SHORT SELLING WITH MARGIN RISK AND RECALL RISK

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ABSTRACT. To investigate the effect of short-selling constraints on investor behaviour, we formulate an optimal stopping model in which the decision to cover a short position is affected by two short sale-specific frictions—margin risk and recall risk. Margin risk is introduced by assuming that a short seller is forced to close out their position involuntarily if they cannot fund margin calls (since short sales are collateralised transactions). Recall risk is introduced by permitting the lender to recall borrowed stock at any time, once again triggering an involuntary close-out. Examining the effect of these frictions on the optimal close-out strategy and associated value function, we finding that the optimal behaviour can be qualitatively different in their presence. Moreover, these frictions lead to a substantial loss in value, relative to the first-best situation without them (a reduction of approximately 17% for our conservative base-case parameters). This significant effect has important implications for many familiar no-arbitrage identities, which are predicated on the assumption of unfettered short selling.

1. INTRODUCTION

Short sales facilitate negative exposures to financial securities. The salient features of a short sale in the equity market may be summarised as follows.[1] First, a prospective short seller identifies a willing lender of the desired stock. He then borrows the stock, sells it in the market, and posts collateral equal to its market value plus a haircut with the lender.[2] The collateral is marked to market daily, so that an increase in the stock price prompts a margin call for more collateral, while a decrease entitles the short seller to withdraw some collateral. The lender invests the collateral at the prevailing interest rate, and pays part of the resulting income to the short seller, in the form of a negotiated rebate.[3] In return, the short seller compensates the lender for the dividends forgone

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[1]See Cohen et al. (2004), D'Avolio (2002), and Reed (2013) for comprehensive overviews of the short selling process and the equity lending market.

[2]The haircut is usually set at 2% of the market value of the stock.

[3]The difference between the interest rate and the rebate rate represents the lending fee.

during the life of the loan. To complete the transaction, the short seller repurchases the stock, and returns it to the lender. Although this is likely to occur at the short seller's discretion, stock loan contracts generally include a recall provision that permits lenders to force short sellers to liquidate their positions involuntarily.

The previous overview highlights several costs and risks associated with short sales, which have no counterparts when stocks are purchased. First, short sellers may incur search costs, since the process of identifying willing lenders can be expensive. Second, the lending fees associated with stock loans impose a cost on short sellers that can be quite significant. Third, short sellers are exposed to margin risk, due to the possibility that successive increases in the price of borrowed stock may generate margin calls that eventually exhaust their collateral budgets. Finally, short sellers face recall risk, since lenders may recall borrowed stock at any time. Collectively, these frictions are referred to as short-selling constraints.

This paper focuses on margin risk and recall risk. To understand the impact of these two frictions on short sales, we formulate an optimal stopping problem where a trader must decide when to close out a short position initiated at time zero. To capture the effect of margin risk, we assume that the trader has a limited collateral budget to fund margin calls, while recall risk is modelled by assuming that forced close out due to stock recall occurs at some independent random time. The collateral constraint introduces a knock-out barrier above the initial stock price, at a level determined by the collateral budget. Immediate close out occurs when the stock price reaches this barrier, since the available collateral will have been depleted by then.

Shleifer and Vishny (1997) first revealed the importance of margin risk as a limit to arbitrage, by demonstrating that a hedge fund may be forced to close out a theoretically profitable arbitrage trade in a loss-making position, if interim losses trigger sufficient investor withdrawals to compromise the fund's ability to meet its margin calls. Liu and Longstaff (2004) further reinforced the intuition that margin risk detracts from the attractiveness of arbitrage, by showing that a collateral-constrained risk-averse trader will underinvest in arbitrage opportunities, due to the possibility that the margin calls arising from interim losses may exhaust their collateral budget before they realise a profit.

As for the impact of recall risk, the empirical study by Chuprinin and Ruf (2017) found that recalls induce profit sharing between short sellers and stock lenders, by forcing the former to liquidate otherwise profitable short positions prematurely. They estimated that informed short sellers lose around 20% of their first-best profits due to recalls, indicating that recall risk is an economically significant short-selling constraint. Engelberg et al. (2018) provided further implicit evidence on the impact of recall risk, by demonstrating that deviations between spot and put-call parity-implied stock prices increase with option maturity. This suggests that short-selling constraints intensify as the time-horizon of trading strategies involving short positions increases, consistent with the nature of recall risk.

Our economic analysis confirms that margin risk and recall risk have a dramatic impact on the value of a short sale and the optimal close-out strategy, effectively driving a wedge between the solutions to the constrained and unconstrained short-selling problems. In particular, these risks can qualitatively change the optimal close-out strategy, in the sense that it can be optimal to close out a position immediately when exposed to margin or recall risk, but not without them. We are also able to quantify the loss in value due to margin risk and recall risk, which we observe to be very sensitive to the assumed drift rate of the stock price. Specifically, for our conservative base-case set of parameters we find that the loss in value from the first-best (unconstrained) situation is 17% when the drift rate of the stock price is -2%, but that this loss increases to 74% when the drift rate is 2%. This fragility creates a practical challenge to a short seller, due to the difficulty of estimating the drift parameter accurately.

The short-selling frictions in our model are also responsible for other surprising effects. For example, unlike the case with unconstrained short sales, the value of the constrained short position is non-monotonic with respect to the volatility of the stock price. The short seller is effectively long volatility close to the initial stock price, but short volatility as the stock price approaches the collateral exhaustion boundary (hence margin risk introduces non-convexity into the value function). The value of the constrained short position is also seen to be an increasing function of the discount rate when the stock price has positive drift, whereas the value of the unconstrained short position is always monotonically decreasing with respect to the discount rate. The intuitions behind these counterintuitive effects are quite subtle, and are discussed in detail in our model analysis.

The significant impact of margin risk and recall risk in our model is consistent with the well-documented effect that short-selling constraints can have on textbook no-arbitrage relationships (since arbitrage portfolios invariably contain long and short positions). For example, Lamont and Thaler (2003) highlight the role of short-selling constraints in preventing the correction of mispriced equity carve-outs, while Mitchell et al. (2002) identify their effect when a company trades at a discount relative to its subsidiaries. Short-selling constraints have also been implicated in mispriced equity index futures by Fung and Draper (1999), and in put-call parity violations by Ofek et al. (2004).

This paper also contributes to the literature on optimal stopping models for the related problem of optimal margin lending (see e.g. Cai and Sun 2014, Dai and Xu 2011, Ekström and Wanntorp 2008, Grasselli and Gómez 2013, Liang et al. 2010, Liu and Xu 2010, Siu et al. 2016, Wong and Wong 2013, Xia and Zhou 2007, Xu and Yi 2020, Yan et al. 2019, Zhang and Zhou 2009). In the case of a margin loan (also known as a stock loan), an investor borrows money from a bank to purchase stock, which the bank then holds as collateral. A fall in the stock price decreases the value of the collateral, triggering a margin call. If the investor fails to meet the margin call, the bank sells the stock to cover the loan. The investor's problem is to choose the optimal time to sell the stock and settle the loan. The similarity between short sales and margin loans rests on the fact that both are collateralised transactions, in which margin risk plays a prominent role.

Compared to the margin lending problem, optimal short selling has received little attention, with the studies by Chung and Tanaka (2015) and Chung (2016) appearing to be the only exceptions. Those articles consider the situation of a trader who must choose when to close out a short position, subject to lending fees, recall risk and liquidity risk. Our model differs from the models presented in those studies, in terms of which constraints are considered. Specifically, while they ignore margin risk (which is an important short sale-specific constraint), we are not concerned with liquidity risk (which is not a short sale-specific friction). As a result, our analysis offers different economic insights.

The remainder of the article is structured as follows. Section 2 models the price of a non-dividend-paying stock as a geometric Brownian motion and recalls several facts about such processes. The problem of when to close out a short position in the stock, in the presence of margin risk and recall risk, is then formulated as an optimal stopping problem. Section 3 derives and solves a related (more general) optimal stopping problem and identifies three different parameter regimes in which the optimal solution is qualitatively different. Section 4 then uses the solution to the more general problem to construct the solution to the original optimal short-selling problem. Finally, Section 5 investigates the economic insights that can be gleaned from our model, in particular by comparing the optimal strategy and value of our (constrained) model versus a benchmark (unconstrained) model. Detailed proofs of the results presented in the body of the paper can be found in the two appendices at the end.

2. Short Selling as an Optimal Stopping Problem

2.1. **Modelling the price of a non-dividend-paying stock.** Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion on a complete filtered probability space $(\Omega, \mathscr{F}, \mathfrak{F}, \mathsf{P})$, whose filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\geq 0}$ satisfies the usual conditions of right continuity and completion by the null-sets of $\mathscr{F}$. We consider a non-dividend-paying stock, whose price $X = (X_t)_{t\geq 0}$ is modelled as the unique strong solution to the stochastic differential equation

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t, \tag{2.1}$$

for all $t \geq 0$, where $X_0 \in (0,\infty)$, $\mu \in \mathbb{R}$ and $\sigma \in (0,\infty)$. That is to say, we model the stock price as a geometric Brownian motion.

The following definitions will also be useful for our subsequent analysis. Given $\alpha > 0$, let $\phi_\alpha, \psi_\alpha \in \mathsf{C}^2(0,\infty)$ denote the unique (up to multiplication by a positive scalar) decreasing and increasing solutions, respectively, to the second-order ordinary differential equation

$$\mathcal{L}_X u(x) \coloneqq \frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) = \alpha u(x), \tag{2.2}$$

for all $x \in (0,\infty)$. These solutions are given explicitly by

$$\phi_\alpha(x) \coloneqq x^{-\sqrt{\nu^2 + 2\alpha/\sigma^2} - \nu} \qquad \text{and} \qquad \psi_\alpha(x) \coloneqq x^{\sqrt{\nu^2 + 2\alpha/\sigma^2} - \nu}, \tag{2.3}$$

for all $x \in (0,\infty)$, where $\nu \coloneqq {}^{\mu}/_{\sigma^2} - {}^{1}/_{2}$ (see Borodin and Salminen 2002, Appendix 1.20).

Let $\mathfrak{S}$ denote the family of all $\mathfrak{F}$-stopping times. These include the first-exit times

$$\hat{\tau}_z \coloneqq \inf\{t \geq 0 \,|\, X_t \geq z\} \qquad \text{and} \qquad \check{\tau}_z \coloneqq \inf\{t \geq 0 \,|\, X_t \leq z\},$$

for all $z \in (0,\infty)$, whose Laplace transforms are given by

$$\mathsf{E}_x\left(\mathrm{e}^{-\alpha\hat{\tau}_z}\right) = \begin{cases} \frac{\psi_\alpha(x)}{\psi_\alpha(z)} & \text{if } x \leq z \\ 1 & \text{if } x \geq z, \end{cases} \tag{2.4a}$$

and

$$\mathsf{E}_x\left(\mathrm{e}^{-\alpha\check{\tau}_z}\right) = \begin{cases} 1 & \text{if } x \leq z \\ \frac{\phi_\alpha(x)}{\phi_\alpha(z)} & \text{if } x \geq z, \end{cases} \tag{2.4b}$$

for all $\alpha > 0$ and all $x, z \in (0, \infty)$ (see Borodin and Salminen 2002, Section II.10). As usual, $\mathsf{E}_x(\,\cdot\,)$ denotes the expected value operator with respect to the probability measure P_x, under which $X_0 = x$.

2.2. **The optimal time to close out a short sale.** We consider a risk-neutral trader with collateral budget $c \geq 0$, who sells the stock short at time zero. Forced close-out of his position due to collateral exhaustion occurs when the price of the stock first exceeds its initial price by more than his budgeted collateral. We denote this stopping time by

$$\zeta := \inf\{t \geq 0 \,|\, X_t = X_0 + c\}. \tag{2.5}$$

In other words, at time ζ the stock price will be high enough to ensure that the trader will have spent his entire collateral budget on margin calls.

Involuntary close-out of the short sale due to stock recall occurs at an exponentially distributed random time $\rho \sim \mathrm{Exp}(\lambda)$, where $\lambda > 0$ is the recall intensity. The recall time is assumed to be $\mathscr{F}$-measurable and independent of $\mathscr{F}_\infty := \bigvee_{t \geq 0} \mathscr{F}_t$, which is to say that the recall event is independent of the stock price. Hence, the distribution of the recall time is given by

$$\mathsf{P}_x(\rho \in \mathrm{d}t) = \lambda \mathrm{e}^{-\lambda t}\,\mathrm{d}t, \tag{2.6}$$

for all $t \geq 0$ and all $x \in (0, \infty)$. We model recall as an independent random event to capture the intuition that the lender may recall the stock at any time.

To maintain tractability, we restrict our attention to the situation when the rebate rate on the stock loan is zero. That is to say, we assume that the stock lending fee is identical to the prevailing interest rate $r \geq 0$. Hence, if the trader initiates the short sale at time zero and closes it out at time $t \geq 0$, the present value of his profit is $\mathrm{e}^{-rt}(X_0 - X_t)$.[4] His objective is to repurchase the stock at a time that maximises the

[4]If the rebate rate was non-zero then the present value of the profit would be $e^{-rt}(X_0 e^{qt} - X_t)$, where q denotes the rebate rate, hence the optimal stopping problem would become time *in*homogenous. Moreover, while it might seem tempting to incorporate a non-zero rebate rate into the drift rate μ via a change of variable ($\widehat{X}_t = e^{-qt} X_t$), the collateral exhaustion boundary defined by $\zeta := \inf\{t \geq 0 \,|\, X_t = X_0 + c\}$ would then become time-dependent. We also note that when $q = r$ the present value of the profit becomes $X_0 - e^{-rt} X_t$, from which it is clear that the short seller should simply stop immediately if $\mu \geq r$ or should never stop if $\mu < r$. However, the case $r = q$ corresponds to a zero lending fee, which is rarely seen empirically.

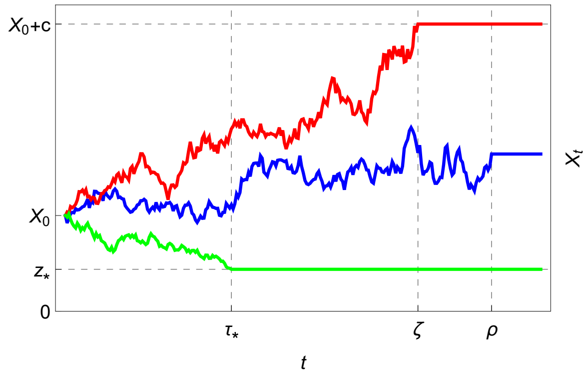


FIGURE 2.1. Three possible outcomes for the short-selling problem (2.7). The lower (green) path illustrates the case when the short sale is closed out optimally at time τ_*, which is the first time the stock price reaches a putative optimal close-out threshold z_*. (At this stage we do not know that the optimal close-out strategy is a threshold strategy, but we shall demonstrate that this is indeed the case.) The upper (red) path illustrates the case when the short sale is closed out due to collateral exhaustion at time ζ, which is the first time the stock price exceeds its initial value X_0 by the collateral budget c. The middle (blue) path illustrates the case when the short sale is closed out due to stock recall at a random time ρ.

expected value of his profit, in present value terms, subject to the frictions outlined above. This gives rise to the following optimal stopping problem:

$$V(X_0) := \sup_{\tau \in \mathfrak{S}} \mathsf{E}_{X_0}\Big(\mathrm{e}^{-r(\tau\wedge\zeta\wedge\rho)}\big(X_0 - X_{\tau\wedge\zeta\wedge\rho}\big)\Big), \tag{2.7}$$

for all $X_0 \in (0, \infty)$. Figure 2.1 illustrates Problem (2.7), by giving an example of a stock price path where the trader closes out the short position optimally, as well as examples of paths where the short sale is closed out involuntarily due to collateral exhaustion and recall.

3. An Auxiliary Optimal Stopping Problem and Its Solution

3.1. **Markovian embedding.** Optimal stopping problems are often solved by reformulating them as free-boundary problems. This approach involves embedding the original problem (2.7) into a Markovian framework where X is allowed to start at an arbitrary point x. However, since the collateral exhaustion time (2.5) depends on the initial stock price X_0, it is important to make the distinction between the arbitrary initial value of the process after embedding, x, and the initial stock price X_0 for the original problem (2.7). For a given short-selling problem, the initial stock price is fixed at X_0, which in turn defines a fixed collateral exhaustion boundary at $X_0 + c$. When this problem is embedded, the initial value X_0 remains fixed, as does the collateral exhaustion boundary $X_0 + c$. However the process X is allowed to start at the arbitrary point x, in general different from X_0. The value of the original problem (2.7) is then found by setting $x = X_0$ in the embedded problem.

With this necessary distinction in mind we formulate a more general optimal stopping problem where collateral exhaustion occurs as soon as the stock price exceeds some fixed level $\kappa > 0$ by the collateral budget.[5] Formally, we define

$$\widetilde{V}(x) := \sup_{\tau \in \mathfrak{S}} J(x,\tau), \tag{3.1a}$$

for all $x \in (0,\infty)$, where

$$\begin{aligned} J(x,\tau) &:= \mathsf{E}_x\Big(\mathrm{e}^{-r(\tau\wedge\hat{\tau}_{\kappa+c}\wedge\rho)}\big(\kappa - X_{\tau\wedge\hat{\tau}_{\kappa+c}\wedge\rho}\big)\Big) \\ &= \mathsf{E}_x\Big(\mathsf{E}_x\Big(\mathbf{1}_{\{\rho\le\tau\wedge\hat{\tau}_{\kappa+c}\}}\mathrm{e}^{-r\rho}(\kappa - X_\rho)\,\big|\,\mathscr{F}_{\tau\wedge\hat{\tau}_{\kappa+c}}\Big) \\ &\qquad + \mathrm{e}^{-r(\tau\wedge\hat{\tau}_{\kappa+c})}\big(\kappa - X_{\tau\wedge\hat{\tau}_{\kappa+c}}\big)\mathsf{P}_x\big(\rho > \tau\wedge\hat{\tau}_{\kappa+c}\,\big|\,\mathscr{F}_{\tau\wedge\hat{\tau}_{\kappa+c}}\big)\Big) \\ &= \mathsf{E}_x\bigg(\int_0^{\tau\wedge\hat{\tau}_{\kappa+c}} \lambda\mathrm{e}^{-(\lambda+r)t}(\kappa - X_t)\,\mathrm{d}t + \mathrm{e}^{-(\lambda+r)(\tau\wedge\hat{\tau}_{\kappa+c})}\big(\kappa - X_{\tau\wedge\hat{\tau}_{\kappa+c}}\big)\bigg), \end{aligned} \tag{3.1b}$$

for all $\tau \in \mathfrak{S}$, upon using (2.6). Hence, since $\zeta = \hat{\tau}_{X_0+c}$, we can recover the solution to Problem (2.7) for a given $X_0 \in (0,\infty)$ via $V(X_0) = \widetilde{V}(X_0)|_{\kappa=X_0}$.[6]

3.2. **An associated free-boundary problem.** The time-homogeneity of Problem (3.1) leads us to conjecture that the optimal stopping time for that problem is a threshold time. That is to say, it is the first time $\check{\tau}_{z_*}$ the stock price crosses some threshold $z_* \in (0,\kappa+c)$ from above. If that is true, we may be able to solve Problem (3.1) by solving the following free-boundary problem for $z_* \in (0,\kappa+c)$ and $\widehat{V} \in \mathsf{C}(0,\infty) \cap \mathsf{C}^2(z_*,\kappa+c)$ (see Peskir and Shiryaev 2006, Chapter III):

$$\mathcal{L}_X\widehat{V}(x) - (\lambda+r)\widehat{V}(x) + \lambda(\kappa - x) = 0, \text{ for all } x \in (z_*,\kappa+c), \tag{3.2a}$$

$$\widehat{V}(x) = \kappa - x, \text{ for all } x \in (0,z_*] \cup [\kappa+c,\infty), \tag{3.2b}$$

$$\widehat{V}'(z_*+) = -1. \tag{3.2c}$$

[5]Once solved we will set $\kappa = X_0$. However, expressing the reference value as an arbitrary constant allows for the additional flexibility of incorporating any difference in the value at which the stock was sold short and the value at which it was liquidated (due to transaction costs for example).

[6]For $t > 0$, the collateral exhaustion boundary remains fixed since it was determined by X_0 at initiation of the trade. In other words, the value of the short-sale for $t \ge 0$ is given by $\widetilde{V}(X_t)|_{\kappa=X_0}$, and the optimal stopping time in (3.1) remains optimal.

In other words, $z_* \in (0, \kappa+c)$ and $\widehat{V} \in \mathsf{C}(0,\infty) \cap \mathsf{C}^2(z_*, \kappa+c)$ must satisfy the ordinary differential equation (3.2a) in the continuation region $(z_*, \kappa+c)$, the instantaneous stopping condition (3.2b) in the stopping region $(0, z_*] \cup [\kappa+c, \infty)$, and the smooth pasting condition (3.2c) at the free boundary z_*.

To analyse Problem (3.2), we first assume that it admits a solution, consisting of a boundary $z_* \in (0, \kappa+c)$ and a function $\widehat{V} \in \mathsf{C}(0,\infty) \cap \mathsf{C}^2(z_*, \kappa+c)$. The latter may be expressed in terms of the general solution to the homogeneous equation (2.2), with $\alpha = \lambda + r$, and a particular solution $\widehat{v} \in \mathsf{C}^2(0,\infty)$ to the inhomogeneous equation (3.2a), as follows:

$$\widehat{V}(x) = A\phi_{\lambda+r}(x) + B\psi_{\lambda+r}(x) + \widehat{v}(x), \tag{3.3}$$

for all $x \in (z_*, \kappa+c)$, where $A, B \in \mathbb{R}$ are constants. Letting $x \downarrow z_*$ and $x \uparrow \kappa+c$ in (3.3), and substituting the resulting expressions for $\widehat{V}(z_*)$ and $\widehat{V}(\kappa+c)$ into (3.2b), allows A and B to be uniquely determined, yielding

$$\begin{aligned}\widehat{V}(x) = \big(\kappa - z_* - \widehat{v}(z_*)\big)\frac{\phi_{\lambda+r}(\kappa+c)\psi_{\lambda+r}(x) - \phi_{\lambda+r}(x)\psi_{\lambda+r}(\kappa+c)}{\phi_{\lambda+r}(\kappa+c)\psi_{\lambda+r}(z_*) - \phi_{\lambda+r}(z_*)\psi_{\lambda+r}(\kappa+c)} \\ + \big(c + \widehat{v}(\kappa+c)\big)\frac{\phi_{\lambda+r}(z_*)\psi_{\lambda+r}(x) - \phi_{\lambda+r}(x)\psi_{\lambda+r}(z_*)}{\phi_{\lambda+r}(\kappa+c)\psi_{\lambda+r}(z_*) - \phi_{\lambda+r}(z_*)\psi_{\lambda+r}(\kappa+c)} + \widehat{v}(x),\end{aligned} \tag{3.4}$$

for all $x \in (z_*, \kappa+c)$. Substituting the derivative of this expression into (3.2c) produces the following implicit characterisation of the free boundary $z_* \in (0, \kappa+c)$:

$$\begin{aligned}\big(\kappa - z_* - \widehat{v}(z_*)\big)\frac{\phi_{\lambda+r}(\kappa+c)\psi'_{\lambda+r}(z_*) - \phi'_{\lambda+r}(z_*)\psi_{\lambda+r}(\kappa+c)}{\phi_{\lambda+r}(\kappa+c)\psi_{\lambda+r}(z_*) - \phi_{\lambda+r}(z_*)\psi_{\lambda+r}(\kappa+c)} \\ + \big(c + \widehat{v}(\kappa+c)\big)\frac{\phi_{\lambda+r}(z_*)\psi'_{\lambda+r}(z_*) - \phi'_{\lambda+r}(z_*)\psi_{\lambda+r}(z_*)}{\phi_{\lambda+r}(\kappa+c)\psi_{\lambda+r}(z_*) - \phi_{\lambda+r}(z_*)\psi_{\lambda+r}(\kappa+c)} + \widehat{v}\,'(z_*) = -1.\end{aligned} \tag{3.5}$$

Finally, given $\alpha > 0$, recall that the resolvent operator $\mathscr{R}_\alpha$ acts on suitable functions $g : (0,\infty) \to \mathbb{R}$ as follows:

$$(\mathscr{R}_\alpha g)(x) \coloneqq \mathsf{E}_x\bigg(\int_0^\infty \mathrm{e}^{-\alpha t} g(X_t)\,\mathrm{d}t\bigg),$$

for all $x \in (0,\infty)$ (see Borodin and Salminen 2002, Section I.7). Moreover, $\mathscr{R}_\alpha g$ satisfies the ordinary differential equation

$$\mathcal{L}_X(\mathscr{R}_\alpha g)(x) - \alpha(\mathscr{R}_\alpha g)(x) + g(x) = 0,$$

for all $x \in (0, \infty)$. By comparing this equation with (3.2a), we see that a particular solution to the latter equation is given by

$$\widehat{v}(x) := \mathsf{E}_x\left(\int_0^\infty \mathrm{e}^{-(\lambda+r)t}\lambda(\kappa - X_t)\,\mathrm{d}t\right) = \frac{\lambda\kappa}{\lambda+r} - \frac{\lambda x}{\lambda+r-\mu}, \tag{3.6}$$

for all $x \in (0, \infty)$.

3.3. **Identifying the parameter regimes.** The previous analysis demonstrates that the existence of a solution $z_* \in (0, \kappa+c)$ to (3.5) is necessary for Problem (3.2) to admit a solution. Conversely, the function $\widehat{V} \in \mathsf{C}(0,\infty)\cap\mathsf{C}^2(z_*, \kappa+c)$ defined by (3.4) satisfies the ordinary differential equation (3.2a), as well as the boundary conditions (3.2b) and (3.2c), if $z_* \in (0, \kappa+c)$ satisfies (3.5). Hence, Problem (3.2) admits a unique solution if and only if the free-boundary equation (3.5) admits a unique solution $z_* \in (0, \kappa+c)$.

The next proposition gives a complete description of the roots of equation (3.5). In so doing, it identifies different parameter regimes that will prove useful for organising the solution to Problem (3.1) in what follows.

Proposition 3.1. *The roots of equation* (3.5) *depend on the model parameters as follows.*

(a) If the parameters satisfy

$$\mu < \frac{rc}{\kappa+c} \qquad \textit{and} \qquad r > 0, \tag{3.7a}$$

then there is a unique solution to (3.5) *given by some value*

$$z_* \in \left(0, \frac{r\kappa}{r-\mu}\right) \subset (0, \kappa+c).$$

(b) If the parameters satisfy

$$\mu \geq \frac{rc}{\kappa+c} \qquad \textit{and} \qquad r > 0, \tag{3.7b}$$

then there is no solution to (3.5) *in the interval* $(0, \kappa+c)$.

(c) If $r = 0$ *then there is no solution to* (3.5), *except if* $\mu = 0$, *in which case all values of* z_* *trivially satisfy* (3.5).

Proof. See Appendix A. □

3.4. **Constructing candidate solutions.** By analysing Proposition 3.1, we can guess the optimal stopping policy for Problem (3.1), under each of the parameter regimes identified there. Based on those guesses, we can derive expressions for the candidate value function under each regime. To simplify the exposition, and because it is the case of economic interest, we will assume that $r > 0$ in what follows.[7]

First, if Condition (3.7a) holds, Problem (3.2) admits a unique solution, comprising a free boundary $z_* \in (0, \kappa + c)$ and a function $\widehat{V} \in \mathsf{C}(0,\infty) \cap \mathsf{C}^2(z_*, \kappa + c)$. In detail, z_* satisfies the free-boundary equation (3.5), where $z_* \in (0, {}^{r\kappa}/_{(r-\mu)}) \subset (0, \kappa+c)$ is the root of (3.5), whose existence was established by Proposition 3.1(a), while $\widehat{V}$ is determined by (3.4) over $(z_*, \kappa + c)$ and $\widehat{V}(x) := \kappa - x$, for all $x \in (0, z_*] \cup [\kappa + c, \infty)$. We speculate that z_* is the optimal stopping threshold for Problem (3.1) and $\widehat{V}$ is the corresponding value function. This is economically plausible, since it implies that Problem (3.1) has a unique non-trivial solution when the drift rate of the stock price is less than the threshold ${}^{rc}/_{(\kappa+c)}$ and the discount rate is positive. A low drift rate ensures that there may be some value in waiting for the stock price to fall before closing out the short position (i.e. immediate stopping is not always optimal), while a positive discount rate ensures that the short seller should not wait forever.

Next, if Condition (3.7b) holds, the free boundary equation (3.5) does not admit a solution in $(0, \kappa + c)$, whence Problem (3.2) does not admit a solution either. However, the upper bound for the root z_* under Condition (3.7a), satisfies ${}^{r\kappa}/_{(r-\mu)} \uparrow \kappa + c$ as $\mu \uparrow {}^{rc}/_{(\kappa+c)}$, for any given $r > 0$. This suggests that the root itself may satisfy $z_* \uparrow \kappa+c$ as $\mu \uparrow {}^{rc}/_{(\kappa+c)}$. Since that root is the candidate optimal stopping threshold for Problem (3.1), under Condition (3.7a), $z_* := \kappa+c$ is the natural candidate optimal stopping threshold, under Condition (3.7b). The candidate value function $\widehat{V} \in \mathsf{C}^2(0,\infty)$ is then determined by $\widehat{V}(x) := \kappa - x$, for all $x \in (0,\infty)$. This seems economically reasonable, since it suggests that the short seller should close out his position immediately if the drift rate of the stock price is large enough (i.e. greater than or equal to ${}^{rc}/_{(\kappa+c)}$) and the discount rate is positive. In other words, waiting for a fall in the stock price destroys value if the stock price is expected to appreciate at a high enough rate.

3.5. **Verifying the candidate solutions.** Finally, we must verify that the candidate optimal stopping policies and the candidate value functions proposed above do indeed solve Problem (3.1), under their associated parameter regimes. A statement of the verification theorem is provided below which also acts to summarize the optimal solution in the different regimes. A detailed proof of the theorem is provided in Appendix B.

[7]If $r = 0$ it can be shown that it is optimal never to stop should $\mu < 0$, to stop immediately should $\mu > 0$, and if $\mu = 0$, then any stopping policy would yield the same value. Further details can be obtained from the authors upon request.

Theorem 3.2. *The optimal stopping time $\tau_* \in \mathfrak{S}$ and the value function $\widetilde{V} \in \mathsf{C}(0,\infty)$ for Problem (3.1) are given by $\tau_* = \check{\tau}_{z_*}$ and $\widetilde{V} = \widehat{V}$, where*

(a) $z_* \in (0, {r\kappa}/{(r-\mu)}) \subset (0, \kappa + c)$ is the solution to (3.5) and $\widehat{V} \in \mathsf{C}(0,\infty) \cap \mathsf{C}^2(z_*, \kappa+c)$ is determined by (3.4) over $(z_*, \kappa+c)$ and $\widehat{V}(x) = \kappa - x$, for all $x \in (0, z_*] \cup [\kappa + c, \infty)$, if Condition (3.7a) holds; and

(b) $z_* = \kappa + c$ (hence $\tau_* = 0$) and $\widehat{V} \in \mathsf{C}^2(0,\infty)$ is determined by $\widehat{V}(x) = \kappa - x$, for all $x \in (0,\infty)$, if Condition (3.7b) holds.

4. The Solution to the Original Short Selling Problem

4.1. **The constrained problem.** The optimal close-out time $\tau_* \in \mathfrak{S}$ and the value function $V \in \mathsf{C}(0,\infty)$ for the original short-selling problem (2.7) are obtained from the solution to the embedded Problem (3.1), with $x = X_0$ and κ set equal to the initial stock price X_0. The solution to that problem is presented in Theorem 3.2 and below we expose the solution to Problem (2.7) explicitly.

Given $X_0 \in (0,\infty)$, suppose (3.5), with $\kappa := X_0$, possesses a solution in the interval $(0, X_0)$. That is to say, the free-boundary equation

$$\begin{aligned} &\big(X_0 - z_* - \widehat{v}(z_*)|_{\kappa=X_0}\big) \tfrac{\phi_{\lambda+r}(X_0+c)\psi'_{\lambda+r}(z_*) - \phi'_{\lambda+r}(z_*)\psi_{\lambda+r}(X_0+c)}{\phi_{\lambda+r}(X_0+c)\psi_{\lambda+r}(z_*) - \phi_{\lambda+r}(z_*)\psi_{\lambda+r}(X_0+c)} \\ &\quad + \big(c + \widehat{v}(X_0+c)|_{\kappa=X_0}\big) \tfrac{\phi_{\lambda+r}(z_*)\psi'_{\lambda+r}(z_*) - \phi'_{\lambda+r}(z_*)\psi_{\lambda+r}(z_*)}{\phi_{\lambda+r}(X_0+c)\psi_{\lambda+r}(z_*) - \phi_{\lambda+r}(z_*)\psi_{\lambda+r}(X_0+c)} + \widehat{v}\,'(z_*)|_{\kappa=X_0} = -1. \end{aligned} \tag{4.1}$$

admits a solution $z_* \in (0, X_0)$. The optimal close-out time is then $\tau_* = \check{\tau}_{z_*}$ and the value function satisfies (3.4), with $\kappa := X_0$. In other words,

$$\begin{aligned} V(X_0) = \widehat{V}(X_0)|_{\kappa=X_0} &= \big(X_0 - z_* - \widehat{v}(z_*)|_{\kappa=X_0}\big) \tfrac{\phi_{\lambda+r}(X_0+c)\psi_{\lambda+r}(X_0) - \phi_{\lambda+r}(X_0)\psi_{\lambda+r}(X_0+c)}{\phi_{\lambda+r}(X_0+c)\psi_{\lambda+r}(z_*) - \phi_{\lambda+r}(z_*)\psi_{\lambda+r}(X_0+c)} \\ &\quad + \big(\widehat{v}(X_0+c)|_{\kappa=X_0} + c\big) \tfrac{\phi_{\lambda+r}(z_*)\psi_{\lambda+r}(X_0) - \phi_{\lambda+r}(X_0)\psi_{\lambda+r}(z_*)}{\phi_{\lambda+r}(X_0+c)\psi_{\lambda+r}(z_*) - \phi_{\lambda+r}(z_*)\psi_{\lambda+r}(X_0+c)} + \widehat{v}(X_0)|_{\kappa=X_0}. \end{aligned} \tag{4.2}$$

On the other hand, if (4.1) does not possess a solution in the interval $(0, X_0)$, the optimal repurchase price is $z_* := X_0$, which implies that the optimal close-out time is $\tau_* = \check{\tau}_{z_*} = 0$ and the value function satisfies $V(X_0) = 0$.

4.2. **The unconstrained problem.** To assess the impact of the collateral constraint and the possibility of recall on Problem (2.7), it is useful to compare the optimal close-out time and value function for that problem with the corresponding data for the unconstrained short-selling problem. That problem is obtained by letting $c \uparrow \infty$ and $\lambda \downarrow 0$ in (2.7), so that $\zeta = \infty$ and $\rho = \infty$ P_{X_0}-a.s., for all $X_0 \in (0,\infty)$. Problem (2.7) then reduces to

$$V(X_0) := \sup_{\tau \in \mathfrak{S}} \mathsf{E}_{X_0}\big(\mathrm{e}^{-r\tau}(X_0 - X_\tau)\big), \tag{4.3}$$

for all $X_0 \in (0,\infty)$. We also note that the distinction between the initial stock price in the original problem (X_0) and the initial value in the embedded problem (x) is no longer required in this limit (since the value X_0 no longer enters into the law of ζ).

Assuming that $r > 0$ and letting $c \uparrow \infty$ and $\lambda \downarrow 0$ in (4.1) gives

$$(X_0 - z_*)\frac{\phi_r'(z_*)}{\phi_r(z_*)} = -1,$$

for all $X_0 \in (0,\infty)$, by virtue of the limits

$$\lim_{\lambda\downarrow 0} \hat{v}(z) = 0 \ (\forall z \geq 0), \ \ \lim_{c\uparrow\infty}\lim_{\lambda\downarrow 0} \phi_{\lambda+r}(X_0 + c) = 0 \ \text{ and } \ \lim_{c\uparrow\infty}\lim_{\lambda\downarrow 0} \frac{c}{\psi_{\lambda+r}(X_0+c)} = 0, \tag{4.4}$$

which follow by inspection of (3.6) and (2.3). Using those limits again, the previous identity provides the following explicit formula for the optimal close-out price for the unconstrained short-selling problem:

$$z_* = \frac{\sqrt{\nu^2 + \frac{2r}{\sigma^2}} + \nu}{1 + \sqrt{\nu^2 + \frac{2r}{\sigma^2}} + \nu} X_0, \tag{4.5}$$

for all $X_0 \in (0,\infty)$. Note that $z_* \in (0, X_0)$, for all $X_0 \in (0,\infty)$, which implies the optimal close-out time $\tau_* = \check{\tau}_{z_*}$ is both strictly positive and finite. In other words, immediate close-out and waiting forever are both suboptimal for the unconstrained short-selling problem, if the discount rate is non-zero. Hence, the addition of the short-selling constraints can lead to a qualitatively different solution to the short-selling problem. Finally, letting $c \uparrow \infty$ and $\lambda \downarrow 0$ in (4.2) gives the following expression for the value function for the unconstrained short-selling problem:

$$V(X_0) = (X_0 - z_*)\frac{\phi_r(X_0)}{\phi_r(z_*)}, \tag{4.6}$$

for all $X_0 \in (0,\infty)$, by virtue of the limits in (4.4), where z_* is determined by (4.5).

5. Model Analysis

5.1. **Methdology and parameter values.** This section analyses the dependence of the optimal repurchase price and value function for the constrained short-selling problem (2.7) on the model parameters. To assess the impact of margin risk and recall risk, we compare the solution to the constrained problem with the solution to the unconstrained problem (4.3). The solid red curves in the figures below illustrate the dependence of the optimal repurchase price and value function for the constrained short-selling problem on one particular parameter, with the remaining parameters fixed. The dashed blue curves correspond to the unconstrained problem.

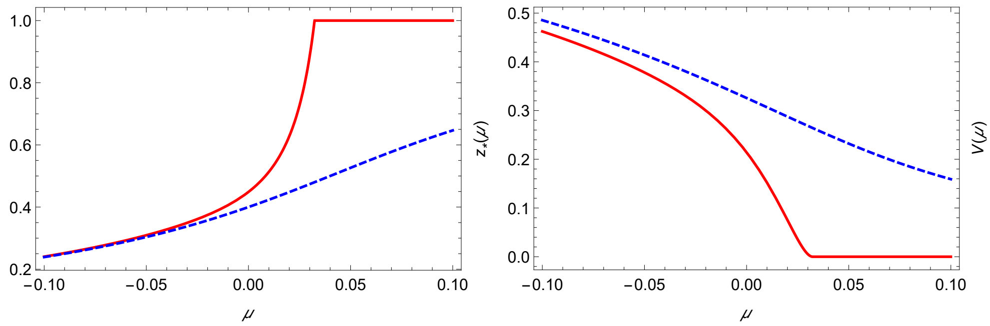
The base-case parameter values used are $\mu = \pm 0.02$, $\sigma = 0.3$, $r = 0.05$, $\lambda = 0.01$, $c = \$50$ and $X_0 = \$1$, and time is measured in years.[8] As we shall see, the solution to the constrained short-selling problem is very sensitive to the sign of the drift rate of the stock price, which is why we consider the modest negative and positive drift scenarios $\mu = -0.02$ and $\mu = 0.02$. The volatility $\sigma = 0.3$ is a reasonable proxy for observed equity implied volatilities, while the discount rate $r = 0.05$ corresponds roughly to the cost of borrowing in a developed market. The recall intensity $\lambda = 0.01$ implies that 1% of stock loans are recalled per year, on average, which is much lower than the observed frequency.[9] Finally, the values $c = \$50$ and $X_0 = \$1$ for the collateral budget and the initial stock price mean that after selling the stock for $\$1$, the trader eventually runs out of collateral when the stock price reaches $\$51$. The base-case recall intensity and collateral budget were chosen to be conservative, to avoid overstating the impact of margin risk and recall risk on the short-seller's problem.

Before we analyse the figures below in detail, we first observe that, in all cases, the optimal repurchase price for the constrained short-selling problem exceeds the optimal repurchase price for the unconstrained problem. In other words, the trader always chooses to close out earlier, when confronted with the possibility of involuntary close-out due to collateral exhaustion or early recall, than he would otherwise. This is because a more conservative strategy for the constrained problem reduces the likelihood of forced close-out, which usually results in a loss. Similarly, we observe that, in all cases, the value function for the unconstrained short-selling problem dominates the value function for the constrained problem. The vertical distance between the two curves represents the loss in value due to the short-selling constraints.

For our conservative base-case parameters described above, and in the negative drift scenario with $\mu = -0.02$, the value of the constrained problem is $\$0.300$, compared to $\$0.363$ for the unconstrained problem. Hence, short-selling constraints account for a loss in value of approximately 17% from first-best in this case. In the positive drift scenario with $\mu = 0.02$, significantly more value is lost. In this case, the value of

[8]Note the values of c and λ are not relevant in the unconstrained case.

[9]In the broker data studied by D'Avolio (2002), around 2% of the stocks on loan were recalled *per month*.



(a) Optimal close-out price $z_*(\mu)$ (b) Value function $V(\mu)$

FIGURE 5.1. The dependence of the optimal close-out price (left) and the value function (right) on the drift rate of the stock price, for the constrained (solid red curve) and unconstrained (dashed blue curve) short-selling problems.

the constrained problem is $\$0.075$, compared to $\$0.288$ for the unconstrained problem, giving a loss of 74%.

5.2. **The impact of a change in the drift rate of the stock price.** Figure 5.1 plots the optimal repurchase price and value function for the constrained and unconstrained short-selling problems, as functions of the drift rate of the stock price. In Figure 5.1(a) we observe that the constrained and unconstrained optimal repurchase prices are very similar when the drift rate is negative, since forced close-out due to collateral exhaustion is unlikely if the stock price is expected to decline over time. However, the optimal close-out policies for the constrained and unconstrained short-selling problems diverge rapidly as the drift rate of the stock price increases beyond zero. In particular, immediate repurchase is optimal ($z_* = \$1 = X_0$) when $\mu > 0.03$, in the case of the constrained problem (which means the trader will not sell the stock short in the first place), while immediate close-out is never optimal for the unconstrained problem. This is because, for such high drift rates, there is a high likelihood that the constrained trader will run out of collateral and be forced to close-out in a loss-making position, while the unconstrained trader has the luxury of waiting for the stock price to fall.

As expected, Figure 5.1(b) shows that the loss in value due to margin risk and recall risk is relatively small when the drift rate of the stock price is negative. Since the stock price is expected to decline over time in that case, the collateral constraint is unlikely to bind. Hence, the only real effect is due to the likelihood of recall (which does not depend on the drift rate). By contrast, the likelihood of forced close-out due to collateral exhaustion has a substantial impact on the constrained value function as the drift rate of the stock price increases beyond zero, resulting in a large loss of value relative to the unconstrained problem. In particular, the value of the constrained short sale is zero when $\mu > 0.03$, since the trader will not sell the stock short in the first place, while the value of the unconstrained short sale is always strictly positive.

Figure 5.1 reveals that the constrained short-selling problem is much more sensitive to the drift rate of the stock price than the unconstrained problem. This enhanced sensitivity for the constrained problem is a source of fragility. Since estimates of the drift rate are accompanied by large standard errors, in practice, the trader is likely to misestimate it substantially, causing him to pursue a materially suboptimal close-out strategy. In detail, suppose the trader has observed the stock price at $n \in \mathbb{N}$ regular intervals over the period $[0, T]$, for some $T > 0$. The sequence of observed prices is thus $(S_{i\Delta t})_{0\leq i\leq n}$, where $\Delta t := {}^{T}\!/_{n}$. Based on those observations, the maximum likelihood estimator $\hat{\mu}$ of the drift rate is determined by

$$\begin{aligned}\hat{\mu} - \frac{1}{2}\sigma^2 = \frac{1}{n\Delta t}\sum_{i=1}^{n}\ln\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} &= \frac{1}{T}\sum_{i=1}^{n}\Big(\Big(\mu - \frac{1}{2}\sigma^2\Big)\Delta t + \sigma\big(B_{i\Delta t} - B_{(i-1)\Delta t}\big)\Big)\\ &= \mu - \frac{1}{2}\sigma^2 + \frac{\sigma}{T}B_T\end{aligned}$$

(see Campbell et al. 1997, Section 9.3.2), whence

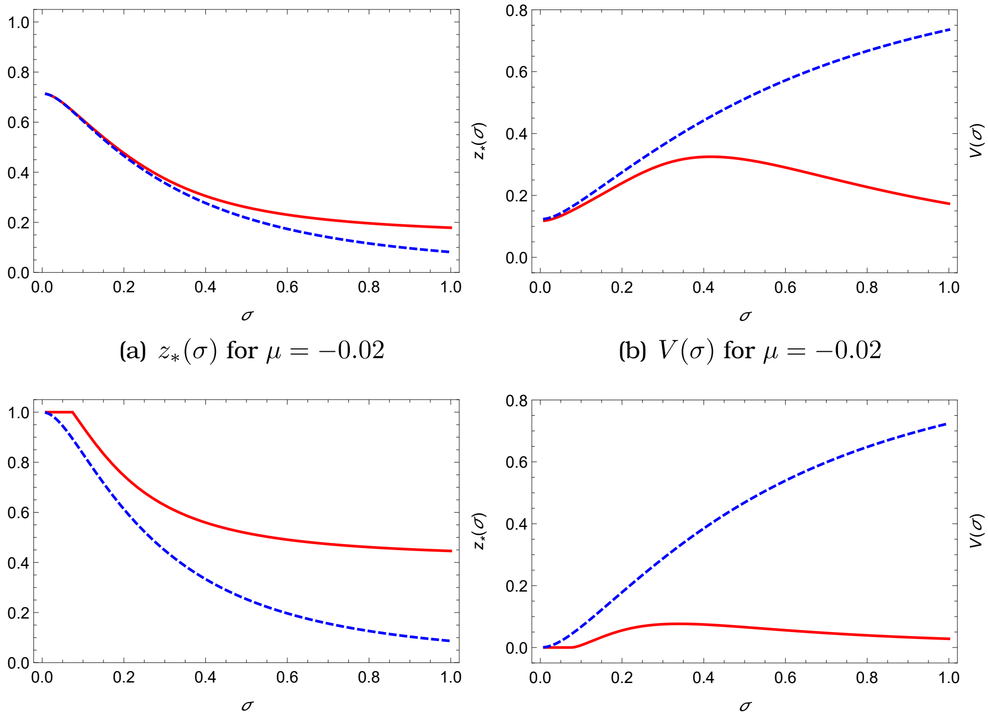
$$\hat{\mu} = \mu + \frac{\sigma}{T}B_T \sim \mathcal{N}\Big(\mu, \frac{\sigma^2}{T}\Big).$$

Hence, if the drift rate and volatility of the stock price are $\mu = 0.04$ and $\sigma = 0.3$, respectively, and if the trader has 25 years of price data, the probability that he will estimate a non-positive drift rate is

$$\mathsf{P}(\hat{\mu} \leq 0) = \mathsf{P}\Big(\mu + \frac{\sigma}{\sqrt{T}}Z \leq 0\Big) = \mathsf{P}\Big(Z \leq -\frac{\mu}{\sigma}\sqrt{T}\Big) = \mathsf{P}\Big(Z \leq -\frac{2}{3}\Big) \approx 0.2525,$$

where $Z \sim \mathcal{N}(0, 1)$. With reference to Figure 5.1(a), this implies that there is a 25% chance the trader will sell the stock short and maintain the position until its price reaches some level below $\$0.449$ (the optimal repurchase price for the constrained short sale when $\mu = 0$), instead of correctly recognising that the drift rate is too high to sell it short in the first place.[10]

[10]A possible refinement to our model would be to incorporate parameter uncertainty and learning into the trader's close-out decision, along the lines followed by Ekström and Lu (2011). They considered an investor whose choice of when to liquidate a pre-existing stock holding is confounded by uncertainty about the drift rate of the stock price. In their model, the investor updates their prior belief about the drift rate by observing the stock price over time. They showed that the investor's optimal strategy is



(a) $z_*(\sigma)$ for $\mu = -0.02$ (b) $V(\sigma)$ for $\mu = -0.02$

(c) $z_*(\sigma)$ for $\mu = 0.02$ (d) $V(\sigma)$ for $\mu = 0.02$

FIGURE 5.2. The dependence of the optimal close-out price (left) and the value function (right) on the volatility of the stock price, for the constrained (solid red curve) and unconstrained (dashed blue curve) short-selling problems. The top figures correspond to the negative drift scenario and the bottom figures to the positive drift scenario.

5.3. **The impact of a change in the volatility of the stock price.** The dependence of the optimal repurchase price and value function on the volatility of the stock price, for the constrained and unconstrained short-selling problems, is illustrated by Figure 5.2. Figures 5.2(a) and 5.2(c) show that the optimal repurchase price for both problems is inversely related to volatility. In essence, a higher volatility increases the trader's incentive to wait for the stock price to reach a lower level before closing out, since it increases the likelihood that a lower level will be reached quickly. This is analogous to the situation with American puts, where higher volatilities correspond to lower early exercise boundaries.[11] However, the optimal repurchase prices for the two problems diverge as the stock price volatility increases, since a higher volatility in the constrained case will also increase the likelihood that the collateral boundary will be breached

to liquidate as soon as the stock price falls below a certain time-dependent boundary (to be obtained numerically).

[11]To justify this analogy, note that the constrained short-selling problem (2.7) is similar to the pricing problem for an up-and-out at-the-money perpetual American put, while the unconstrained short-selling problem (4.3) is similar to the pricing problem for a vanilla at-the-money perpetual American put.

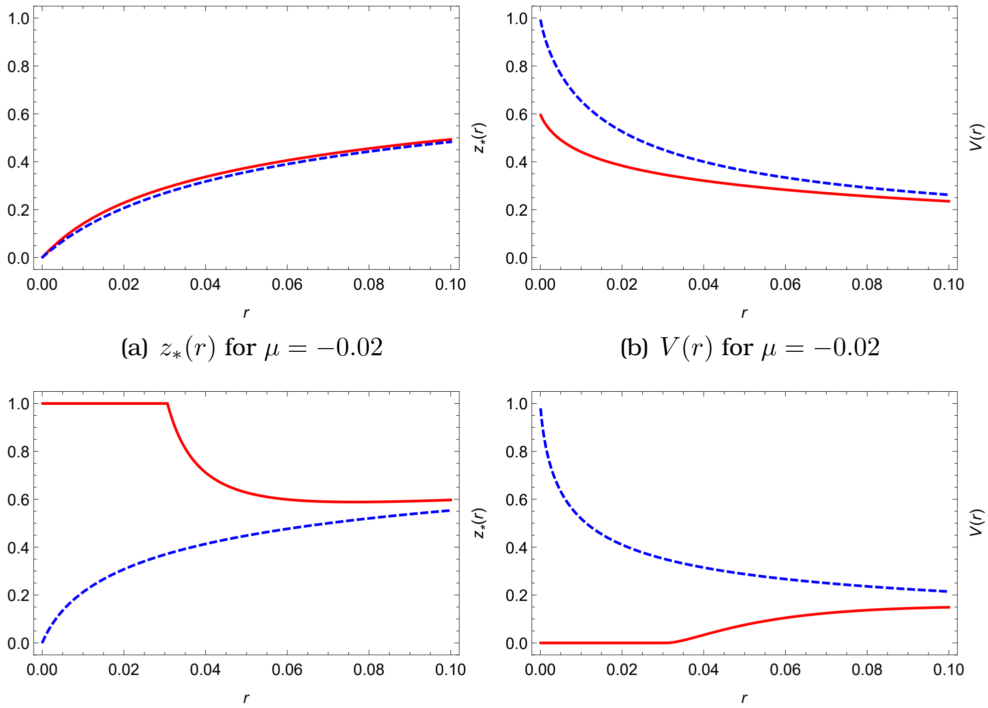
before the short sale can be closed out profitably. In addition, volatility appears to drive a larger wedge between optimal close-out strategies when the drift rate is positive than when it is negative. This indicates that high volatility levels exacerbate the effect of a positive drift, by making it more likely that the constrained trader will run out of collateral. As a result, the optimal close-out strategy for the constrained short-selling problem is more conservative relative to the optimal close-out strategy for the unconstrained problem.

Moreover, in the case of the constrained short-selling problem, Figure 5.2(c) indicates that the position should be closed out immediately ($z_* = \$1 = X_0$) when the drift rate of the stock price is positive and its volatility is small. If the stock is sold short under those circumstances, the positive drift rate will dominate the small volatility, causing a large proportion of stock price paths to reach the collateral barrier before the position can be closed out profitably. On the other hand, it is always optimal to wait before closing out the short position in the unconstrained case, even when the drift rate is positive and the volatility is small.

In Figures 5.2(b) and 5.2(d) we see that the value function for the unconstrained short-selling problem increases monotonically as the volatility of the stock price increases. This is analogous to the positive dependence of American option prices on the volatilities of their underlying assets (see Ekström 2004). By contrast, the value function for the constrained short-selling problem initially increases as the volatility of the stock price increases, before subsequently decreasing. This reflects a tradeoff, where a higher volatility increases the likelihood of the optimal repurchase price being reached quickly, while simultaneously increasing the likelihood that the trader will run out of collateral. The former effect dominates when the volatility of the stock price is low, while the latter effect dominates when it is high. The prices of knock-out barrier options exhibit a similar non-monotonic dependence on the the volatilities of their underlying assets, for the same reason (see Derman and Kani 1996).[12]

When comparing Figures 5.2(b) and 5.2(d), we observe that the value of the constrained short sale is small relative to its value when the drift rate is negative, due to the fact that collateral exhaustion is much more likely to occur in the positive drift scenario. By contrast, the unconstrained short-selling problem displays a modest decline in value across all volatilities, when the negative drift scenario is compared with

[12]Derman and Kani (1996) express this nicely by observing that "the owner of a barrier option is long volatility at the strike ... and short volatility at an out barrier."



(a) $z_*(r)$ for $\mu = -0.02$ (b) $V(r)$ for $\mu = -0.02$

(c) $z_*(r)$ for $\mu = 0.02$ (d) $V(r)$ for $\mu = 0.02$

FIGURE 5.3. The dependence of the optimal close-out price (left) and the value function (right) on the discount rate, for the constrained (solid red curve) and unconstrained (dashed blue curve) short-selling problems. The top figures correspond to the negative drift scenario and the bottom figures to the positive drift scenario.

the positive drift scenario. As expected from our analysis of the optimal close-out strategies for the constrained and unconstrained short-selling problems, we see that volatility drives a larger wedge between their value functions when the drift rate is positive, than when it is negative.

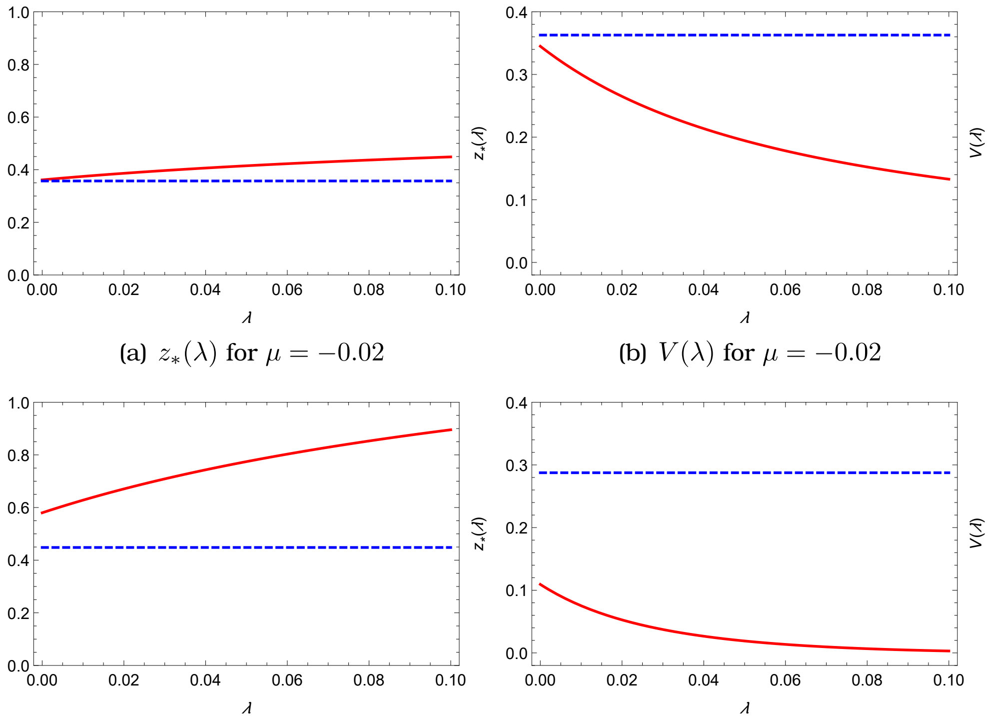
5.4. **The impact of a change in the discount rate.** In Figure 5.3 we see how the optimal repurchase price and value function depend on the discount rate, for the constrained and unconstrained short-selling problems. Figure 5.3(a) reveals that, when the drift rate is negative, there is little difference between the optimal close-out strategies for the two problems, irrespective of the discount rate. In both problems, the optimal repurchase price increases monotonically with respect to the discount rate, since a higher discount rate imposes a larger penalty for waiting. Figure 5.3(c), however, shows that the optimal repurchase price behaviour for the constrained problem is *qualitatively* different when the drift rate is positive, compared to when it is negative. Specifically, when $r \leq 0.03$, immediate close-out is optimal ($z_* = \$1 = X_0$), but waiting is optimal when $r > 0.03$. Moreover, as the discount rate increases further, the optimal repurchase prices for the constrained and unconstrained short-selling problems appear to converge.

To explain these features, note that a positive drift rate ensures a high likelihood of running out of collateral before the constrained short sale can be closed out profitably, resulting in a substantial loss. When the discount rate is low, that loss is very significant, in present value terms. However, as the discount rate increases, its present value becomes less significant, since the stock price generally takes a long time to reach the collateral barrier. Ultimately, the loss due to collateral exhaustion becomes so small, in present value terms, that it plays no role in determining the optimal close-out strategy for the constrained short-selling problem.

Figures 5.3(b) and 5.3(d) also indicate that the unconstrained short sale is considerably more valuable than the constrained short sale when the discount rate is small, but the difference becomes smaller as the discount rate increases. The reason is that losses due to collateral exhaustion are more costly, in present value terms, when the discount rate is low than when it is high. Given that it generally takes a long time for the stock price to reach the collateral barrier, a higher discount rate means the loss incurred when the collateral barrier is eventually reached is less significant, in present value terms.

When comparing Figures 5.3(b) and 5.3(d), we also see that the value function for the constrained short-selling problem in the positive drift scenario behaves very differently than in the negative drift scenario. In particular, since immediate close-out is optimal for a positive drift rate when the discount rate is small, the value function for the constrained problem is equal to zero here. On the other hand, for the unconstrained problem the value is at its largest for low discount rates. As the discount rate increases, however, the optimal close-out policies for the two problems converge and their value functions converge too.

5.5. **The impact of a change in the recall intensity.** In regard to the effect of recall risk on the optimal close-out strategy and value, we first note that the delineation of the different parameter regimes in Proposition 3.1 (and hence Theorem 3.2) do not depend on the recall intensity λ. Hence, at first blush, recall risk may seem to have a rather benign effect on the optimal close-out strategy. However, as we shall see below, the quantitative effect of even a small recall intensity can be rather large.



(a) $z_*(\lambda)$ for $\mu = -0.02$ (b) $V(\lambda)$ for $\mu = -0.02$

(c) $z_*(\lambda)$ for $\mu = 0.02$ (d) $V(\lambda)$ for $\mu = 0.02$

FIGURE 5.4. The dependence of the optimal close-out price (left) and the value function (right) on the recall intensity, for the constrained (solid red curve) and unconstrained (dashed blue curve) short-selling problems. The top figures correspond to the negative drift scenario and the bottom figures to the positive drift scenario.

The optimal repurchase price and value function for the constrained and unconstrained short-selling problems are displayed as functions of the recall intensity in Figure 5.4. The optimal repurchase price for the unconstrained problem is naturally unaffected by the recall intensity. However, the optimal repurchase price for the constrained problem increases as the recall intensity increases, as is evident in Figures 5.4(a) and 5.4(c). Essentially, a higher recall intensity increases the likelihood of recall at an inopportune time. Consequently, the optimal repurchase price for the constrained short-selling problem becomes more conservative as the recall intensity increases, which drives a wedge between the optimal close-out strategies for the two problems.

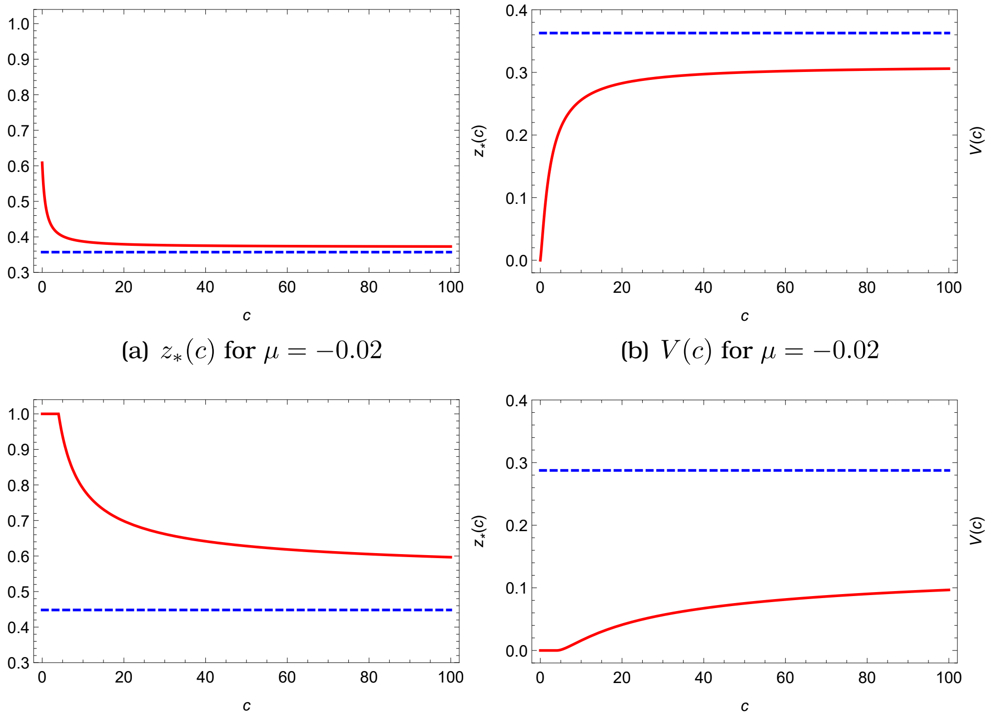
In Figures 5.4(b) and 5.4(d) we see that the value function for the constrained problem is monotonically decreasing with respect to the recall intensity. This reflects the fact that a higher recall intensity forces the trader to adopt a less 'optimal' repurchase price, relative to the unconstrained problem, with a corresponding loss in value.

To quantify the loss due to the possibility of recall, we observe from Figure 5.4(b) that the difference between the value functions for the two problems is approximately $\$0.363 - \$0.345 = \$0.018$ when $\lambda = 0$, and increases to around $\$0.363 - \$0.133 = \$0.230$ when $\lambda = 0.1$. Hence, all else being equal, the value of the short sale in the negative drift scenario is reduced by approximately $61\%[= (\$0.345 - \$0.133)/\$0.345]$ as the recall intensity λ increases from zero to 0.1. Even for our conservative base-case value of $\lambda = 0.01$, the value of the short sale is reduced by approximately 13% $[=(\$0.345 - \$0.300)/\$0.345]$, a substantial reduction in value.

For the positive drift scenario, we observe that the difference between the two value functions is approximately $\$0.288 - \$0.109 = \$0.179$ when $\lambda = 0$, and increases to about $\$0.288 - \$0.003 = \$0.285$ when $\lambda = 0.1$. Hence, all else being equal, the value of the short sale in the positive drift scenario is virtually destroyed (reducing from $\$0.109$ to $\$0.003$) as the recall intensity λ increases from zero to 0.1. Even for our conservative base case value of $\lambda = 0.01$, the value of the short sale is reduced by approximately 31% $[=(\$0.109 - \$0.075)/\$0.109]$, a substantial reduction in value, and an even greater (percentage) reduction when compared to the same change in the negative drift case.

Finally, we note that the difference between the value functions for the two problems with $\lambda = 0$ is attributable only to the collateral constraint, since the stock will never be recalled for $\lambda = 0$. As such, we can see that only $\$0.018$ ($= \$0.363 - \$0.345$) is lost for the negative drift scenario, compared to $\$0.179$ ($= \$0.288 - \$0.109$) for the positive drift scenario. This is clearly due to the increased likelihood of collateral exhaustion for a positive, versus negative, drift.

5.6. **The impact of a change in the collateral budget.** Figure 5.5 illustrates the dependence of the optimal repurchase price and value function for the constrained and unconstrained short-selling problems on collateral availability. Figures 5.5(a) and 5.5(c) confirm that the collateral budget has no impact on the optimal close-out strategy for the unconstrained problem, but we observe a dramatic impact on the optimal close-out strategy for the constrained problem. In particular, for the negative drift scenario, Figure 5.5(a) shows that it is optimal to close the constrained short sale out as soon as the stock price reaches $z_* = \$0.609$, if the margin calls arising from potential future stock price increases cannot be funded at all (i.e., $c = \$0$), but the optimal repurchase price declines rapidly as the collateral budget increases. Note that, as the collateral amount grows large, margin risk becomes unimportant and the difference between the



(a) $z_*(c)$ for $\mu = -0.02$

(b) $V(c)$ for $\mu = -0.02$

(c) $z_*(c)$ for $\mu = 0.02$

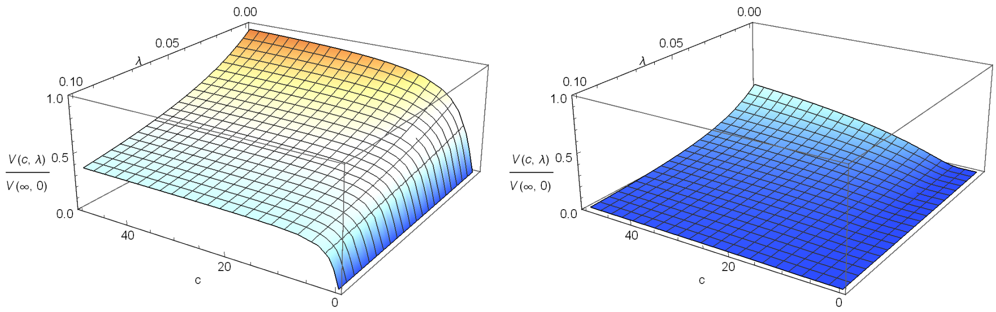
(d) $V(c)$ for $\mu = 0.02$

FIGURE 5.5. The dependence of the optimal close-out price (left) and the value function (right) on the collateral budget, for the constrained (solid red curve) and unconstrained (dashed blue curve) short-selling problems, when the drift rate of the stock price is negative. The top figures correspond to the negative drift scenario and the bottom figures to the positive drift scenario.

optimal repurchase prices for the two problems can be attributed almost exclusively to the possibility of early recall.

For the positive drift scenario, Figure 5.5(c) once again shows that the optimal repurchase price for the constrained problem is very sensitive to the collateral budget. If $c \leq \$4$, we see that immediate close-out ($z_* = \$1 = X_0$) of the constrained short sale is optimal, since the collateral budget is insufficient to fund the margin calls that are likely to occur before the position can be closed-out profitably. However, as the amount of available collateral increases beyond $c = \$4$, the optimal repurchase price for the constrained short-selling problem decreases rapidly, before levelling off.

Figures 5.5(b) and 5.5(d) reveal that margin risk has no impact on the value of the unconstrained short sale, while its impact on the value of the constrained short sale is dramatic. In particular, for the negative drift scenario, the constrained short position is valueless when there is no collateral to fund margin calls, but its value increases rapidly as the collateral budget increases. For the positive drift scenario, we observe



(a) (b)

FIGURE 5.6. The dependence of the value function for the constrained short-selling problem on both the recall intensity and the collateral budget. The negative drift case is shown in (a) and the positive drift case is shown in (b). For comparison, each value function has been scaled by the value of the unconstrained problem for the given value of μ, and the shading and axes are consistent across both plots.

from Figure 5.5(d) that the value of the constrained short sale is zero when $c \leq \$4$ (since immediate close-out is optimal in that case), but it gradually increases as the amount of available collateral increases beyond $c = \$4$.

5.7. **The interaction of recall intensity and collateral budget.** To illustrate the dependency between the two risks, Figure 5.6 plots the value function for the constrained short-selling problem as a function of both the recall intensity λ and the collateral budget c. Specifically, Figure 5.6(a) plots this joint dependency in the case of a negative drift ($\mu = -0.02$) and Figure 5.6(b) in the case of a positive drift ($\mu = 0.02$).

In both cases, we observe that for smaller collateral budgets, the value function becomes less sensitive to the level of the recall intensity. The economic interpretation is that, for lower levels of collateral, there is a smaller probability that the short sale will be recalled before the collateral constraint becomes binding (or the trader has optimally closed out their position). Thus, the impact of recall risk decreases as the collateral constraint becomes more salient. Similarly, we observe that margin risk becomes less important as the likelihood of recall increases. For example, for very large values of the recall intensity, the collateral budget needs to be very small for it to have a significant impact on the value of the short sale (since a healthy collateral budget is unlikely to be exhausted before recall or optimal closure).

Finally, we note that Figure 5.6 allows us to compute the trade-off between both constraints within our model. In particular, the contours of Figure 5.6 can be used to find all the combinations of the recall intensity λ and collateral level c that lead to the same value of the short sale (and hence the same reduction in value relative to the unconstrained case).

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Appendix A. Proof of Proposition 3.1

Proof. Firstly, we recall the definitions of the scale function and the Wronskian associated with the process (2.1), which will be used extensively in the analysis below. The scale function is given by

$$\mathfrak{s}(x) := \begin{cases} -\frac{x^{-2\nu}}{2\nu} & \text{if } \nu \neq 0; \\ \ln x & \text{if } \nu = 0 \end{cases} \tag{A.1}$$

for all $x \in (0, \infty)$ and the Wronskian by

$$w_\alpha := \frac{\phi_\alpha(x)\psi'_\alpha(x) - \phi'_\alpha(x)\psi_\alpha(x)}{\mathfrak{s}'(x)} = 2\sqrt{\nu^2 + \frac{2\alpha}{\sigma^2}}, \tag{A.2}$$

which is independent of $x \in (0, \infty)$ (see Borodin and Salminen 2002, Appendix 1.20).

Next, to evaluate the roots of equation (3.5) we introduce the function $H \in \mathrm{C}^2(0, \infty)$, given by

$$H(z) := \frac{F'(z)G(z) - F(z)G'(z)}{w_{\lambda+r}\mathfrak{s}'(z)} + F(\kappa + c), \tag{A.3}$$

for all $z \in (0, \infty)$ and $F, G \in \mathrm{C}^2(0, \infty)$ are given by

$$F(z) := \widehat{v}(z) - (\kappa - z) = \frac{r - \mu}{\lambda + r - \mu} z - \frac{r\kappa}{\lambda + r}$$

and

$$G(z) := \phi_{\lambda+r}(\kappa + c)\psi_{\lambda+r}(z) - \phi_{\lambda+r}(z)\psi_{\lambda+r}(\kappa + c),$$

for all $z \in (0, \infty)$. It follows that $z_* \in (0, \kappa + c)$ satisfies (3.5) if and only if $H(z_*) = 0$, by virtue of the identity

$$w_{\lambda+r}\mathfrak{s}'(z) = \phi_{\lambda+r}(z)\psi'_{\lambda+r}(z) - \phi'_{\lambda+r}(z)\psi_{\lambda+r}(z),$$

for all $z \in (0, \infty)$.

From (A.3) we have

$$\begin{aligned}
H'(z) &= \frac{F''(z)G(z) - F(z)G''(z) + \frac{2\mu z}{\sigma^2 z^2}\big(F'(z)G(z) - F(z)G'(z)\big)}{w_{\lambda+r}\mathfrak{s}'(z)} \\
&= \frac{F''(z)G(z) - \frac{2}{\sigma^2 z^2}F(z)\big((\lambda+r)G(z) - \mu z G'(z)\big) + \frac{2\mu z}{\sigma^2 z^2}\big(F'(z)G(z) - F(z)G'(z)\big)}{w_{\lambda+r}\mathfrak{s}'(z)} \\
&= \frac{2}{\sigma^2 z^2}\frac{\mathcal{L}_X F(z) - (\lambda+r)F(z)}{w_{\lambda+r}\mathfrak{s}'(z)}G(z) \\
&= \frac{2}{\sigma^2 z^2}\frac{\mathcal{L}_X \widehat{v}(z) - (\lambda+r)\widehat{v}(z) + \mu z + (\lambda+r)(\kappa - z)}{w_{\lambda+r}\mathfrak{s}'(z)}G(z) \\
&= \frac{2}{\sigma^2 z^2}\frac{r\kappa + (\mu - r)z}{w_{\lambda+r}\mathfrak{s}'(z)}G(z),
\end{aligned}$$

for all $z \in (0,\infty)$. Note that the first equality above follows from the identity $\mathfrak{s}''(z) = -\frac{2\mu}{\sigma^2 z}\mathfrak{s}'(z)$ (see Borodin and Salminen 2002, Section II.9), the second equality follows from the fact that G satisfies (2.2) with $\alpha = \lambda + r$, and the final equality follows since $\widehat{v}$ satisfies (3.2a). Note that $G(\kappa + c) = 0$ implies that $H'(\kappa + c) = 0$. Also note that $G(\kappa + c) = 0$ and $G'(\kappa + c) = w_{\lambda+r}\mathfrak{s}'(\kappa + c)$ imply that $H(\kappa + c) = 0$. That is to say, H has a root at $\kappa + c$, which is also a stationary point. However, $\kappa + c$ is not a solution to (3.5), since the left-hand side of that equation is not well-defined if $z_* = \kappa + c$. Next, since $\phi_{\lambda+r}$ and $\psi_{\lambda+r}$ are strictly decreasing and increasing, respectively, it follows that G is strictly increasing. Hence, $G(z) < 0$, for all $z \in (0, \kappa + c)$; $G(\kappa + c) = 0$; and $G(z) > 0$, for all $z \in (\kappa + c, \infty)$. Finally, note that $\mathfrak{s}'(z) > 0$, for all $z \in (0,\infty)$.

We now consider the roots of $H(z) = 0$ in each of the three parameter regimes (a)–(c).

(a): Suppose Condition (3.7a) holds, in which case

$$0 < r - \frac{rc}{\kappa + c} = \frac{r\kappa}{\kappa + c} < r - \mu,$$

so that the point $\bar{z} := {r\kappa}/{(r-\mu)}$ satisfies $\bar{z} \in (0, \kappa + c)$. Observe that $r\kappa + (\mu - r)z > 0$, for all $z \in (0, \bar{z})$; $r\kappa + (\mu - r)\bar{z} = 0$; and $r\kappa + (\mu - r)z < 0$, for all $z \in (\bar{z}, \infty)$. Combined with the properties of G and $\mathfrak{s}'$ described earlier, this implies that $H'(z) < 0$, for all $z \in (0, \bar{z})$; $H'(\bar{z}) = 0$; $H'(z) > 0$, for all $z \in (\bar{z}, \kappa + c)$; $H'(\kappa + c) = 0$; and $H'(z) < 0$, for all $z \in (\kappa + c, \infty)$. In particular, H has stationary points at $\bar{z}$ and $\kappa + c$, with former being a local minimum and the latter being a local maximum. Since $H(\kappa + c) = 0$ and H is strictly increasing over $(\bar{z}, \kappa + c)$, it follows that $H(\bar{z}) < 0$. Furthermore, $H(\kappa + c) = 0$ rules out the existence of roots in the intervals $(\bar{z}, \kappa + c)$ and $(\bar{z}, \kappa + c)$ and $(\kappa + c, \infty)$, since H is strictly increasing over the former interval and strictly decreasing over the latter interval. Finally, we use (2.3) to write

$$\begin{aligned} H(z) &= \frac{r-\mu}{\lambda+r-\mu}\frac{\phi_{\lambda+r}(\kappa+c)\psi_{\lambda+r}(z)-\phi_{\lambda+r}(z)\psi_{\lambda+r}(\kappa+c)}{w_{\lambda+r}\mathfrak{s}'(z)} \\ &\quad + \left(\frac{r\kappa}{\lambda+r}-\frac{r-\mu}{\lambda+r-\mu}z\right)\frac{\phi_{\lambda+r}(\kappa+c)\psi'_{\lambda+r}(z)-\phi'_{\lambda+r}(z)\psi_{\lambda+r}(\kappa+c)}{w_{\lambda+r}\mathfrak{s}'(z)} + F(\kappa+c) \\ &= \frac{r-\mu}{\lambda+r-\mu}\frac{\phi_{\lambda+r}(\kappa+c)}{w_{\lambda+r}}\left(\frac{z}{\sqrt{\nu^2+\frac{2(\lambda+r)}{\sigma^2}}-\nu}+\frac{r\kappa}{\lambda+r}\frac{\lambda+r-\mu}{r-\mu}-z\right)\frac{\psi'_{\lambda+r}(z)}{\mathfrak{s}'(z)} \\ &\quad + \frac{r-\mu}{\lambda+r-\mu}\frac{\psi_{\lambda+r}(\kappa+c)}{w_{\lambda+r}}\left(\frac{z}{\sqrt{\nu^2+\frac{2(\lambda+r)}{\sigma^2}}+\nu}-\frac{r\kappa}{\lambda+r}\frac{\lambda+r-\mu}{r-\mu}+z\right)\frac{\phi'_{\lambda+r}(z)}{\mathfrak{s}'(z)} \\ &\quad + F(\kappa+c), \end{aligned}$$

for all $z \in (0,\infty)$. Now, since

$$\lim_{z\downarrow 0}\frac{\phi'_{\lambda+r}(z)}{\mathfrak{s}'(z)} = \lim_{z\downarrow 0}\left(-\sqrt{\nu^2+\frac{2(\lambda+r)}{\sigma^2}}-\nu\right)z^{-\sqrt{\nu^2+2(\lambda+r)/\sigma^2}+\nu} = -\infty$$

and

$$\lim_{z\downarrow 0}\frac{\psi'_{\lambda+r}(z)}{\mathfrak{s}'(z)} = \lim_{z\downarrow 0}\left(\sqrt{\nu^2+\frac{2(\lambda+r)}{\sigma^2}}-\nu\right)z^{\sqrt{\nu^2+2(\lambda+r)/\sigma^2}+\nu} = 0,$$

by virtue of (2.3) and (A.1), it follows that $H(0+) = \infty$.[13] This, in turn, ensures the existence of a unique root $z_* \in (0,\bar{z})$, since H is strictly decreasing over $(0,\bar{z})$, with $H(\bar{z}) < 0$.

(b): Suppose Condition (3.7b) holds. In this case, either $\mu = {rc}/{(\kappa+c)}$, ${rc}/{(\kappa+c)} < \mu < r$, or $\mu \geq r$, and in each of these three cases the function H has qualitatively different behaviour.

Firstly, if $\mu = {rc}/{(\kappa+c)}$, then $r-\mu = {r\kappa}/{(\kappa+c)} > 0$. Consequently, $r\kappa + (\mu-r)z > 0$, for all $z \in (0,\kappa+c)$; $r\kappa+(\mu-r)(\kappa+c) = 0$; and $r\kappa+(\mu-r)z < 0$, for all $z \in (\kappa+c,\infty)$. Combined with the properties of G and $\mathfrak{s}'$ described earlier, this implies that $H'(z) < 0$, for all $z \in (0,\kappa+c)$; $H'(\kappa+c) = 0$; and $H'(z) < 0$, for all $z \in (\kappa+c,\infty)$. That is to say, H is strictly decreasing over $(0,\kappa+c)$ and $(\kappa+c,\infty)$, with an inflection point at $\kappa+c$. Since $H(\kappa+c) = 0$, it follows that $\kappa+c$ is the only root of H.

Secondly, if ${rc}/{(\kappa+c)} < \mu < r$ holds, then

$$0 < r-\mu < r-\frac{rc}{\kappa+c} = \frac{r\kappa}{\kappa+c},$$

[13]Note that the values for the two limits above can also be inferred from the fact that the origin is a natural boundary for X (see Borodin and Salminen 2002, Section II.10).

so that the point $\bar{z} := {}^{r\kappa}\!/\!_{(r-\mu)}$ satisfies $\bar{z} \in (\kappa + c, \infty)$. Observe that $r\kappa + (\mu - r)z > 0$, for all $z \in (0, \bar{z})$; $r\kappa + (\mu - r)\bar{z} = 0$; and $r\kappa + (\mu - r)z < 0$, for all $z \in (\bar{z}, \infty)$. Combined with the properties of G and $\mathfrak{s}'$ described earlier, this implies that $H'(z) < 0$, for all $z \in (0, \kappa + c)$; $H'(\kappa + c) = 0$; $H'(z) > 0$, for all $z \in (\kappa + c, \bar{z})$; $H'(\bar{z}) = 0$; and $H'(z) < 0$, for all $z \in (\bar{z}, \infty)$. In particular, H has stationary points at $\kappa + c$ and $\bar{z}$, with former being a local minimum and the latter being a local maximum. Since $H(\kappa + c) = 0$ and H is strictly increasing over $(\kappa + c, \bar{z})$, it follows that $H(\bar{z}) > 0$. Furthermore, $H(\kappa + c) = 0$ rules out the existence of roots in the intervals $(0, \kappa + c)$ and $(\kappa + c, \bar{z})$, since H is strictly decreasing over the former interval and strictly increasing over the latter interval. Finally, we use (2.3) to write

$$\begin{aligned}
H(z) &= \frac{r-\mu}{\lambda+r-\mu}\frac{\phi_{\lambda+r}(\kappa+c)\psi_{\lambda+r}(z) - \phi_{\lambda+r}(z)\psi_{\lambda+r}(\kappa+c)}{w_{\lambda+r}\mathfrak{s}'(z)} \\
&\quad + \left(\frac{r\kappa}{\lambda+r} - \frac{r-\mu}{\lambda+r-\mu}z\right)\frac{\phi_{\lambda+r}(\kappa+c)\psi'_{\lambda+r}(z) - \phi'_{\lambda+r}(z)\psi_{\lambda+r}(\kappa+c)}{w_{\lambda+r}\mathfrak{s}'(z)} + F(\kappa+c) \\
&= \frac{r-\mu}{\lambda+r-\mu}\frac{\phi_{\lambda+r}(\kappa+c)}{w_{\lambda+r}} \\
&\qquad \times \left(1 + \left(\sqrt{\nu^2 + \frac{2(\lambda+r)}{\sigma^2}} - \nu\right)\left(\frac{r\kappa}{\lambda+r}\frac{\lambda+r-\mu}{r-\mu}\frac{1}{z} - 1\right)\right)\frac{\psi_{\lambda+r}(z)}{\mathfrak{s}'(z)} \\
&\quad - \frac{r-\mu}{\lambda+r-\mu}\frac{\psi_{\lambda+r}(\kappa+c)}{w_{\lambda+r}} \\
&\qquad \times \left(1 - \left(\sqrt{\nu^2 + \frac{2(\lambda+r)}{\sigma^2}} + \nu\right)\left(\frac{r\kappa}{\lambda+r}\frac{\lambda+r-\mu}{r-\mu}\frac{1}{z} - 1\right)\right)\frac{\phi_{\lambda+r}(z)}{\mathfrak{s}'(z)} \\
&\quad + F(\kappa+c),
\end{aligned}$$

for all $z \in (0, \infty)$. Note that

$$(\nu+1)^2 - \nu^2 = 2\nu + 1 = \frac{2\mu}{\sigma^2} < \frac{2(\lambda+r)}{\sigma^2}$$

since $\mu < r < \lambda + r$, from which it follows that

$$\sqrt{\nu^2 + \frac{2(\lambda+r)}{\sigma^2}} > \nu + 1. \tag{A.4}$$

Consequently,

$$\lim_{z\uparrow\infty}\left(\sqrt{\nu^2 + \frac{2(\lambda+r)}{\sigma^2}} - \nu\right)\left(\frac{r\kappa}{\lambda+r}\frac{\lambda+r-\mu}{r-\mu}\frac{1}{z} - 1\right) < -1.$$

Moreover, (2.3) and (A.1) give

$$\lim_{z\uparrow\infty}\frac{\phi_{\lambda+r}(z)}{\mathfrak{s}'(z)} = \lim_{z\uparrow\infty} z^{-\sqrt{\nu^2+2(\lambda+r)/\sigma^2}+\nu+1} = 0$$

and

$$\lim_{z\uparrow\infty}\frac{\psi_{\lambda+r}(z)}{\mathfrak{s}'(z)} = \lim_{z\uparrow\infty} z^{\sqrt{\nu^2+2(\lambda+r)/\sigma^2}+\nu+1} = \infty,$$

where the first limit follows from (A.4). Putting all of this together gives $H(\infty-) = -\infty$. This, in turn, ensures the existence of a unique root $z_* \in (\bar{z}, \infty)$, since H is strictly decreasing over $(\bar{z}, \infty)$ with $H(\bar{z}) > 0$.

Thirdly, if $\mu \geq r$ holds, then $r\kappa + (\mu - r)z \geq r\kappa > 0$, for all $z \in (0, \infty)$. Combined with the properties of G and $\mathfrak{s}'$ described earlier, this implies that $H'(z) < 0$, for all $z \in (0, \kappa + c)$; $H'(\kappa + c) = 0$; and $H'(z) > 0$, for all $z \in (\kappa + c, \infty)$. That is to say, H achieves a unique global minimum at $\kappa + c$. Moreover, since $H(\kappa + c) = 0$, it follows that $H(z) > 0$, for all $z \in (0, \kappa + c) \cup (\kappa + c, \infty)$. In other words, H has a unique root at $\kappa + c$.

(c): Firstly, if $r = \mu = 0$ then $F(z) = 0$, and consequently $H(z) = 0$, for all $z \in (0, \infty)$. Next, if $r = 0$ and $\mu < 0$ (resp. $\mu > 0$) then $r\kappa + (\mu - r)z = \mu z < 0$ (> 0), for all $z \in (0, \infty)$. Combined with the properties of G and $\mathfrak{s}'$ described earlier, this implies that $H'(z) > 0$ (< 0), for all $z \in (0, \kappa + c)$; $H'(\kappa + c) = 0$; and $H'(z) < 0$ (> 0), for all $z \in (\kappa + c, \infty)$. That is to say, H achieves a unique global maximum (minimum) at $\kappa + c$. Since $H(\kappa + c) = 0$, it follows that $\kappa + c$ is the only root of H.

□

Appendix B. Proof of Theorem 3.2

Firstly, we provide the following lemma required for the proof of Theorem 3.2. It derives a lower bound for the candidate value function under conditions (3.7a).

Lemma B.1. *Suppose Condition (3.7a) holds. Let* $z_* \in (0, {r\kappa}/{(r-\mu)}) \subset (0, \kappa + c)$ *be the unique solution to (3.5), whose existence is established by Proposition 3.1(a), and let* $\widehat{V} \in \mathsf{C}(0, \infty) \cap \mathsf{C}^2(z_*, \kappa + c)$ *be determined by (3.4) over* $(z_*, \kappa + c)$ *and* $\widehat{V}(x) := \kappa - x$*, for all* $x \in (0, z_*] \cup [\kappa + c, \infty)$*. Then* $\widehat{V}(x) > \kappa - x$*, for all* $x \in (z_*, \kappa + c)$*.*

Proof. Define the function $U \in \mathsf{C}(0, \infty) \cap \mathsf{C}^2(z_*, \kappa + c)$, by setting $U(x) := \widehat{V}(x) - (\kappa - x)$, for all $x \in (0, \infty)$. We shall demonstrate that $U(x) > 0$, for all $x \in (z_*, \kappa + c)$. To begin with, observe that

$$\begin{aligned}\mathcal{L}_X U(x) - (\lambda + r)U(x) &= \mathcal{L}_X \widehat{V}(x) - (\lambda + r)\widehat{V}(x) + \mu x + (\lambda + r)(\kappa - x) \\ &= -\lambda(\kappa - x) + \mu x + (\lambda + r)(\kappa - x) = r\kappa - (r - \mu)x,\end{aligned} \tag{B.1}$$

for all $x \in (z_*, \kappa + c)$, by virtue of (3.2a). In particular,

$$\begin{aligned}\frac{1}{2}\sigma^2 z_*^2 U''(z_*+) &= \frac{1}{2}\sigma^2 z_*^2 U''(z_*+) + \mu z_* U'(z_*+) - (\lambda + r)U(z_*+) \\ &= \mathcal{L}_X U(z_*+) - (\lambda + r)U(z_*+) = r\kappa - (r - \mu)z_* > 0,\end{aligned} \tag{B.2}$$

since $U(z_*+) = U(z_*) = 0$ and $U'(z_*+) = 0$, due to (3.2b), (3.2c) and the continuity of $\widehat{V}$, and since $z_* < {}^{r\kappa}/_{(r-\mu)}$. So, $U''(z_*+) > 0$, which together with $U'(z_*+) = U(z_*+) = 0$, ensures the existence of some $\varepsilon > 0$, such that $U(x) > U(z_*) = 0$, for all $x \in (z_*, z_* + \varepsilon)$. In particular, given any $x \in (z_*, {}^{r\kappa}/_{(r-\mu)}]$, it follows that

$$\max_{\xi \in [z_*, x]} U(\xi) > U(z_*) = 0,$$

whence U has a positive maximum over $[z_*, x]$, which is realised either at an interior point of the interval or at the right end-point. However, given any $x \in (z_*, {}^{r\kappa}/_{(r-\mu)}]$, (B.1) yields

$$\mathcal{L}_X U(\xi) - (\lambda + r)U(\xi) = r\kappa - (r - \mu)\xi > 0,$$

for all $\xi \in (z_*, x) \subseteq (z_*, {}^{r\kappa}/_{(r-\mu)})$. Based on this differential inequality, the maximum principle (see Protter and Weinberger 1967, Theorem 1.3) asserts that U cannot realise its maximum in the interior of $[z_*, x]$, for any $x \in (z_*, {}^{r\kappa}/_{(r-\mu)}]$, since it is a non-constant function with a non-negative maximum over the interval. Consequently,

$$U(x) = \max_{\xi \in [z_*, x]} U(\xi) > U(z_*) = 0,$$

for all $x \in (z_*, {}^{r\kappa}/_{(r-\mu)}]$. Next, observe that

$$\max_{x \in [\frac{r\kappa}{r-\mu}, \kappa + c]} -U(x) \geq -U(\kappa + c) = 0,$$

by virtue of (3.2b). That is to say, $-U$ has a non-negative maximum over $[\frac{r\kappa}{r-\mu}, \kappa + c]$, which is realised either at an interior point of the interval or at the right end-point, since we have already established that $-U({}^{r\kappa}/_{(r-\mu)}) < 0$. Another application of (B.1) gives

$$\mathcal{L}_X(-U)(x) - (\lambda + r)(-U)(x) = -\big(\mathcal{L}_X U(x) - (\lambda + r)U(x)\big) = (r - \mu)x - r\kappa > 0,$$

for all $x \in ({}^{r\kappa}/(r-\mu), \kappa{+}c)$. Once again, the maximum principle ensures that $-U$ cannot achieve its maximum in the interior of $[{}^{r\kappa}/(r-\mu), \kappa{+}c]$, since it is a non-constant function with a non-negative maximum over the interval. It follows that $-U$ must have a unique maximum at the right end-point of the interval, which implies that $U(x) > U(\kappa{+}c) = 0$, for all $x \in [{}^{r\kappa}/(r-\mu), \kappa + c)$. $\square$

We now proceed with the proof of Theorem 3.2 in the parameter regimes (a) and (b).

Proof. (a): Suppose Condition (3.7a) holds, in which case $z_* \in (0, \kappa + c)$ is the unique solution to (3.5) and $\widehat{V} \in \mathsf{C}(0, \infty) \cap \mathsf{C}^2(z_*, \kappa + c)$ is determined by (3.4) over $(z_*, \kappa + c)$ and $\widehat{V}(x) = \kappa - x$, for all $x \in (0, z_*] \cup [\kappa + c, \infty)$. Note that $\widehat{V} \in \mathsf{C}^1(0, \kappa + c]$, since (3.2c) ensures that $\widehat{V}'$ is continuous at z_* and $|\widehat{V}'((\kappa + c)-)| < \infty$, by inspection of (3.4). On the other hand, $\widehat{V} \notin \mathsf{C}^2(0, \kappa + c]$, since $\widehat{V}''(z_*+) > 0 = \widehat{V}''(z_*-)$, as remarked in the discussion following Lemma B.1. Consequently, the standard Itô formula cannot be applied to the process

$$\mathbb{R}_+ \ni t \mapsto \mathrm{e}^{-(\lambda+r)(t\wedge\hat{\tau}_{\kappa+c})}\widehat{V}(X_{t\wedge\hat{\tau}_{\kappa+c}}).$$

However, $\widehat{V} \in \mathsf{C}^2(0, z_*] \cap \mathsf{C}^2[z_*, \kappa + c]$, since $|\widehat{V}''(z_*+)| < \infty$ and $|\widehat{V}''((\kappa + c)-)| < \infty$, by inspection of (3.4). Hence, we may apply the local time-space formula of Peskir (2005) to the above-mentioned process, to get

$$\begin{aligned}
\mathrm{e}^{-(\lambda+r)(t\wedge\hat{\tau}_{\kappa+c})}\widehat{V}(X_{t\wedge\hat{\tau}_{\kappa+c}}) &= \widehat{V}(X_0) - \int_0^{t\wedge\hat{\tau}_{\kappa+c}} (\lambda + r)\mathrm{e}^{-(\lambda+r)s}\widehat{V}(X_s)\,\mathrm{d}s \\
&\quad + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\widehat{V}'(X_s)\,\mathrm{d}X_s + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathbf{1}_{\{X_s\neq z_*\}}\mathrm{e}^{-(\lambda+r)s}\frac{1}{2}\widehat{V}''(X_s)\,\mathrm{d}\langle X\rangle_s \\
&\quad + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathbf{1}_{\{X_s=z_*\}}\mathrm{e}^{-(\lambda+r)s}\frac{1}{2}\big(\widehat{V}'(X_s+) - \widehat{V}'(X_s-)\big)\,\mathrm{d}\ell_s^{z_*}(X),
\end{aligned}$$

for all $t \geq 0$, where the local time process $\ell^{z_*}(X)$ quantifies the time X spends in the immediate vicinity of z_* (see Peskir and Shiryaev 2006, Section 3.5). Note that the final term above is zero, since (3.2c) ensures that $\widehat{V}'(z_*+) = \widehat{V}'(z_*-)$, while the identities

$$\begin{aligned}
\widehat{V}(X_s) &= \mathbf{1}_{\{X_s\leq z_*\}}(\kappa - X_s) + \mathbf{1}_{\{X_s>z_*\}}\widehat{V}(X_s), \\
\widehat{V}'(X_s) &= -\mathbf{1}_{\{X_s\leq z_*\}} + \mathbf{1}_{\{X_s>z_*\}}\widehat{V}'(X_s),
\end{aligned}$$

and

$$\mathbf{1}_{\{X_s\neq z_*\}}\widehat{V}''(X_s) = \mathbf{1}_{\{X_s>z_*\}}\widehat{V}''(X_s),$$

for all $s \geq 0$, follow from the definition of $\widehat{V}$. We therefore obtain

$$
\begin{aligned}
&\mathrm{e}^{-(\lambda+r)(t\wedge\hat{\tau}_{\kappa+c})}\widehat{V}(X_{t\wedge\hat{\tau}_{\kappa+c}}) \\
&= \widehat{V}(X_0) + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathbf{1}_{\{X_s\leq z_*\}}\mathrm{e}^{-(\lambda+r)s}\big(-\mu X_s - (\lambda+r)(\kappa - X_s)\big)\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathbf{1}_{\{X_s> z_*\}}\mathrm{e}^{-(\lambda+r)s}\Big(\mu X_s\widehat{V}'(X_s) + \frac{1}{2}\sigma^2X_s^2\widehat{V}''(X_s) - (\lambda+r)\widehat{V}(X_s)\Big)\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\sigma X_s\widehat{V}'(X_s)\,\mathrm{d}B_s \\
&= \widehat{V}(X_0) + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathbf{1}_{\{X_s\leq z_*\}}\mathrm{e}^{-(\lambda+r)s}\big((r-\mu)X_s - r\kappa - \lambda(\kappa - X_s)\big)\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathbf{1}_{\{X_s> z_*\}}\mathrm{e}^{-(\lambda+r)s}\big(\mathcal{L}_X\widehat{V}(X_s) - (\lambda+r)\widehat{V}(X_s)\big)\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\sigma X_s\widehat{V}'(X_s)\,\mathrm{d}B_s \\
&\leq \widehat{V}(X_0) - \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \lambda\mathrm{e}^{-(\lambda+r)s}(\kappa - X_s)\,\mathrm{d}s + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\sigma X_s\widehat{V}'(X_s)\,\mathrm{d}B_s,
\end{aligned}
$$

for all $t \geq 0$, where the inequality follows from

$$
\mathbf{1}_{\{X_s\leq z_*\}}\big((r-\mu)X_s - r\kappa - \lambda(\kappa - X_s)\big) \leq -\mathbf{1}_{\{X_s\leq z_*\}}\lambda(\kappa - X_s),
$$

for all $s \geq 0$, since Proposition 3.1(a) established that $z_* < {}^{r\kappa}/_{(r-\mu)}$, together with

$$
\mathbf{1}_{\{X_s> z_*\}}\big(\mathcal{L}_X\widehat{V}(X_s) - (\lambda+r)\widehat{V}(X_s)\big) = -\mathbf{1}_{\{X_s> z_*\}}\lambda(\kappa - X_s),
$$

for all $s \geq 0$, since $\widehat{V}$ satisfies (3.2a) over $(z_*, \kappa + c)$. Next, observe that $\widehat{V}'$ is bounded over $(0, \kappa + c]$, since $\widehat{V} \in \mathsf{C}^1(0, \kappa + c]$ and $\widehat{V}'(0+) = -1$. This is sufficient to ensure that the local martingale

$$
\mathbb{R}_+ \ni t \mapsto \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\sigma X_s\widehat{V}'(X_s)\,\mathrm{d}B_s
$$

is in fact a uniformly integrable martingale. Given an arbitrary stopping time $\tau \in \mathfrak{S}$, an application of the optional sampling theorem then yields

$$
\begin{aligned}
\widehat{V}(x) &\geq \mathsf{E}_x\bigg(\int_0^{\tau\wedge\hat{\tau}_{\kappa+c}} \lambda\mathrm{e}^{-(\lambda+r)s}(\kappa - X_s)\,\mathrm{d}s + \mathrm{e}^{-(\lambda+r)(\tau\wedge\hat{\tau}_{\kappa+c})}\widehat{V}\big(X_{\tau\wedge\hat{\tau}_{\kappa+c}}\big)\bigg) \\
&\geq \mathsf{E}_x\bigg(\int_0^{\tau\wedge\hat{\tau}_{\kappa+c}} \lambda\mathrm{e}^{-(\lambda+r)t}(\kappa - X_t)\,\mathrm{d}t + \mathrm{e}^{-(\lambda+r)(\tau\wedge\hat{\tau}_{\kappa+c})}\big(\kappa - X_{\tau\wedge\hat{\tau}_{\kappa+c}}\big)\bigg) = J(x,\tau),
\end{aligned}
$$

for all $x \in (0, \infty)$, since Lemma B.1 established that $\widehat{V}(x) \geq \kappa - x$. This implies that $\widehat{V}(x) \geq \widetilde{V}(x)$, for all $x \in (0, \infty)$. On the other hand, the function $(0, \infty) \ni x \mapsto J(x, \check{\tau}_{z_*})$ is the unique solution to the Dirichlet problem with data (3.2a) and (3.2b), due to the stochastic representation theorem for the solutions to Dirichlet problems (for a precise formulation of the relevant result applicable to our setting, see Vigo-Aguiar et al. 2005, Theorem 1). Consequently, $\widehat{V}(x) = J(x, \check{\tau}_{z_*}) \leq \widetilde{V}(x)$, for all $x \in (0, \infty)$, since $\widehat{V}$ satisfies (3.2a) and (3.2b) by construction.

(b): Suppose Condition (3.7b) holds, in which case $z_* = \kappa + c$ and $\widehat{V} \in \mathsf{C}^2(0, \infty)$ is determined by $\widehat{V}(x) = \kappa - x$, for all $x \in (0, \infty)$. Itô's formula then gives

$$\begin{aligned}
\mathrm{e}^{-(\lambda+r)(t\wedge\hat{\tau}_{\kappa+c})}(\kappa - X_{t\wedge\hat{\tau}_{\kappa+c}}) &= \mathrm{e}^{-(\lambda+r)(t\wedge\hat{\tau}_{\kappa+c})}\widehat{V}(X_{t\wedge\hat{\tau}_{\kappa+c}}) \\
&= \widehat{V}(X_0) - \int_0^{t\wedge\hat{\tau}_{\kappa+c}} (\lambda+r)\mathrm{e}^{-(\lambda+r)s}\widehat{V}(X_s)\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\widehat{V}'(X_s)\,\mathrm{d}X_s + \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\frac{1}{2}\widehat{V}''(X_s)\,\mathrm{d}\langle X\rangle_s \\
&= \widehat{V}(X_0) - \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\big(\mathcal{L}_X\widehat{V}(X_s) - (\lambda+r)\widehat{V}(X_s)\big)\,\mathrm{d}s \\
&\qquad\qquad\qquad\qquad - \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\sigma X_s\,\mathrm{d}B_s \\
&\leq \widehat{V}(X_0) - \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \lambda\mathrm{e}^{-(\lambda+r)s}(\kappa - X_s)\,\mathrm{d}s - \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\sigma X_s\,\mathrm{d}B_s,
\end{aligned} \tag{B.3}$$

for all $t \geq 0$. To justify the inequality above, note that

$$0 < r - \mu \leq r - \frac{rc}{\kappa + c} = \frac{r\kappa}{\kappa + c},$$

since $\mu \geq {}^{rc}/_{(\kappa+c)} > 0$, by assumption. Consequently,

$$\begin{aligned}
\mathcal{L}_X\widehat{V}(x) - (\lambda+r)\widehat{V}(x) = -\mu x - (\lambda+r)(\kappa - x) &= (r-\mu)x - r\kappa - \lambda(\kappa - x) \\
&\leq (r-\mu)(\kappa+c) - r\kappa - \lambda(\kappa - x) \\
&\leq -\lambda(\kappa - x),
\end{aligned}$$

for all $x \in (0, \kappa + c]$. Note that the process

$$\mathbb{R}_+ \ni t \mapsto \int_0^{t\wedge\hat{\tau}_{\kappa+c}} \mathrm{e}^{-(\lambda+r)s}\sigma X_s\,\mathrm{d}B_s$$

is a uniformly integrable martingale. Given an arbitrary stopping time $\tau \in \mathfrak{S}$, an application of the optional stopping theorem then gives

$$\widehat{V}(x) \geq \mathsf{E}_x\biggl(\int_0^{\tau\wedge\hat{\tau}_{\kappa+c}} \lambda \mathrm{e}^{-(\lambda+r)s}(\kappa - X_s)\,\mathrm{d}s + \mathrm{e}^{-(\lambda+r)(\tau\wedge\hat{\tau}_{\kappa+c})}(\kappa - X_{\tau\wedge\hat{\tau}_{\kappa+c}})\biggr) = J(x,\tau),$$

for all $x \in (0,\infty)$. This implies that $\widehat{V}(x) \geq \widetilde{V}(x)$, for all $x \in (0,\infty)$. On the other hand, $\widehat{V}(x) = J(x, \hat{\tau}_{\kappa+c}) \leq \widetilde{V}(x)$, for all $x \in (0,\infty)$. □

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