European Journal of Physics

PAPER

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To cite this article: Qiaochu Wang *et al* 2018 *Eur. J. Phys.* **39** 015701

IOP Publishing European Journal of Physics
Eur. J. Phys. **39** (2018) 015701 (11pp) https://doi.org/10.1088/1361-6404/aa83d5

A new method of measuring gravitational acceleration in an undergraduate laboratory program

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Received 27 June 2017, revised 28 July 2017
Accepted for publication 3 August 2017
Published 29 November 2017

Abstract

This paper presents a high accuracy method to measure gravitational acceleration in an undergraduate laboratory program. The experiment is based on water in a cylindrical vessel rotating about its vertical axis at a constant speed. The water surface forms a paraboloid whose focal length is related to rotational period and gravitational acceleration. This experimental setup avoids classical source errors in determining the local value of gravitational acceleration, so prevalent in the common simple pendulum and inclined plane experiments. The presented method combines multiple physics concepts such as kinematics, classical mechanics and geometric optics, offering the opportunity for lateral as well as project-based learning.

Keywords: gravitational acceleration, undergraduate education, multi-concept pedagogy

(Some figures may appear in colour only in the online journal)

1. Introduction

Gravitational acceleration is one of the basic Earth-inherent parameters, so closely related to daily life, as well as a fundamental defining property in many physics experiments. The

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precise measurement of gravitational acceleration is especially important in the fields of aerospace, engineering, navigation, manufacturing and many other fields in daily life and research.

There are many ways to measure gravitational acceleration in an undergraduate or high school setting. The swinging simple pendulum is probably the most used experiment to determine gravitational acceleration in an educational setting since little conceptual physics background is needed to conduct the experiment and evaluate the measured data set. All that is required is to measure the simple pendulum's string length and its swing period and then to evaluate the data according to a given or derived formula. Apart from the experimental errors, additional errors in determining the gravitational acceleration arise through mathematical approximations that have been made in the model as well as experimental assumptions. For instance, one cannot be sure that the pendulum always swings in the same plane due to the asymmetry of the force when releasing the pendulum bob. Air resistance adds another element of uncertainty, which has a measurable impact in particular for a long pendulum string. As a result, the pendulum approximates a classical conical pendulum. The swing period of the conical pendulum is

$$\mathrm{T} = 2\pi\sqrt{\frac{L\cos\vartheta}{g}} \tag{1}$$

with ϑ being the tilt angle, which is different from the swing period of the simple pendulum. Moreover, in a simple pendulum experiment we assume [1] that the swing angle $\theta \approx \sin\theta$ and assume that the surface of the Earth is an infinite plane, neglecting the Earth's limited radius [2] as well as ignoring the mass of the rope. The factors above all exacerbate the error in determining the local gravitational acceleration. The alternatives to the simple pendulum are the free fall experiment and weight-based experiments. In the educational free fall experiment air resistance, which is a major factor, is neglected. In the weight-based experiments, despite their simple and convenient operation, the accuracy of a determined gravitational acceleration is much lower due to the limited precision of force meters and force sensors usually available in undergraduate laboratories (2–3 decimal places).

Here we present experiments suitable for high school and undergraduate junior laboratories that are capable of determining the local gravitational acceleration to very high accuracy and, if maintained within the course structure over several years, may monitor the drift of gravitational acceleration over years. Our approach is based on an idea for measuring the gravitational acceleration firstly put forward by W A Marrison in 1939 [3]. Marrison left his design idea as such, i.e. he never managed to actually realize the experiment itself, and thus the experimental accuracy of this approach has not been reported.

The experimental design for measuring the local gravitational acceleration comprises two main components associated with two different physics concepts: a rotating cylindrical vessel filled with water, where the water mass due to gravity forms a parabolic surface; and a collimated light beam that is reflected on the water surface, where the angle of reflection changes depending on the speed with which the cylinder rotates as the parabolic curvature of the water mass's surface changes [4], forming a reflecting paraboloid.

The focal length of the paraboloid is related to the gravitational acceleration and the rotational period of the cylinder. The gravitational acceleration can be determined by measuring the focal length and rotational period of the cylinder. The experimental setup is quite basic and its actual enactment is sufficiently simple to make it suitable for adoption in a high school or undergraduate experimental physics program. In our case, we made use of an existing moment of inertia experiment with a rotating table controlled by an electric motor that drives a belt gear with the rotating table, with adjustable gear transitions. The adjustable

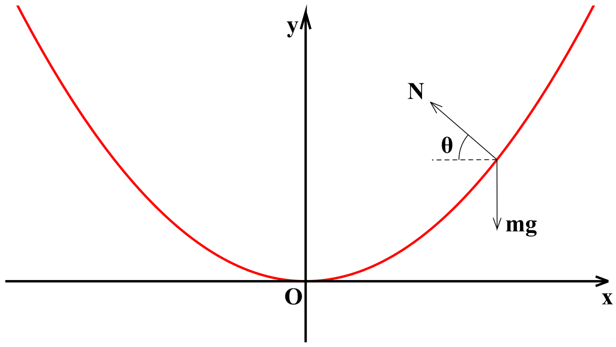

Figure 1. Force analysis of a water droplet on the paraboloid surface.

gear transition allows the table to have a wide range of rotational speed; however, for this experiment this was not essential.

2. Theoretical model

We assume the cylindrical vessel to be filled with an amount of water such that when the highest allowed rotational speed is reached, the resulting paraboloid does not reach the upper rim of the vessel and does not touch the bottom of the cylindrical vessel, where it would then flatten out and no longer present a perfect paraboloid [5]. In order to establish a relationship between the water surface angle relative to the vessel's rotational axis and the gravitational acceleration, we consider the force exerted on a water droplet on the surface of the water.

The net forces in the x- and y-direction are

$$\mathrm{N}\cos\theta = \mathrm{m}\omega^2\mathrm{x} \tag{2.1}$$

$$\mathrm{N}\sin\theta = \mathrm{mg}. \tag{2.2}$$

Dividing equation (2.1) by equation (2.2) and realizing that the gradient at the surface location is related to the tangent at that point,

$$\tan\theta = \frac{g}{\omega^2\mathrm{x}} = \frac{\mathrm{dx}}{\mathrm{dy}}\,. \tag{2.3}$$

Figure 1 shows a cross section of the water paraboloid when the cylinder is under rotation. Newton's second law and the rotational kinematics of a uniform circulation present an expression for the focal point of the paraboloid and its relation to the gravitational acceleration.

Separating the variables in equation (2.3) and integrating

$$\int_o^y \frac{g}{\omega^2}dy = \int_o^x xdx \tag{2.4}$$

presents the equation for the cross section of the paraboloid shown in figure 1 and the arc in figure 2,

$$\mathrm{x}^2 = \frac{2g}{\omega^2}y, \tag{2.5}$$

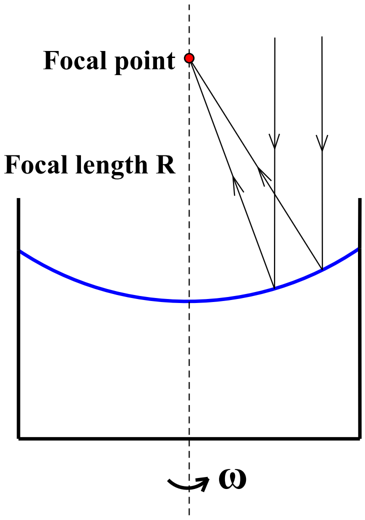

Figure 2. The schematic diagram of the experiment. The lines with arrows stand for the light path. The blue arc stands for the water surface.

confirming the water surface is a paraboloid. The focal length of the paraboloid (figure 2) is

$$\mathrm{R} = \frac{g}{2\omega^2}. \tag{2.6}$$

Substituting the angular velocity $\omega = \dfrac{2\pi}{\mathrm{T}}$ in equation (2.6) gives the desired expression for the gravitational acceleration as a function of the paraboloid focal length and cylinder rotation period:

$$\mathrm{g} = 8\pi^2 \cdot \frac{\mathrm{R}}{\mathrm{T}^2}. \tag{2.7}$$

Thus, a experiment can be conducted where the body of water forms a paraboloid and the focal length R and the rotational period T are measured to determine the local gravitational acceleration g.

The focal length R can be determined as shown in figure 2. Two rays of light (parallel to the *y*-axis in figure 1) reflect on the parabolic water surface and converge at the focal point of the paraboloid. A transparent optical screen can be used to find the focal point and then measure the distance R from the screen to the nadir of the water. The cylinder's rotational period is measured using a photogate.

3. Experimental method

The experimental setup (figure 3) comprises a rotating table (in our case recycled from a moment of inertia experiment), a photogate with digital timer, a cylindrical vessel, a DC electric motor with its controller and transmission belt, three lasers and a vertical vernier caliper.

A transmission belt connects the low-speed DC motor and the transmission axis of the rotating table. The low-speed DC motor in this study allowed speeds between zero and 120 rev/min. Two vertical lasers are fixed on a sliding support above the water surface. Another laser pointing horizontally fixed on a vertical vernier caliper is used to determine the

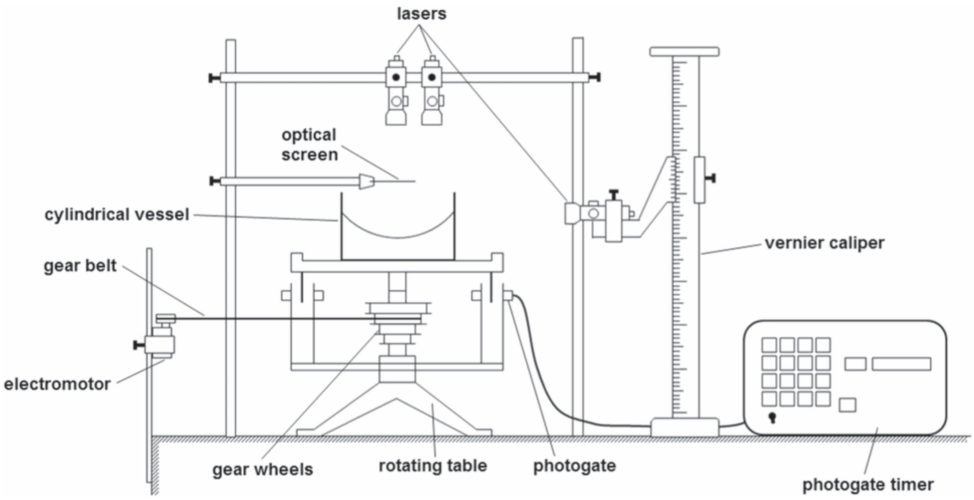

Figure 3. Schematic of the experimental setup.

relative vertical height of the optical screen, which may be moved up and down to locate the focal point and bottom of the paraboloid, i.e. the focal length R.

A photogate is connected to the digital timer, which records the time each time the rotating table's two flags, mounted at opposite ends, pass the gate. The first time a flag passes the photogate is recorded as T_0, which serves as the benchmark zero time for all subsequent measurements. The time the second flag passes the photogate is then recorded as T_1, which is the time it takes for the table to complete half of a rotation. The third time is recorded as T_2 and so on. We use the time T_{10}, which is the time for completing five revolutions. The average period T is then $\frac{T_{10}}{5}$.

The focal length was measured for eight different rotational speeds, where measurements were repeated six times for each rotational speed to determine respective average period. For comparison, a simple pendulum experiment was carried using eight pendulum string lengths ranging from 0.6048 m to 0.9458 m with a metal sphere as a bob attached at their ends. The average swing period of the pendulum was calculated from six swings for each pendulum length.

4. Results and discussion

Resulting average focal lengths and pendulum period data (inset) are shown in figure 4.

Of interest in the linear-regression analysis of the focal lengths of the rotating water paraboloid in figure 4 is the gradient of R versus T^2, from which with equation (2.7) the gravitational acceleration can be calculated. The regression analysis finds a gradient $b = R/T^2$ of $0.124\,26\ \mathrm{m\,s^{-2}}$ with a standard error of $\pm 1.0100 \times 10^{-4}\ \mathrm{m\,s^{-2}}$. The regression analysis for the pendulum data finds a respective gradient of $0.251\,51\ \mathrm{m\,s^{-2}}$ with a standard error of $\pm 0.001\,51\ \mathrm{m\,s^{-2}}$.

Substituting b into equation (2.7) yields a local gravitational acceleration of

$$g = 8\pi^2 \cdot \frac{R}{T^2} = 8\pi^2 \cdot 0.124\,26 = 9.8112\ \mathrm{m\ s^{-2}}$$

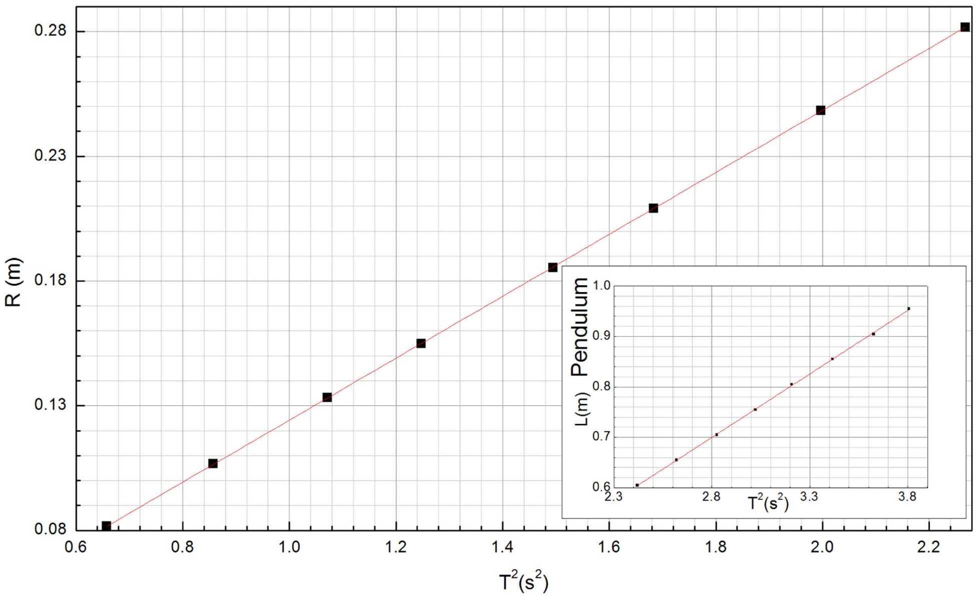

Figure 4. Linear-regression of focal length R and T^2 using the rotating water method and the analysis of T^2 and L using the pendulum method for determining the local gravitational acceleration.

with an uncertainty range of g: $\mathrm{u}(g) = 8\pi^2 \cdot \mathrm{u(b)} = \pm 0.008\ \mathrm{m\ s^{-2}}$, i.e. $g_w = 9.811(\pm 0.008)\ \mathrm{m\ s^{-2}}$.

With the gradient from the pendulum experiment we find the local gravitational acceleration $g = 4\pi^2 \cdot \dfrac{\mathrm{L}}{\mathrm{T}^2} = 9.9292\ \mathrm{m\ s^{-2}}$ with an uncertainty $\mathrm{u}(g) = 4\pi^2 \cdot \mathrm{u(b)} = 0.06\ \mathrm{m\ s^{-2}}$, i.e. $g_p = 9.93(\pm 0.06)\ \mathrm{m\ s^{-2}}$.

Analyzing the experimental execution of the pendulum model and the theoretical model for calculating a respective gravitational acceleration more closely, we find a number of additional uncertainties contributing to the overall error.

The pendulum string length was measured using a steel tape ruler with a measurement uncertainty of 0.2 mm. The dimension of the weight bob at its end was measured using a vernier caliper with an uncertainty of 0.02 mm.

An error analysis of the model approximations, i.e. neglecting the inertia of the string and air resistance, reveals a considerable error contribution in the small angle $\sin\theta \approx \theta$ assumption of the model [1]. The complete pendulum equation of motion $\frac{d^2\vartheta}{dt^2} + \frac{g}{l}\sin\theta = 0$ which results in a gravitational acceleration $g = \frac{4\pi^2 L}{T^2}\left(1 + \frac{1}{2}\sin^2\frac{\theta}{2}\right)$ with an uncertainty $\Delta g = \frac{2\pi^2 L}{T^2}\sin^2\frac{\theta}{2}$. In our experiment we used a small tilt angle of 5.5°, which yields the relative error of $\frac{\Delta g}{g} = 0.115\%$. Accounting for the mass of rope m′ by writing it as a multiple of the mass m of the spherical weight at its end, i.e. $\mathrm{m}' = \mathrm{km}$, the simple model gravitational acceleration becomes $g = \frac{4\pi^2 L}{T^2}\left(1 + \frac{k}{4}\right)$ with a relative error $\frac{\Delta g_k}{g} = \frac{k}{4+k} = 0.005\%$. Compared to the order of magnitudes for uncertainties mentioned here, we may safely neglect the error due to air resistance because of the string lengths and low speeds of the swinging bob in these experiments [1].

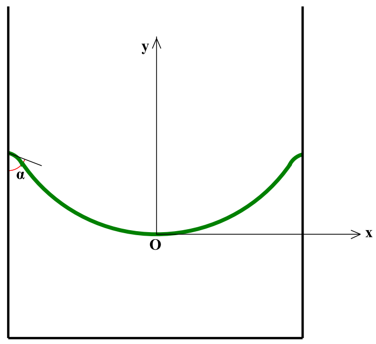

Figure 5. The surface of water when considering the surface tension.

A more serious source of error entering the simple pendulum experiment arises from getting the pendulum swing started, which in a high school or undergraduate environment is usually done by hand. Quite commonly there is an asymmetry of force when releasing the weight, which leads to an off-plane movement of the simple pendulum thus turning it into a conical pendulum. The swing period of a conical pendulum is $\mathrm{T} = 2\pi\sqrt{\frac{L\cos\vartheta}{g}}$ (ϑ is the off-plane tilt angle). The gravitational acceleration calculated from this is then $g = \frac{4\pi^2 L\cos\vartheta}{T^2}$. The respective relative error is $\frac{\Delta g_c}{g} = 1 - \cos\vartheta = 0.46\%$. In reality, though, the actual movement of the pendulum has neither a planar motion nor a conical motion; instead, it is moving between these two model idealizations. The relative error can then be expected to be somewhat less than 0.46%.

For the rotating column of water experiment with the paraboloidal water surface serving as parabolic mirror, experimental and model errors are more subtle and limited.

The water surface forms a thin layer at the interphase between water and air, the so called superficial layer. In this layer, molecules are sparser placed than in the bulk water. When a light ray is incident of the water surface, part of it will be reflected and another part will enter the superficial layer and then be reflected by the lower boundary of the layer. The thickness of this layer is about 10^{-9} m. With the lasers used in undergraduate experiments, this will not cause clearly separable rays to be seen reflecting from the surface, resulting in two dots on the screen. We may safely ignore here any uncertainty originating in double reflection at the water–air interphase.

In addition, because the molecules are sparser, the attraction between molecules is larger than repulsion, thus creating surface tension. In this experiment, we ignore the presence of surface tension.

Considering the surface tension of water, we note that the rotating water column does not produce an ideal paraboloid.

Capillary effects at the contact with the vessel boundary (figure 5) affect the contact angle between the liquid and the vessel wall. The contact angle α is determined by [5] the solid/vapor σ_{sv}, liquid/vapor surface σ_{lv} and solid/liquid σ_{sl} interface energies,

$$\cos\alpha = \frac{\sigma_{sv} - \sigma_{sl}}{\sigma_{lv}}.$$

The contact angle is independent of g and ω. The angle α does not decrease with the increasing angular velocity, as it does in the absence of surface tension, because it is a local material property, governed by the surface tensions alone, unaffected by the external fields due to gravity and rotation [5]. As long as the vertical lasers (figure 4) are kept well away from the vessel wall, variance in the surface tension due to contact forces will not affect the measuring result.

The laser beams have a certain diameter, resulting in small dots on the screen, which is used to determine the focal length. The dot diameters of somewhat smaller than 1 mm were measured with a vernier caliper ($\pm$0.02 mm). The transparent optical screen to determine the focal length is sufficiently thin so that beam fraction at the screen can be safely neglected since the change it may cause is well below what can be measured with our instruments.

Rotation table vibrations due to environmental effects or due to electromotor and driving belt vibrations may cause vibration of the laser dots on the optical screen. In that case the overlap middle of the crossing beams is then taken as the focal point location.

How does our local gravitational acceleration model and experiments compare to state-of-the-art models and experiments Until 1960, the most common model to approximate a local gravitational acceleration is the smoothed Earth spheroid model by Helmert [6] with

$$g = 978.0495\,(1 + 0.005\,2892\sin^2\Phi - 0.000\,0073\sin^2 2\Phi)$$

and Φ being the angle of latitude. The local latitude at our institute is $\Phi = 39.940\,714\,571\,0438$ which, with Helmert's formula, yields a local gravitational acceleration of 9.780 773 m s^{-2}.

A later refinement, based on the WGS84 Earth ellipsoid model, is the WGS (World Geodetic System) 1984 Ellipsoidal Gravity Formula [7]:

$$g(\Phi) = 9.780\,325\,3359\left[\frac{1 + 0.001\,931\,852\,652\,41\sin^2\Phi}{\sqrt{1 - 0.006\,694\,379\,990\,13\sin^2\Phi}}\right]$$

which with our local latitude yields a local gravitational acceleration of 9.815 194 m s^{-2}.

WolframAlpha's method for calculating the local gravitational field strength is based on the EGM2008 12th order model. Assuming a 52-meter altitude above sea level, the local gravitational acceleration is calculated as 9.802 67 m s^{-2} with a down component of 9.802 61 m s^{-2}, West component of 3.6×10^{-4} m s^{-2} and South component of 0.032 39 m s^{-2} [8].

In reality, the shape of the Earth is far from a geometrical ideal such as an ellipsoid or spheroid. The Gravity Recovery And Climate Experiment (GRACE), a joint survey satellite mission of NASA and the German Aerospace Center, was launched in 2002 and since then has been producing detailed measurements of the Earth's gravity field's anomalies and its irregular shape [9].

Figure 6 shows the satellite measured gravity anomalies, i.e. deviations from the standard model of gravitation in units of mgal (1 mgal = 10^{-5} m s^{-2}) [10]. The red and yellow areas indicate a larger than standard model gravitation whereas the blue areas are below the standard. The Beijing area shows an above standard gravitation, which is well within range of the undergraduate experiment presented here.

Gravitation may also change measurably over time at some locations due to earthquakes, magma movement, the change of buried matter, or other large-scale underground movement. It is interesting to see whether our experiment, if repeated in an undergraduate experimental program year after year, can reflect changes in gravity triggered by external local events. Zheng *et al* [11] discussed the relationship between the change of a local gravity field and the underground mass migration in the Beijing area by means of using a point-source disturbed body as a substitute for the mass migration underground (figure 7). Zheng *et al* found that the

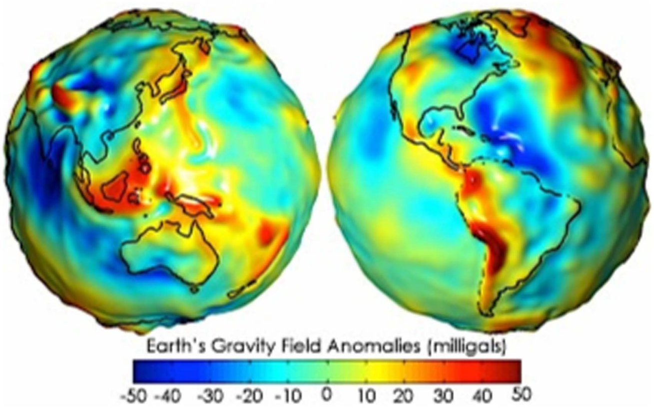

Figure 6. Earth gravity map (1 mgal $= 10^{-5}$ m s^{-2}). Image courtesy of NASA's Jet Propulsion Laboratory [10].

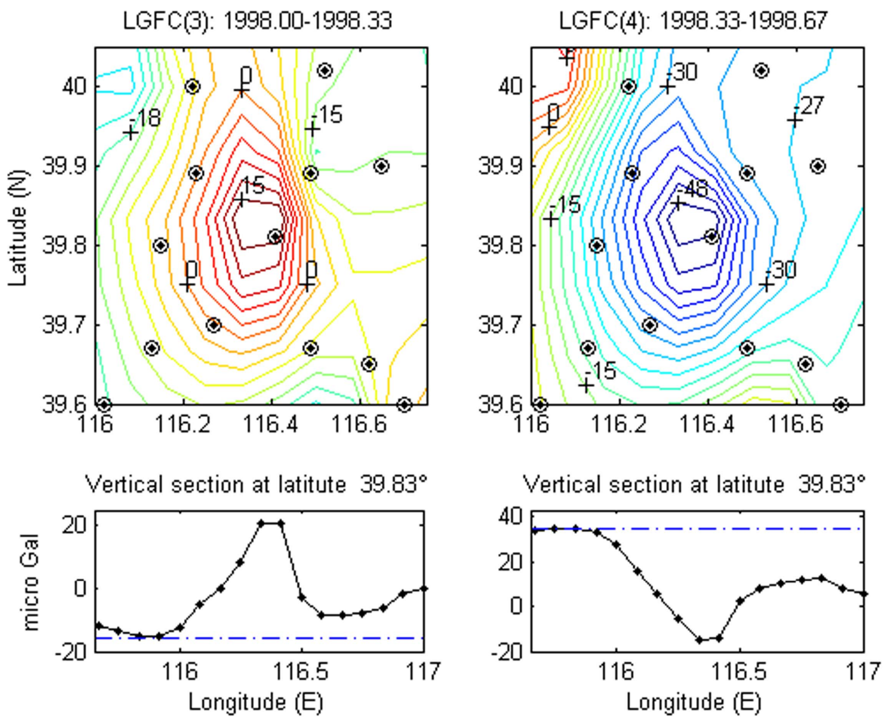

Figure 7. The left two images are the local gravity field change (LGFC) before the earthquake near Beijing in 1998, and the right two images are the ones after the earthquake [11].

timing of the two pair local gravity field change (LGFC) can be attributed to two earthquakes in the Beijing region and a related underground mass migration of up to 20 km.

For our longitude 116.13° and latitude 39.94° Zheng *et al* estimate a correction of the local vertical component of the gravitation (figure 8) due to underground mass migration of

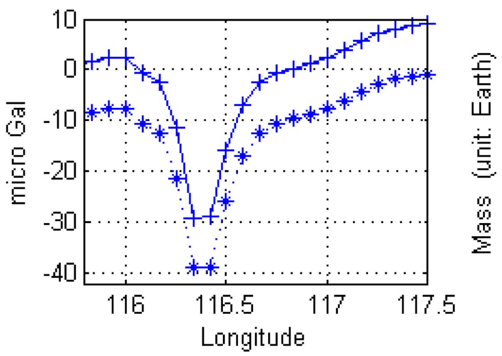

Figure 8. Uncorrected (+) and corrected (*) vertical part of the local gravity field change (LGFC) [11].

about 10^{-7} m s^{-2}, which is two orders of magnitude smaller that what can be measured with the experiment presented here in its current setup. It should be noted that Zheng *et al*'s estimates are based on a sparse distribution of measuring nodes, which as highlighted [10] may not account for all mass migration and earthquakes in the Beijing region.

For the time being we may safely neglect local underground mass migration until our spare parts assembled experiment is turned into a purpose built setup that may reach a higher accuracy. Nevertheless, our measurement of the local gravitational acceleration using the presented rotating water column method presents a genuine step up in accuracy from how gravity is usually measured in experimental undergraduate programs. Our measurements deviate from the established EGM2008 12th order model by only 0.09% while the conventional simple pendulum method was within 1.30% of the expected value and the uncertainty range.

5. Conclusions

We develop a new method for accurately measuring the local gravitational acceleration suitable for a high school or undergraduate physics laboratory program. The measurement of the local gravitational acceleration is based on a rotating water column and the establishment of the focal length of the resulting inner column water paraboloid. The presented experiment is well suited for students working in small teams and opens up an opportunity to implement a multi-concept pedagogy where multiple, not obviously related physics concepts need to be considered. The experiment draws on laws of inertia, uniform rotational motion, geometric optics and applied geometry, and encourages students to research the local geography in their region.

Comparable measurements of the local gravitational acceleration using the classical simple pendulum approach showed that the rotating water column approach is ten times more accurate than the pendulum approach. The rotating water column experiment came within 0.09% of the local gravitational acceleration, as calculated with the EGM2008 12th order model, while the pendulum experiment came within only 1.3% of the theoretically predicted value. Although the rotating water column proved to be highly accurate irrespective of its experimental simplicity, the experiment in its current spare parts assembly is not suitable to establish local gravity changes due to, for example, underground mass migration as they occur from time to time in the Beijing area. However, it may be sufficiently accurate to compare local gravity with schools in other geographic areas.

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