\xi,Y) = \frac{1}{zv_n(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \times \exp\left\{-\frac{[\ln\frac{SY}{a(\xi)z}+(r_n(\tau-\xi)-q-\frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\},$$

where $r_n(\tau)$ and $v_n(\tau)$ are given by Proposition 5.1, and thus $I(S,z,\tau,\xi)$ becomes

$$I(S,z,\tau,\xi) = \frac{1}{zv_n(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \times\frac{1}{\delta\sqrt{2\pi}}\int_0^z\frac{1}{Y}\exp\left\{-\frac{[\ln Y-(\gamma-\frac{\delta^2}{2})]^2}{2\delta^2}\right\} \times \exp\left\{-\frac{[\ln\frac{SY}{a(\xi)z}+(r_n(\tau-\xi)-q-\frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\}dY.$$

Finally, we again apply (77) to $I(S,z,\tau,\xi)$ and obtain

$$\begin{aligned} I(S,z,\tau,\xi) &= \frac{1}{z v_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}}\mathcal{N}[f(z)] \\ &\quad \times \exp\left\{ -\frac{[\ln \frac{S}{a(\xi)z} + (r_{n+1}(\tau-\xi) - q - \frac{v_{n+1}^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_{n+1}^2(\tau-\xi)(\tau-\xi)} \right\}, \end{aligned}$$

where

$$\begin{aligned} f(z) &= \frac{\delta^2 \ln \frac{S}{za(\xi)} + \left[(\ln z)v_{n+1}^2(\tau-\xi) + \delta^2[r_n(\tau-\xi) - q] - \gamma v_n^2(\tau-\xi)\right](\tau-\xi)}{v_n(\tau-\xi)v_{n+1}(\tau-\xi)\delta(\tau-\xi)} \\ &\equiv D(S/a(\xi), z, r_n(\tau-\xi), q, v_n(\tau-\xi), v_{n+1}(\tau-\xi), \tau-\xi, \gamma, \delta). \end{aligned}$$

Substituting for $I(S,z,\tau,\xi)$ into (31) and combining this with the results in Proposition 5.1, we arrive at equation (47) of Proposition 5.2.

References

Abramowitz, M. & Stegun, I. A. (1970), *Handbook of Mathematical Functions*, Dover, New York.

Ahn, C. M. & Thompson, H. E. (1992), 'The Impact of Jump Risks on Nominal Interest Rates and Foreign Exchange Rates', *Review of Quantitative Finance and Accounting* **2**, 17–31.

Alobaidi, G. & Mallier, R. (2002), 'Laplace Transform and the American Straddle', *Journal of Applied Mathematics* **2**, 121–129.

Amin, K. I. (1993), 'Jump Diffusion Option Valuation in Discrete Time', *Journal of Finance* **48**(5), 1833–1863.

Andersen, L. & Andreasen, J. (2000), 'Jump-Diffusion Processes: Volatility Smile Fitting and Numerical Methods for Option Pricing', *Review of Derivatives Research* **4**, 231–262.

Ball, C. & Torous, W. (1985), 'On Jumps in Common Stock Prices and their Impact on Call Option Pricing', *Journal of Finance* **40**, 155–173.

Barone-Adesi, G. (2005), 'The Saga of the American Put', *Journal of Banking and Finance* **29**, 2909–2918.

Bates, D. S. (1996), 'Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutshe Mark Options', *Review of Financial Studies* **9**, 69–107.

Briani, M., Chioma, C. L. & Natalini, R. (2004), 'Convergence of Numerical Schemes for Viscosity Solutions to Integro-Differential Degenerate Parabolic Problems arising in Financial Theory ', *Numerische Mathematik* **98**(4), 607–646.

Broadie, M. & Yamamoto, Y. (2003), 'Application of the Fast Gauss Transform to Option Pricing', *Management Science* **49**(8), 1071–1088.

Carr, P. & Hirsa, A. (2003), 'Why be Backward? Forward Equations for American Options', *Risk* **16**(1), 103–107.

Carr, P. & Madan, D. B. (1999), 'Option Valuation using the Fast Fourier Transform', *Journal of Computational Finance* **2**(4), 61–73.

Cheang, G., Chiarella, C. & Ziogas, A. (2006), 'American-style Options on Two Assets under Jump-Diffusion Processes'. Working Paper, University of Technology, Sydney.

Chen, X. & Chadam, J. (2007), 'A Mathematical Analysis of the Optimal Exercise Boundary for American Put Options', *SIAM Journal on Mathematical Analysis* **38**, 1613–1641.

Chiarella, C., Kucera, A. & Ziogas, A. (2004), 'A Survey of the Integral Representation of American Option Prices', *Quantitative Finance Research Centre, University of Technology Sydney* . Research Paper, No. 118.

Chiarella, C. & Ziogas, A. (2005), 'Evaluation of American Strangles', *Journal of Economic Dynamics and Control* **29**, 31–62.

Chiarella, C. & Ziogas, A. (2006), 'A Fourier Transform Analysis of the American Call Option on Assets driven by Jump-Diffusion Processes', *Quantitative Finance Research Centre, University of Technology Sydney* . Research Paper, No. 174.

Debnath, L. (1995), *Integral Transforms and their Applications*, CRC Press, Baton Rouge.

d'Halluin, Y., Forsyth, P. & Labahn, G. (2004), 'A penalty method for American options with jump diffusion processes', *Numerische Mathematik* **97**, 321–352.

Geske, R. & Johnson, H. E. (1984), 'The American Put Option Valued Analytically', *Journal of Finance* **39**, 1511–1524.

Gukhal, C. R. (2001), 'Analytical Valuation of American Options on Jump-Diffusion Processes', *Mathematical Finance* **11**, 97–115.

Jacka, S. D. (1991), 'Optimal Stopping and the American Put', *Mathematical Finance* **1**, 1–14.

Jamshidian, F. (1992), 'An Analysis of American Options', *Review of Futures Markets* **11**, 72–80.

Jamshidian, F. (2006), 'The Duality of Optimal Exercise and Domineering Claims: A Doob-Meyer Decomposition Approach to the Snell Envelope'. Working Paper, http://wwwhome.math.utwente.nl/~jamshidianf/pdf/FJ_DomineeringClaims_15May06.pdf.

Jarrow, R. A. & Rosenfeld, E. (1984), 'Jump Risks and the Intertemporal Capital Asset Pricing Model', *Journal of Business* **57**, 337–351.

Jorion, P. (1988), 'On Jump Processes in the Foreign Exchange and Stock Markets', *Review of Financial Studies* **1**, 427–455.

Kallast, S. & Kivinukk, A. (2003), 'Pricing and Hedging American Options using Approximations by Kim Integral Equations', *European Finance Review* **7**, 361–383.

Karatzas, I. (1988), 'On the Pricing of American Options', *Applied Mathematics and Optimization* **17**, 37–60.

Kim, I. J. (1990), 'The Analytic Valuation of American Options', *Review of Financial Studies* **3**, 547–572.

Kou, S. G. (2002), 'A Jump-Diffusion Model for Option Pricing', *Management Science* **48**(8), 1086–1101.

Kuske, R. A. & Keller, J. B. (1998), 'Optimal Exercise Boundary for an American Put Option', *Applied Mathematical Finance* **5**, 107–116.

Lee, R. W. (2004), 'Option Pricing by Transform Methods: Extensions, Unification and Error Control', *Journal of Computational Finance* **7**(3), 51–86.

Lewis, A. L. (2000), *Option Valuation under Stochastic Volatility*, Finance Press, California.

McKean, H. P. (1965), 'Appendix: A Free Boundary Value Problem for the Heat Equation Arising from a Problem in Mathematical Economics', *Industrial Management Review* **6**, 32–39.

Merton, R. C. (1976), 'Option Pricing when Underlying Stock Returns are Discontinuous', *Journal of Financial Economics* **3**, 125–144.

Meyer, G. H. (1998), 'The Numerical Valuation of Options with Underlying Jumps', *Acta Mathematica* **47**, 69–82.

Myneni, R. (1992), 'The Pricing of the American Option', *The Annals of Applied Probability* **2**, 1–23.

Pelsser, A. (2000), 'Pricing Double Barrier Options using Laplace Transforms', *Finance and Stochastics* **4**, 95–104.

Peskir, G. (2005), 'On the American Option Problem', *Mathematical Finance* **15**(1), 169–181.

Pham, H. (1997), 'Optimal Stopping, Free Boundary, and American Option in a Jump-Diffusion Model', *Applied Mathematics and Optimization* **35**, 145–164.

Scott, L. O. (1997), 'Pricing Stock Options in a Jump-Diffusion Model with Stochastic Volatility and Interest Rates: Applications of Fourier Inversion Methods', *Mathematical Finance* **7**(4), 413–426.

Wilmott, P., Dewynne, J. & Howison, S. (1993), *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press.

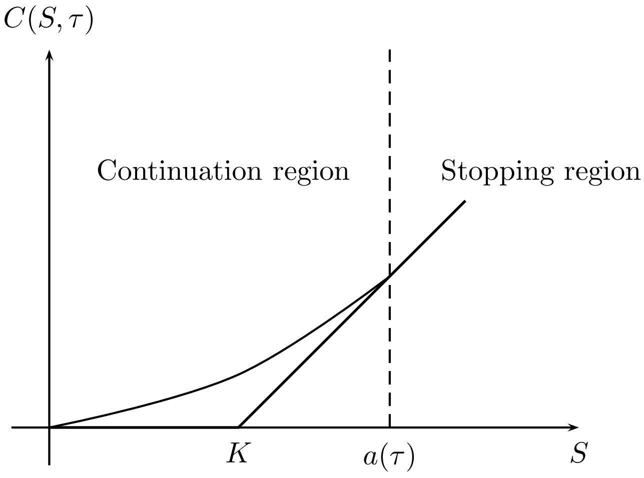


FIGURE 1. Continuation region for the American call option.

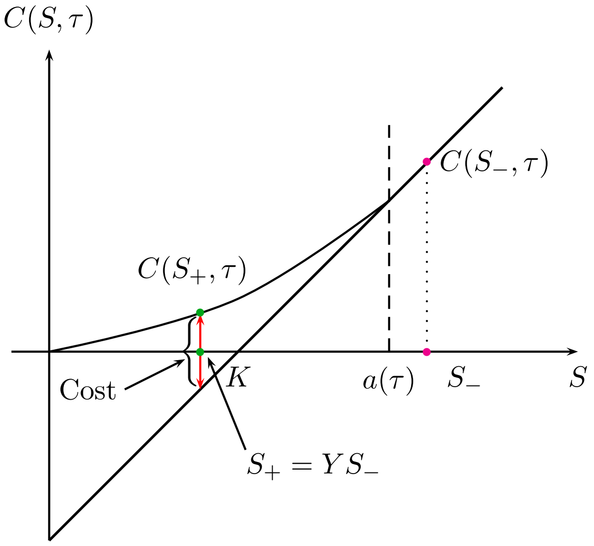


FIGURE 2. Cost incurred by the investor from downward jumps in S.

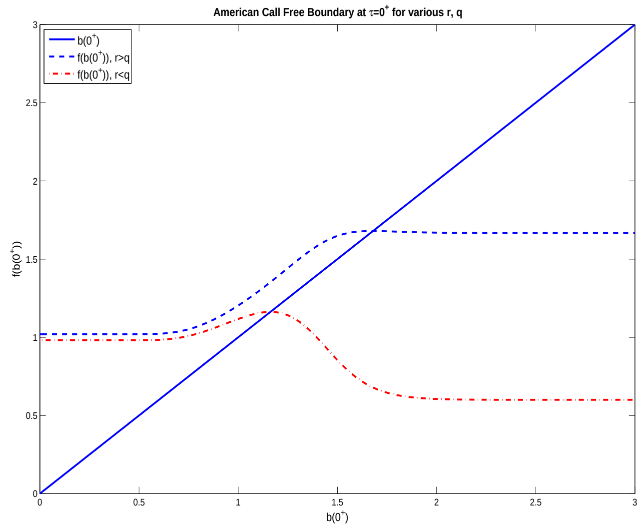


FIGURE 3. Behaviour of equation (54) when $\lambda = 1$, $\gamma = 0$ and $\delta = 0.2$. When $r > q$ we set $r = 0.05$, $q = 0.03$, and $r = 0.03$, $q = 0.05$ when $r < q$.

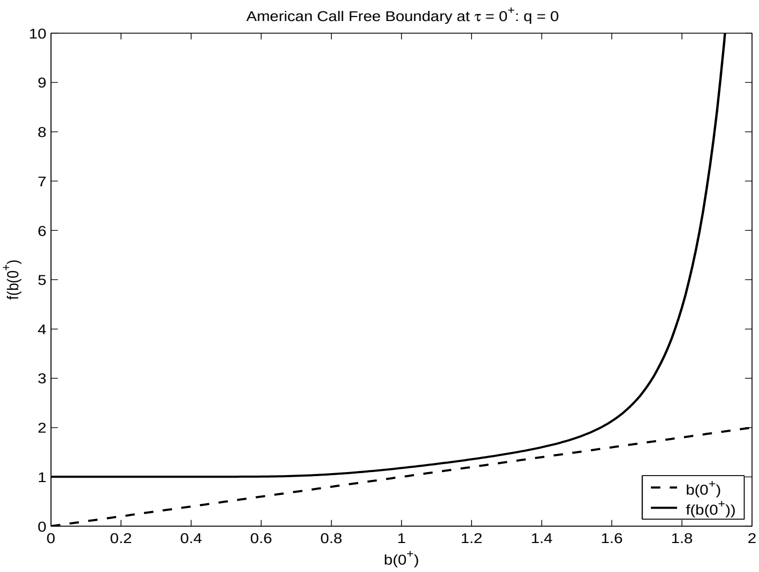
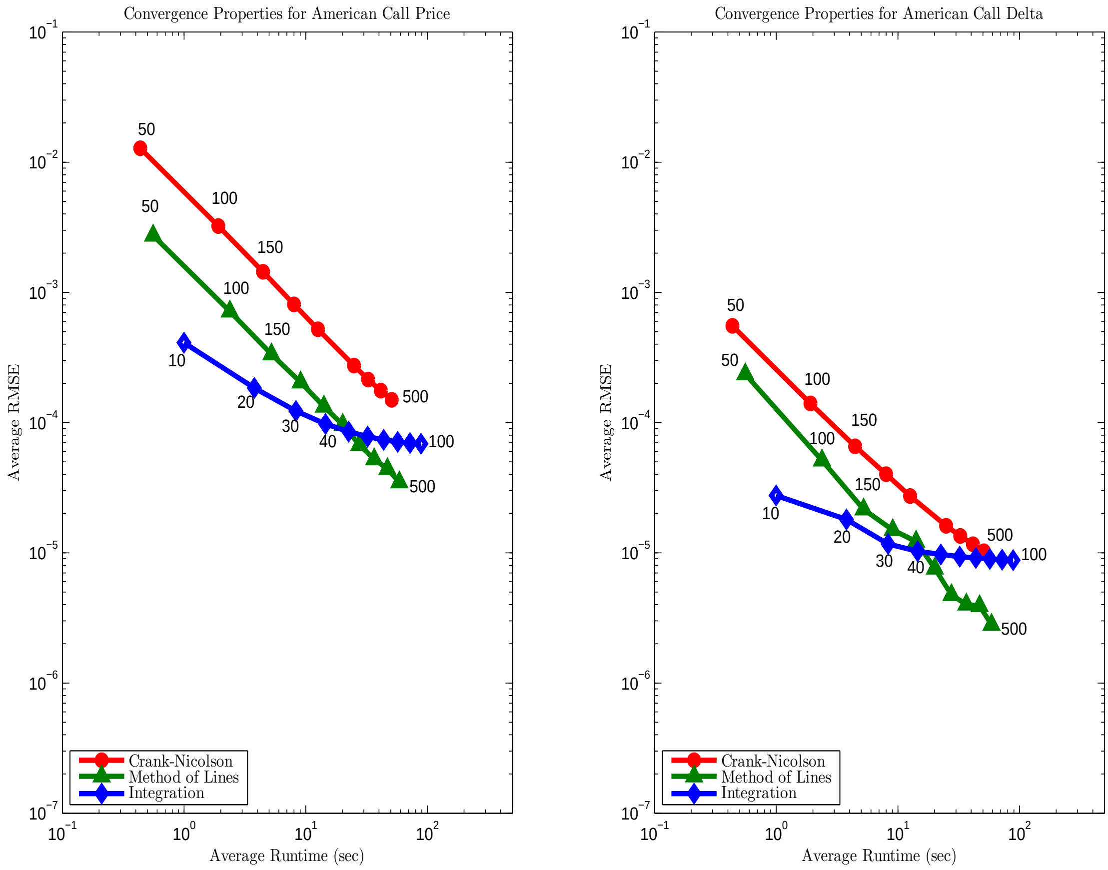


FIGURE 4. Behaviour of equation (54) when $q = 0$. Other parameter values are $r = 0.03$, $\lambda = 10$, $\gamma = 0$ and $\delta = 0.2$.



(a) (b)

FIGURE 5. Comparing the efficiency of numerical integration, Crank-Nicolson and the method of lines for the price and delta of American call options with log-normal jump sizes. Fixed parameters are $K = 100$, $T - t = 0.50$ and $\lambda = 1$. RMSE is found using $S = 80, 90, 100, 110$ and 120. Average RMSE and runtimes found using 6 parameter sets, with $r = 3\%$, $q = 5\%$, and $r = 5\%$, $q = 3\%$, along with $e^{\gamma} = 0.95, 1.00$ and 1.04. Figure 5(a) displays the price efficiency, and Figure 5(b) shows the delta efficiency.

Numbers on the plot indicate the time steps associated with a given point. Crank-Nicolson space steps are set equal to double the number of time steps. Method of lines space steps are set equal to 5 times the number of time steps. Note that the reported runtimes indicate the total time required to find the free boundary, price and delta for the American call. Both axes are given in log-scale.

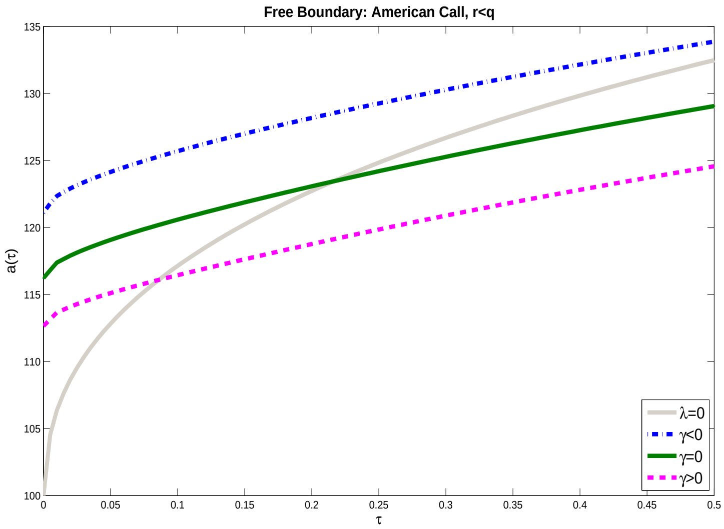


FIGURE 6. Early exercise boundaries for the American call option, in the case where $r < q$, for a range of γ values, compared with the pure diffusion case of $\lambda = 0$. The numerical integration scheme uses 100 time steps, with 20 integration points for the Gauss-quadrature component. Other parameter values are $K = 100$, $T = 0.5$, $r = 3\%$, $q = 5\%$, $\lambda = 1$ and $\delta^2 = 0.04$. See Table 1 for further details.

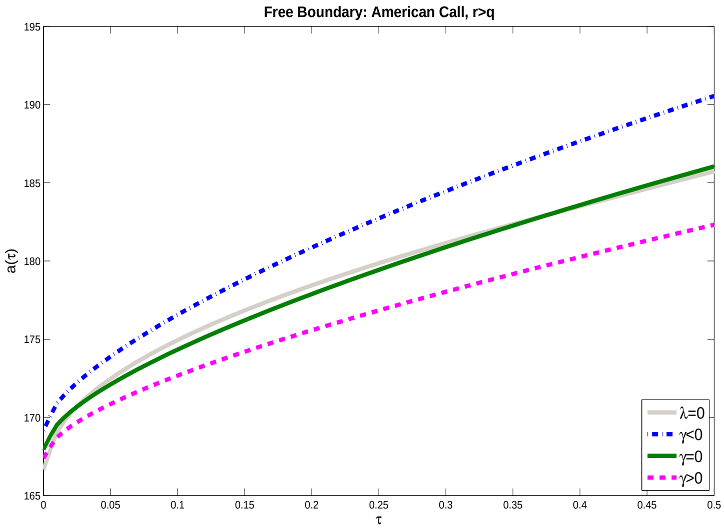


FIGURE 7. Early exercise boundaries for the American call option, in the case where $r > q$, for a range of γ values, compared with the pure diffusion case of $\lambda = 0$. The numerical integration scheme uses 100 time steps, with 20 integration points for the Gauss-quadrature component. Other parameter values are $K = 100$, $T = 0.5$, $r = 5\%$, $q = 3\%$, $\lambda = 1$ and $\delta^2 = 0.04$. See Table 1 for further details.

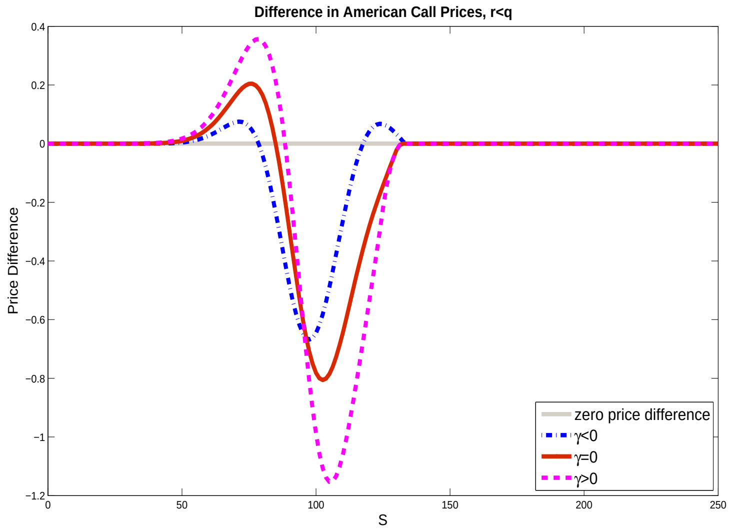


FIGURE 8. Price differences between the pure diffusion American call and the corresponding contract under jump-diffusion, in the case where $r < q$, for various values of γ. Other parameter values are $K = 100$, $T = 0.5$, $r = 3\%$, $q = 5\%$, $\lambda = 1$ and $\delta^2 = 0.04$. See Table 1 for further details.

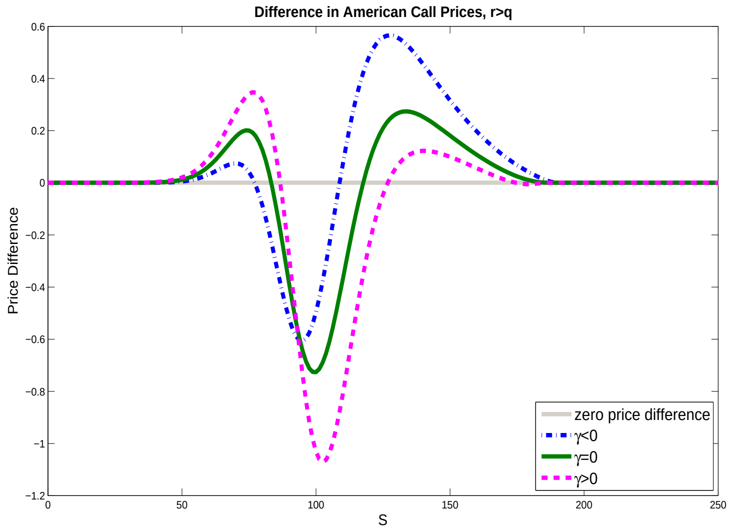


FIGURE 9. Price differences between the pure diffusion American call and the corresponding contract under jump-diffusion, in the case where $r > q$, for various values of γ. Other parameter values are $K = 100$, $T = 0.5$, $r = 5\%$, $q = 3\%$, $\lambda = 1$ and $\delta^2 = 0.04$. See Table 1 for further details.

σ^2	λ	e^γ	δ^2
0.0593	0	-	-
0.0200	1.00	0.95	0.04
0.0185	1.00	1.00	0.04
0.0136	1.00	1.04	0.04

TABLE 1. Parameter values used for the diffusion variance and jump component. The global variance is fixed at $s^2 = 5.93\%$, determined by $s^2 = \sigma^2 + \lambda[e^{2\gamma+\delta^2} - 2e^\gamma + 1]$.

Prices - American Call Option (True Solution)							
	Method	80	90	100	110	120	RMSD
$r > q$							
$e^{\gamma} = 0.95$	CN	0.69400	2.38087	6.73070	13.82696	22.39488	
	MOL	0.69402	2.38088	6.73068	13.82690	22.39482	4.0×10^{-5}
$e^{\gamma} = 1.00$	CN	0.90678	2.51510	6.49996	13.38593	21.99352	
	MOL	0.90678	2.51510	6.49997	13.38593	21.99351	6.3×10^{-6}
$e^{\gamma} = 1.04$	CN	1.09662	2.62043	6.19455	12.94557	21.67626	
	MOL	1.09677	2.62065	6.19466	12.94549	21.67616	1.4×10^{-4}
$r < q$							
$e^{\gamma} = 0.95$	CN	0.57779	1.95176	5.70386	12.30535	20.70560	
	MOL	0.57781	1.95178	5.70385	12.30529	20.70554	4.0×10^{-5}
$e^{\gamma} = 1.00$	CN	0.78198	2.14003	5.56774	11.92355	20.38397	
	MOL	0.78198	2.14002	5.56775	11.92355	20.38394	1.5×10^{-5}
$e^{\gamma} = 1.04$	CN	0.96480	2.30626	5.36027	11.50789	20.13332	
	MOL	0.96494	2.30648	5.36043	11.50786	20.13324	1.4×10^{-4}
						Average RMSD:	4.3×10^{-5}

TABLE 2. Demonstrating the accuracy of the true American call prices found using the Crank-Nicolson method with 10,000 time steps and 5,000 space steps. The method of lines solution uses 1,000 time steps and 5,000 space steps. The values used for r and q were 3% and 5%, with $T-t = 0.5$ and $K = 100$. Additional parameter values are given in Table 1.

Deltas - American Call Option (True Solution)							
	Method	80	90	100	110	120	RMSD
$r > q$							
$e^{\gamma} = 0.95$	CN	0.086567	0.281168	0.589532	0.804381	0.895046	
	MOL	0.086567	0.281167	0.589530	0.804380	0.895047	6.4×10^{-7}
$e^{\gamma} = 1.00$	CN	0.094375	0.254185	0.553839	0.798772	0.904862	
	MOL	0.094374	0.254185	0.553839	0.798772	0.904863	8.3×10^{-7}
$e^{\gamma} = 1.04$	CN	0.102141	0.223464	0.517908	0.804882	0.918433	
	MOL	0.102140	0.223463	0.517908	0.804881	0.918434	5.9×10^{-7}
$r < q$							
$e^{\gamma} = 0.95$	CN	0.070887	0.231935	0.527693	0.770657	0.895029	
	MOL	0.070887	0.231932	0.527688	0.770656	0.895031	5.4×10^{-7}
$e^{\gamma} = 1.00$	CN	0.081431	0.213015	0.490691	0.762207	0.912965	
	MOL	0.081430	0.213015	0.490690	0.762206	0.912964	7.4×10^{-7}
$e^{\gamma} = 1.04$	CN	0.091457	0.192099	0.449855	0.763778	0.941484	
	MOL	0.091456	0.192101	0.449853	0.763776	0.941486	1.4×10^{-6}
						Average RMSD:	7.9×10^{-7}

TABLE 3. Demonstrating the accuracy of the true American call deltas found by applying a central difference approximation to the prices estimates given by the Crank-Nicolson method with 10,000 time steps and 5,000 space steps. The method of lines solution uses 1,000 time steps and 5,000 space steps. The values used for r and q were 3% and 5%, with $T-t = 0.5$ and $K = 100$. Additional parameter values are given in Table 1.