SOME RECENT DEVELOPMENTS IN THE THEORY OF LIE GROUP SYMMETRIES FOR PDES

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Abstract. Lie group symmetry methods provide a powerful tool for the analysis of PDEs. Over the last thirty years, considerable progress has been made in the development of this field. In this article, we provide a brief introduction to the method developed by Lie for the systematic computation of symmetries, then move on to a survey of some of the more recent developments. Our focus is on the use of Lie symmetry methods to construct fundamental solutions of partial differential equations of parabolic type. We will show how recent work has uncovered an intriguing connection between Lie symmetry analysis and the theory of integral transforms. Fundamental solutions of families of PDEs which arise in various applications, can be obtained by exploiting this connection. The major applications we give will be in financial mathematics. We will illustrate our results with the problem of pricing a so called zero coupon bond, as well as giving some applications to option pricing. We also discuss some results on group invariant solutions and show how an important PDE in nilpotent harmonic analysis can be studied via its group invariant solutions. Finally, the connection between Lie symmetry methods and the theory of group representations is presented. This development has the potential to allow for a rich interplay between Lie symmetry group theory and representation theory.

1. Introduction

The purpose of this article is to present a survey of some current research in Lie symmetry analysis for partial differential equations. We will begin with a brief introduction to symmetry methods for the benefit of the reader unfamiliar with these techniques. Then we discuss some recent developments in the theory. Our focus will be on symmetry analysis for a single PDE, but symmetry methods are also very effective

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for the study of systems of PDEs as well as for the solution of ordinary differential equations.

Sophus Lie [17], [15], [16] first developed and applied the theory of symmetry groups to differential equations in the 19th century. The applications of Lie symmetry groups include problems in a diverse range of fields such as control theory, quantum mechanics, relativity and mathematical finance. The references in the bibliography contain numerous applications of the theory. These methods provide a systematic, mechanical computational algorithm which allows us to determine explicitly the symmetry group of any system of differential equations. In fact, software packages exist which partially automate such computations. See for example the Mathematica package in Cantwell's book [2]. Olver [18] and Bluman and Kumei [1] also provide a rigorous account of Lie group methods for differential equations.

We will consider a single PDE of order n in m variables, defined on a simply connected subset $\Omega \subseteq \mathbb{R}^m$. The PDE takes the general form

$$P(x, D^\alpha u) = 0,$$

where P is a differential operator on $\Omega \times \mathbb{R}$,

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_m}},$$

and $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi-index.

Definition 1.1. A symmetry group of a system of differential equations is a local group of transformations G acting on the independent and dependent variables of the system such that it maps solutions of the equations to other solutions. More precisely, let $\mathcal{H}_P$ denote the space of all solutions of the system of PDEs

$$P_\nu(x, D^\alpha u) = 0, \quad \nu = 1, 2, ..., p.$$

A symmetry $\mathcal{S}$ is a mapping of $\mathcal{H}_P$ into itself. i.e $\mathcal{S} : \mathcal{H}_P \to \mathcal{H}_P$. Thus if $u \in \mathcal{H}_P$, then we must have $\mathcal{S}u \in \mathcal{H}_P$.

Example 1.1. Consider the one dimensional heat equation $u_t = u_{xx}$. If $u(x, t)$ is a solution of the one dimensional heat equation then $u(x+\epsilon, t)$ is also a solution, at least for ϵ sufficiently small. This is a symmetry. As we will see below, the heat equation has more interesting symmetries than this simple example.

Though there are symmetries which do not possess group properties, the symmetries which we will be interested in do. In fact these symmetries typically form a Lie group.

Definition 1.2. An r-parameter Lie group is a group which is also an r-dimensional smooth manifold, such that the group operation

$$m : G \times G \to G, \qquad m(g, h) = g \cdot h, \qquad g, h \in G,$$

and the inversion

$$i : G \to G, \qquad i(g) = g^{-1}, \qquad g \in G,$$

are smooth maps between manifolds. The dimension of the Lie group is the dimension of G as a manifold.

Lie groups often arise as groups of transformations on manifolds. Elementary examples of transformations which are associated with Lie groups are scalings, translations and rotations. There are also more complex examples of group transformations.

2. Basic Lie Theory

We require a method for the systematic computation of all Lie group symmetries of a PDE. We will only be considering so called *point symmetries* in this article. A point symmetry is one in which the transformations act only on the independent variables x and dependent variable u of the PDE. There exist *generalized symmetries* which involve transformations that act on the derivatives of the dependent variable. We refer the reader to the discussion in Olver's book for a more extensive treatment of generalized symmetries. We will encounter group symmetries which are not point symmetries in Section 8.

To compute point symmetries, we introduce the notion of a vector field. For our purposes, a vector field is a first order differential operator, which can be written,

$$\mathbf{v}(f) = \sum_{k=1}^{m} \xi_k(x,u)\frac{\partial f}{\partial x_k} + \phi(x,u)\frac{\partial f}{\partial u}. \tag{2.1}$$

A slightly more general form of a vector field is used in Lie's theorem and the prolongation formula given below, but it is only vector fields of the form (2.1) which we will need. The vector fields we deal with will all be *right invariant*. See the remark following the definition below.

A *Lie algebra* $\mathbf{g}$ is a vector space which is closed under a certain kind of multiplication, namely the *Lie bracket*. We will take the definition of a Lie algebra to be the following.

Definition 2.1 (Lie algebras)**.** Consider a collection of n linearly independent vector fields $\mathfrak{V} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$. Define the Lie bracket of $\mathbf{v}_i$ and $\mathbf{v}_k$ by

$$[\mathbf{v}_i, \mathbf{v}_k]f = \mathbf{v}_i(\mathbf{v}_k f) - \mathbf{v}_k(\mathbf{v}_i f).$$

Suppose further that for any vectors $\mathbf{v}_i, \mathbf{v}_j$, $1 \le i \le j \le n$ we have

$$[\mathbf{v}_i, \mathbf{v}_j] = \sum_{k=1}^{n} c_{i,j}^k \mathbf{v}_k$$

for some constants $c_{i,j}^k$. Denote the linear span of $\mathfrak{V}$ by $\mathbf{g}$. Then $\mathbf{g}$ is an n dimensional *Lie algebra*, with basis vectors $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. The numbers

$c_{i,j}^k$ are known as the *structure constants* of the Lie algebra. The Lie bracket of $\mathbf{v}$ and $\mathbf{w}$ satisfies

$$[\mathbf{v}, a\mathbf{w} + b\mathbf{z}] = a[\mathbf{v}, \mathbf{w}] + b[\mathbf{v}, \mathbf{z}] \tag{2.2}$$

$$[a\mathbf{v} + b\mathbf{w}, \mathbf{z}] = a[\mathbf{v}, \mathbf{z}] + b[\mathbf{w}, \mathbf{z}] \tag{2.3}$$

$$[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}] \tag{2.4}$$

$$[\mathbf{v}, [\mathbf{w}, \mathbf{z}]] + [\mathbf{z}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{w}, [\mathbf{z}, \mathbf{v}]] = 0. \tag{2.5}$$

This last equation is known as *Jacobi's identity.*

Lie algebras generate Lie groups via the so called exponential map, introduced below. First we make two remarks.

Remark 2.2. If we consider a Lie group G then it is natural to identify its Lie algebra with $T_e(G)$, the tangent space at the identity.

Remark 2.3. The standard approach to Lie algebras requires that the the vector fields defining the Lie algebra be *left* invariant under the action of G. That is, if $g \in G$ and $L_g : h \to gh$, then the differential map dL_g satisfies

$$dL_g(\mathbf{v}|_h) = \mathbf{v}|_{gh}.$$

In contrast, Lie group symmetry techniques use *right* invariant vector fields. i.e. If $R_g : h \to hg$, then for the differential map we have

$$dR_g(\mathbf{v}|_h) = \mathbf{v}|_{hg}.$$

This will not be of any consequence for our work, but it does alter some standard results slightly. For example, the adjoint representation for right invariant vector fields becomes $ad_X(Y) = [Y, X]$, which is the negative of the usual result. For the reason why right invariant vector fields are used, the reader should consult the first chapter of Olver's book, [18]. We refer readers to a standard text such as Jacobson [12], for the general theory of Lie algebras.

Every vector field $\mathbf{v}$ with sufficiently well behaved coefficients generates a one parameter local Lie group, which is called the *flow* of $\mathbf{v}$. The flow of $\mathbf{v}$ is often written as $\exp(\epsilon\mathbf{v})$, since the action generated by $\mathbf{v}$ can be obtained by summing the so called *Lie series*:

$$f(\exp(\epsilon\mathbf{v})x) = f(x) + \epsilon\mathbf{v}(f) + \frac{\epsilon^2}{2!}\mathbf{v}^2(f) + \cdots = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\mathbf{v}^n(f).$$

The mapping which sends $\mathbf{v}$ to $\exp(\mathbf{v})$ is called the exponential map. In general this series is not easily summed.

A more practical method for computing the flow is the following. Consider a function u which depends upon x. If we have a vector field of the form (2.1), then the group it generates can be thought of as acting on the graph of u, which we write (x, u). That is, the flow acts on the graph (x, u), transforming it in some manner. It is not hard to determine exactly how x and u are transformed. If we denote the transformed variables by $(\tilde{x}, \tilde{u})$, then we have

$$\frac{d\tilde{x}_k}{d\epsilon} = \xi_k(\tilde{x}, \tilde{u}), \quad k = 1, ..., m, \quad \frac{d\tilde{u}}{d\epsilon} = \phi(\tilde{x}, \tilde{u}),$$

and $\tilde{x}_k(0) = x_k,\ k = 1, ..., m$ and $\tilde{u}(0) = u$. It is standard practice to write $\exp(\epsilon\mathbf{v})(x, u) = (\tilde{x}, \tilde{u})$. It is this sense in which a vector field generates a one parameter group action. In this case, the parameter is ϵ.

We turn now to the problem of computing symmetries. Let $\mathcal{G}$ denote the group generated by $\mathbf{v}$. We introduce the *nth prolongation* of $\mathcal{G}$, denoted $\mathrm{pr}^n\mathcal{G}$. It is the natural extension of the action of $\mathcal{G}$, to (x, u), and all the derivatives of u, up to order n. In other words the nth prolongation acts on the collection $(x, u, u_{x_1}, ..., u_{x_m,...,x_m})$, where the order of the highest derivative is n.

More formally, we define the nth prolongation as follows.

Definition 2.4. To determine $\mathrm{pr}^n\mathcal{G}$, let $\mathcal{D}^n$ be the mapping

$$\mathcal{D}^n : (x, u) \longmapsto (x, u, u_{x_1}, ..., u_{x_m,...,x_m}) = (x, u^{(n)}).$$

Then the n-th prolongation must satisfy

$$\mathcal{D}^n \circ \mathcal{G} = \mathrm{pr}^n\mathcal{G} \circ \mathcal{D}^n.$$

The technical definition of the nth prolongation is essentially a statement that the action of the group and differentiation commute with one another. That is, if we act with $\mathcal{G}$ on x and $u(x)$, then write down all the derivatives of the new function $\tilde{u}(\tilde{x})$ up to order n, the result should be the same as writing down the derivatives of $u(x)$ up to order n, then acting on this set with the nth prolongation of $\mathcal{G}$. This condition requires that the chain rule of multi-variable calculus holds.

The nth prolongation of $\mathcal{G}$ also has an infinitesimal generator, which we denote $\mathrm{pr}^n\mathbf{v}$.

Definition 2.5. Let $\mathbf{v}$ be a vector field with corresponding local one-parameter group $\exp(\epsilon\mathbf{v})$. The n-th prolongation of $\mathbf{v}$, denoted $\mathrm{pr}^{(n)}\mathbf{v}$ is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $\mathrm{pr}^{(n)}[\exp(\epsilon\mathbf{v})]$. That is,

$$\mathrm{pr}^{(n)}\mathbf{v}|_{(x,u^{(n)})} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathrm{pr}^{(n)}[\exp(\epsilon\mathbf{v})](x, u^{(n)}).$$

The central result of the theory of symmetry groups is Lie's theorem. It provides the necessary and sufficient conditions for a Lie group $\mathcal{G}$, with infinitesimal generator (2.1), to be a symmetry group.

Theorem 2.6 (Lie)**.** *Let*

$$P_q(x, D^\alpha u) = 0, \qquad q = 1, ..., d \tag{2.6}$$

be a system of d, n-th order partial differential equations. Let $\mathbf{v}$ *be a vector field of the form*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Then $\mathbf{v}$ *generates a one parameter group of symmetries of (2.6) if and only if*

$$\mathrm{pr}^n\mathbf{v}[P(x,D^\alpha u)] = 0 \tag{2.7}$$

whenever $P(x,D^\alpha u) = 0$.

A proof of this result may be found in Olver's book [18]. Applying Lie's Theorem to a system of PDEs yields a set of determining equations for the functions ξ_k and ϕ. These determining equations may usually be solved by inspection. The vector fields satisfying (2.7) are referred to as *infinitesimal symmetries.*

2.0.1. *The Prolongation Formula.* The determination of the symmetry group of a system of differential equations relies upon the prolongation of a group action to the dependent and independent variables of the system as well as the derivatives of the system. An explicit formula for $\mathrm{pr}^n\mathbf{v}$, due to Olver is also to be found in [18].

Theorem 2.7 (Olver; The General Prolongation Formula)**.** *Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{2.8}$$

be a vector field defined on an open subset $M \subset X \times U$. The n-th prolongation of $\mathbf{v}$ *is the vector field*

$$\mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{\boldsymbol{J}} \phi_\alpha^{\boldsymbol{J}}(x,u^{(n)})\frac{\partial}{\partial u_{\boldsymbol{J}}^\alpha} \tag{2.9}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J = (j_1,\ldots,j_k)$, with $1 \le j_k \le p$, $1 \le k \le n$. The coefficient functions $\phi_\alpha^{\boldsymbol{J}}$ of $\mathrm{pr}^{(n)}\boldsymbol{v}$ are given by the following formula:

$$\phi_\alpha^{\boldsymbol{J}}(x,u^{(n)}) = D_{\boldsymbol{J}}\big(\phi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha\big) + \sum_{i=1}^{p}\xi^i u_{\boldsymbol{J},i}^\alpha, \tag{2.10}$$

where $u_i^\alpha = \dfrac{\partial u^\alpha}{\partial x^i}$, and $u_{\boldsymbol{J},i}^\alpha = \dfrac{\partial u_{\boldsymbol{J}}^\alpha}{\partial x^i}$, and $D_{\boldsymbol{J}}$ is the total differentiation operator.

Although the details involved in the construction of Lie's theory of symmetry groups are quite technical, the application of symmetry methods to PDEs is straightforward.

Let us illustrate the prolongation formula by an example. We will make use of the formulae that we obtain. Let $\mathbf{v}$ be a general vector field of the form (2.1) on $X \times U \simeq \mathbb{R}^2 \times \mathbb{R}$. We take $p = 2$ and $q = 1$ in the prolongation formula (2.9), giving

$$\mathrm{pr}^1\mathbf{v} = \mathbf{v} + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t}.$$

This is the first prolongation of $\mathbf{v}$.

The determining functions ϕ^x and ϕ^t are found using (2.10). This is simply an exercise in differentiation. We have

$$\begin{aligned}
\phi^x &= D_x\left(\phi - \xi u_x - \tau u_t\right) + \xi u_{xx} + \tau u_{xt} \\
&= \big(\phi_x + \phi_u u_x - \xi_x u_x - \xi_u u_x^2 - \xi u_{xx} - \tau_x u_t - \tau_u u_x u_t - \tau u_{xt}\big) \\
&\quad + \xi u_{xx} + \tau u_{xt} \\
&= \phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_x u_t - \tau_u u_x u_t.
\end{aligned}$$

Similarly

$$\phi^t = \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2.$$

Subscripts denote partial differentiation.

Further, the second prolongation of $\mathbf{v}$ is the vector field

$$\mathrm{pr}^2\mathbf{v} = \mathbf{v} + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}, \tag{2.11}$$

where for example

$$\begin{aligned}
\phi^{xx} &= D_{xx}\left(\phi - \xi u_x - \tau u_t\right) + \xi u_{xxx} + \tau u_{xxt} \\
&= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx} u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu} u_x u_t \\
&\quad - \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} \\
&\quad - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}.
\end{aligned}$$

The coefficients ϕ^{tt} and ϕ^{xt} can be computed by the same method.

Before we present an example of a symmetry calculation, we point out that the set of all infinitesimal generators of symmetries for a PDE forms a Lie algebra. Again we refer the reader to [18] for a proof.

Theorem 2.8 (Lie)**.** *Let*

$$P_q(x, D^\alpha u) = 0 \qquad q = 1, ...d, \tag{2.12}$$

be a system of d, n-th order partial differential equations. Let the set of all infinitesimal generators of symmetries of (2.12) be $\boldsymbol{g}$. Then $\boldsymbol{g}$ is a Lie algebra.

3. Computing Lie Symmetries

Consider the one dimensional heat equation

$$u_t = u_{xx}. \tag{3.1}$$

To compute the symmetries of (3.1), let $\mathbf{v}$ be given by (2.1), where we seek to determine all possible coefficient functions ξ, τ and ϕ such the the corresponding one-parameter group $\exp(\epsilon\mathbf{v})$ is a symmetry group of the heat equation.

The heat equation is second order, so we need the 2nd prolongation of $\mathbf{v}$, which is given by (2.11). By Lie's theorem, we obtain the following condition;

$$\mathrm{pr}^2\mathbf{v}[u_{xx} - u_t] = 0,$$

whenever $u_{xx} = u_t$.

Applying $\mathrm{pr}^2\mathbf{v}$ to (3.1), we obtain

$$\phi^t = \phi^{xx}, \tag{3.2}$$

which must be satisfied whenever $u_t = u_{xx}$.

Substituting the formulae for ϕ^t and ϕ^{xx} above into (3.2), and replacing u_t with u_{xx}, we obtain the following system of defining equations for the functions ξ, τ and ϕ.

$$\begin{aligned}
&\phi_t - \xi_t u_x + (\phi_u - \tau_t)u_{xx} - \xi_u u_x u_{xx} - \tau_u u_{xx}^2 \\
&= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_{xx} \\
&+ (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_{xx} - \xi_{uu}u_x^3 \\
&- \tau_{uu}u_x^2 u_{xx} + (\phi_u - 2\xi_x)u_{xx} \\
&- 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_{xx}u_{xx} - 2\tau_u u_x u_{xt}.
\end{aligned}$$

To solve this system, we equate the coefficients of the monomials in the zeroth, first and second order partial derivatives of u. We then obtain a set of determining equations for the symmetry group of the heat equation. From the monomial 1, we match the coefficients to obtain $\phi_t = \phi_{xx}$.

Looking at the coefficient of u_{xx}^2, gives $\tau_u = 0$. Also, the coefficient of u_{xt} tells us that $2\tau_x = 0$. Hence τ cannot depend upon x or u.

We have $\xi_u = 0$, and $\xi_x = \frac{1}{2}\tau_t$. Thus ξ depends only on x and t, and so $\xi(x,t) = \frac{1}{2}x\tau_t + \rho(t)$ for some arbitrary function ρ of t.

Continuing in this way, we are able to determine all the coefficient functions ξ, τ and ϕ for the infinitesimal symmetry of the heat equation. A convenient representation is

$$\begin{aligned}
&\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt, \ \tau = c_2 + 2c_4 t + 4c_6 t^2 \\
&\phi = (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \beta(x,t),
\end{aligned}$$

where $c_1, \ldots, c_6$ are arbitrary constants of integration, and $\beta(x,t)$ is an arbitrary solution of the heat equation. Writing out the most general form of $\mathbf{v}$ with these coefficients, and using the numbering of the constants to determine the grouping, we now see that a basis for the Lie algebra of symmetries is given by the six vector fields

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \quad \mathbf{v}_2 = \frac{\partial}{\partial t}, \quad \mathbf{v}_3 = u\frac{\partial}{\partial u}, \ \mathbf{v}_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u},$$
$$\mathbf{v}_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \ \mathbf{v}_6 = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)u\frac{\partial}{\partial u}.$$

There is also an infinite dimensional *ideal* within the Lie algebra consisting of vector fields of the form $\mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u}$. Here β is an arbitrary solution of the heat equation. By the term ideal we mean the following: $\mathcal{J}$ is an ideal in the Lie algebra $\mathbf{g}$ if: (1) $\mathcal{J} \subset \mathbf{g}$ and; (2) for all $X \in \mathbf{g}$ and $Y \in \mathcal{J}$ we have $[X,Y] \in \mathcal{J}$.

The symmetry produced by $\mathbf{v}_\beta$ is simply a superposition of solutions. That is, if u is a solution of the heat equation, and β is a solution, then so is $u+\epsilon\beta$. These symmetries are often thought of as trivial symmetries, because all linear PDEs have them. They are usually considered to be of little interest, however, we will see below that they can be extremely useful.

Given the six vector fields $\mathbf{v}_i, \ i = 1, ..., 6$, calculated above, the next step is to determine the one parameter symmetry groups G_i. If $u(x,t)$ is a solution, then the new solution is denoted by $\tilde{u}(\tilde{x},\tilde{t})$, generated by the exponentiation $\exp(\epsilon\mathbf{v}_i(x,t,u)) = (\tilde{x},\tilde{t},\tilde{u})$.

To determine the group action arising from $\mathbf{v}_6$, we exponentiate the infinitesimal symmetry. We must solve the following system of first order ordinary differential equations

$$\frac{d\tilde{x}}{d\epsilon} = 4\tilde{x}\tilde{t}, \quad \frac{d\tilde{t}}{d\epsilon} = 4\tilde{t}^2, \quad \frac{d\tilde{u}}{d\epsilon} = -(\tilde{x}^2+2\tilde{t})\tilde{u},$$

with $\tilde{x}(0) = x, \quad \tilde{t}(0) = t, \quad \tilde{u}(0) = u.$

We solve the equation for $\tilde{t}$ first to obtain

$$\tilde{t} = \frac{t}{1-4\epsilon t}.$$

We then use this to solve the first equation for $\tilde{x}$. This gives

$$\tilde{x} = \frac{x}{1-4\epsilon t}.$$

The final equation for $\tilde{u}$ may now be solved. The ODE is separable and we have,

$$\tilde{u} = u\sqrt{1-4\epsilon t}\exp\left\{\frac{-\epsilon x^2}{1-4\epsilon t}\right\}.$$

Inverting the group transformations for $\tilde{x}$ and $\tilde{t}$ gives

$$x = \frac{\tilde{x}}{1+4\epsilon\tilde{t}}, \quad t = \frac{\tilde{t}}{1+4\epsilon\tilde{t}}.$$

Now let $u = u(x,t)$ be a solution of the heat equation. The new solution produced by the group action is $\tilde{u}(\tilde{x},\tilde{t})$. Substituting $\tilde{x}$ and $\tilde{t}$ into the expression for $\tilde{u}$ gives

$$\tilde{u} = \frac{1}{\sqrt{1+4\epsilon\tilde{t}}} \exp\left\{\frac{-\epsilon\tilde{x}^2}{1+4\epsilon\tilde{t}}\right\} u\left(\frac{\tilde{x}}{1+4\epsilon\tilde{t}}, \frac{\tilde{t}}{1+4\epsilon\tilde{t}}\right).$$

Since the distinction between the new and old variables is no longer necessary, we drop the tildes from the independent variables. We have thus obtained the group transformation which comes from $\mathbf{v}_6$.

The remaining group actions can be determined by the same methodology. Let us denote the action of the symmetry group generated by $\mathbf{v}_i$ on a solution u by $\rho(\exp(\epsilon\mathbf{v}_i))u(x,t)$. For the infinitesimal symmetries of the one dimensional heat equation, we have

$$\begin{aligned}
\rho(\exp(\epsilon\mathbf{v}_1))u(x,t) &= u(x-\epsilon,t),\\
\rho(\exp(\epsilon\mathbf{v}_2))u(x,t) &= u(x,t-\epsilon),\\
\rho(\exp(\epsilon\mathbf{v}_3))u(x,t) &= e^{\epsilon}u(x,t),\\
\rho(\exp(\epsilon\mathbf{v}_4))u(x,t) &= e^{-\frac{1}{2}\epsilon}u(e^{\epsilon}x, e^{2\epsilon}t),\\
\rho(\exp(\epsilon\mathbf{v}_5))u(x,t) &= e^{-\epsilon x+\epsilon^2 t}u(x-2\epsilon t,t),
\end{aligned}$$

$$\rho(\exp(\epsilon\mathbf{v}_6))u(x,t) = \frac{1}{\sqrt{1+4\epsilon t}} \exp\left\{\frac{-\epsilon x^2}{1+4\epsilon t}\right\} u\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right),$$

and $\rho(\exp(\epsilon\mathbf{v}_\beta))u(x,t) = u(x,t) + \epsilon\beta(x,t)$.

These formulae are to be interpreted as follows. If $u(x,t)$ is a solution of the heat equation, then by symmetry, so are

$$\tilde{u}(\tilde{x},\tilde{t}) = \rho(\exp(\epsilon\mathbf{v}_i))u(x,t)$$

for $i = 1, ..., 6$.

Now let us make an important observation. It is possible to obtain the fundamental solution of the heat equation by symmetry. Consider the symmetry generated by $\mathbf{v}_6$. Take the solution $u = 1$. We obtain the solution

$$\tilde{u} = \frac{1}{\sqrt{1+4\epsilon t}} \exp\left\{\frac{-\epsilon x^2}{1+4\epsilon t}\right\}.$$

If we translate t by $-1/4\epsilon$ and set $\epsilon = \pi$, then we obtain the fundamental solution of the heat equation,

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}.$$

from the constant solution, $u = 1$, by group transformation.

There are many other examples of PDEs whose fundamental solution can be computed by symmetry methods. See for example [5], [3] and [4]. We will consider this problem in more detail in the rest of the chapter. In particular, we will consider how the relationship between Lie symmetries and integral transforms can be used to compute fundamental solutions.

4. Classifying PDEs With Non Trivial Symmetries

The *group classification problem* refers to the task of determining the most general equation which admits a certain group as its Lie symmetry group. This is an important problem, particularly in the application of symmetries to ordinary differential equations. Olver's book contains a discussion of this problem and its connection to the solution of ODEs by quadrature.

A closely related problem is determining the largest class of equations of a certain type that possess non trivial symmetries. In this section we present an example of this and we exploit it to produce a new method for the computation of fundamental solutions. We will be concerned with the symmetries of a PDE which we refer to as a generalized bond pricing equation, since the $\mu = 1$ case arises in the theory of bond pricing:

$$\frac{\partial u}{\partial t} = \sigma x \frac{\partial^2 u}{\partial x^2} + f(x)\,\frac{\partial u}{\partial x} - \mu x u, \tag{4.1}$$

The PDE (4.1) has time translation symmetries; that is, if $u(x,t)$ is a solution then so is $u(x, t+\varepsilon)$. It also has multiplicative symmetries: if $u(x,t)$ is a solution, so is $cu(x,t)$, where c is some constant. We consider these symmetries as trivial, since every PDE of the form (4.1) will possess them. We therefore seek to identify drift functions f for which there exist non trivial symmetries for (4.1); that is, symmetries that are not time translations or multiplication by scalars.

We sketch the proof of the following result.

Proposition 4.1. *The partial differential equation (4.1) has a non trivial Lie algebra of symmetries if and only if the drift function f is a solution of one of the following families of Riccati equations*

$$\begin{aligned}
\sigma x f' - \sigma f + \frac{1}{2}f^2 + 2\mu\sigma x^2 &= Ax + B,\\
\sigma x f' - \sigma f + \frac{1}{2}f^2 + 2\mu\sigma x^2 &= Ax^2 + Bx + C,\\
\sigma x f' - \sigma f + \frac{1}{2}f^2 + 2\mu\sigma x^2 &= Ax^2 + Bx^{\frac{3}{2}} + Cx - \frac{3}{8}\sigma^2.
\end{aligned}$$

Proof. The full proof of this result may be found in [14]. We only do the case of the first Riccati equation, since the other two cases require essentially the same calculation. By Lie's theorem, $\mathbf{v}$ generates symmetries of (4.1) if and only if

$$\mathrm{pr}^2\mathbf{v}[u_t - (\sigma x u_{xx} + f(x)u_x - \mu x u)] = 0$$

whenever $u_t = \sigma x u_{xx} + f(x)u_x - \mu x u$. We take the usual form for $\mathbf{v}$, requiring its second prolongation.

The defining equations for the symmetries are given by

$$\begin{aligned} &\phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t = \\ &\quad \sigma x\big(\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + \phi_{uu}u_x^2 + (\phi_u - 2\xi_x)u_{xx}\big) \\ &\quad +\sigma\xi u_{xx} + \big(\phi_x + (\phi_u - \xi_x)u_x\big)f(x) + \xi f'(x)u_x - \phi\mu x - \xi\mu u. \end{aligned} \tag{4.2}$$

Substituting $u_t = \sigma x u_{xx} + f(x)u_x - \mu x u$ into (4.2), and matching the coefficients on both sides of the equation, we solve for τ, ξ and ϕ. We see that τ is a function of t only, ϕ is linear in u and the function ξ has the form $\xi = x\tau_t + \sqrt{x}\rho$. The function ϕ has the form

$$\phi = \alpha(x,t)u + \beta(x,t),$$

where α and β are arbitrary functions of x and t. It is easy to show that

$$\alpha_t = \sigma x\alpha_{xx} + \alpha_x f(x) - 2\mu x\tau_t - \mu\sqrt{x}\rho \tag{4.3}$$

and $\beta_t = \sigma x\beta_{xx} + f(x)\beta_x - \mu x\beta$. We have no more information about β. Hence it is an arbitrary solution of the generalized bond pricing equation.

Rearranging our expression for α and then integrating, we obtain

$$\alpha = -\frac{x}{2\sigma}\tau_{tt} - \frac{\sqrt{x}}{\sigma}\rho_t - \frac{1}{2\sigma}f(x)\tau_t + \frac{1}{2\sqrt{x}}\Big(\frac{1}{2} - \frac{f(x)}{\sigma}\Big)\rho + \eta(t), \tag{4.4}$$

where η is an arbitrary function of t. Also, we have $\phi = \alpha u + \beta$.

Substituting these functions into the coefficient of u in our defining equations, we produce the final condition

$$\begin{aligned} &-\frac{x}{2\sigma}\tau_{ttt} - \frac{\sqrt{x}}{\sigma}\rho_{tt} + \eta_t = -\frac{1}{2\sigma}(\sigma x f'' + ff' + 4\sigma\mu x)\tau_t \\ &+\rho\Big(\frac{3\sigma^2 x - 8x^2(\sigma x f'' + ff' + 4\sigma\mu x) + 8x(\sigma x f' - \sigma f + \frac{1}{2}f^2 + 2\sigma\mu x^2)}{16\sigma x^{5/2}}\Big). \end{aligned} \tag{4.5}$$

We now have to analyse this set of equations. It is clear that we will obtain non trivial symmetries only when f satisfies one of the Riccati equations in the proposition. We illustrate this for the first Riccati equation.

4.1. **Case 1.** Let

$$\sigma x f'' + f f' + 4\sigma\mu x = A \tag{4.6}$$

Integrating this by parts with respect to x gives

$$\sigma x f' - \sigma f + \frac{1}{2} f^2 + 2\sigma\mu x^2 = Ax + B. \tag{4.7}$$

From (4.5), we have

$$-\frac{x}{2\sigma}\tau_{ttt} - \frac{\sqrt{x}}{\sigma}\rho_{tt} + \eta_t = -\frac{1}{2\sigma}A\tau_t + \rho\Big(\frac{3\sigma^2 + 8B)}{16\sigma x^{3/2}}\Big). \tag{4.8}$$

We naturally have two subcases, which we consider in turn.

4.1.1. *Subcase 1a.* If $3\sigma^2 + 8B \neq 0$, then

$$\tau_{ttt} = 0, \qquad \rho = 0, \qquad \eta_t = -\frac{1}{2\sigma}A\tau_t.$$

Integration then gives

$$\tau = \frac{1}{2}c_1 t^2 + c_2 t + c_3, \quad \eta = -\frac{1}{4\sigma}Ac_1 t^2 - \frac{1}{2\sigma}Ac_2 t + c_4$$

and $\xi = c_1 x t + c_2 x$, where $c_1, ..., c_4$ are constants of integration.

We now substitute the relevant expressions above into the equation for (4.4), which gives

$$\alpha = -\frac{1}{2\sigma}c_1(x + tf(x) + \frac{1}{2}At^2) - \frac{1}{2\sigma}c_2(f(x) + At) + c_4. \tag{4.9}$$

Substituting the expressions for ξ, τ and ϕ into (2.1) gives us the most general vector field which generates symmetries of our generalised bond pricing PDE, when the drift f satisfies the Riccati equation:

$$\sigma x f' - \sigma f + \frac{1}{2} f^2 + 2\sigma\mu x^2 = Ax + B.$$

The resulting vector field is

$$\begin{aligned}
\mathbf{v} &= \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \alpha u\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial u} \\
&= (c_1 x t + c_2 x)\frac{\partial}{\partial x} + (\frac{1}{2}c_1 t^2 + c_2 t + c_3)\frac{\partial}{\partial t} \\
&+ \Big(-\frac{1}{2\sigma}c_1(x + tf(x) + \frac{1}{2}At^2) - \frac{1}{2\sigma}c_2(f(x) + At) + c_4\Big)u\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial u}.
\end{aligned} \tag{4.10}$$

Then the Lie algebra of infinitesimal symmetries, obtained by grouping the constants in (4.10), is spanned by the four following vector fields

$$\mathbf{v}_1 = xt\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - \frac{1}{2\sigma}(x + tf(x) + \frac{1}{2}At^2)u\frac{\partial}{\partial u},$$
$$\mathbf{v}_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2\sigma}(f(x) + At)u\frac{\partial}{\partial u},\ \mathbf{v}_3 = \frac{\partial}{\partial t},\ \mathbf{v}_4 = u\frac{\partial}{\partial u},$$

and there is an infinite-dimensional subalgebra spanned by vector fields of the form

$$\mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u},$$

where β is an arbitrary solution of the PDE.

4.1.2. *Subcase 1b.* If $3\sigma^2+8B=0$, then the Lie algebra of infinitesimal symmetries contains two additional vector fields

$$\mathbf{v}_5 = \sqrt{x}t\frac{\partial}{\partial x} + \big(-\frac{\sqrt{x}}{\sigma} + \frac{t}{4\sqrt{x}} - \frac{tf(x)}{2\sigma\sqrt{x}}\big)u\frac{\partial}{\partial u},$$
$$\mathbf{v}_6 = \sqrt{x}\frac{\partial}{\partial x} + \frac{1}{2\sqrt{x}}(\frac{1}{2} - \frac{f(x)}{\sigma})u\frac{\partial}{\partial u}.$$

The calculations for the other two cases are similar. □

We illustrate with the calculation of the infinitesimal symmetries for the Cox-Ingersoll-Ross model for interest rates.

Example 4.1. Recall that in the Cox-Ingersoll Ross model, the instantaneous interest rate is modelled by the SDE

$$dr_t = (a - b\,r_t)dt + \sqrt{2\sigma r_t}dW_t, \tag{4.11}$$

where σ and a are nonnegative, $b \in \mathbb{R}$, and W is a standard Wiener process. We have written the SDE in a form consistent with our notation.

Let $f(r) = a - br$. This drift is a solution of the second Riccati equation with $A = b^2 + 4\mu\sigma$, $B = -ab$ and $D = \frac{1}{2}a^2 - \sigma a$. A basis for the Lie Algebra of symmetries is given by the following four vector fields.

$$\mathbf{v}_1 = re^{\sqrt{A}t}\frac{\partial}{\partial r} + \frac{1}{\sqrt{A}}e^{\sqrt{A}t}\frac{\partial}{\partial t} - \frac{1}{2\sigma}e^{\sqrt{A}t}(\sqrt{A}r + f(r) + \frac{B}{\sqrt{A}})u\frac{\partial}{\partial u},$$
$$\mathbf{v}_2 = re^{-\sqrt{A}t}\frac{\partial}{\partial r} - \frac{1}{\sqrt{A}}e^{-\sqrt{A}t}\frac{\partial}{\partial t} + \frac{1}{2\sigma}e^{-\sqrt{A}t}(\sqrt{A}r - f(r) + \frac{B}{\sqrt{A}})u\frac{\partial}{\partial u},$$
$$\mathbf{v}_3 = \frac{\partial}{\partial t},\ \mathbf{v}_4 = u\frac{\partial}{\partial u},$$

and we have the infinite dimensional subalgebra spanned by vector fields of the form $\mathbf{v}_\beta = \beta(r,t)\frac{\partial}{\partial u}$.

We can obtain the corresponding group transformations for each of the vector fields for the Cox-Ingersoll-Ross model by exponentiating the infinitesimal symmetries $\mathbf{v}_1, \ldots, \mathbf{v}_4$.

Let us consider the second vector field $\mathbf{v}_2$. In order to find the transformed variables $\tilde{r}$, $\tilde{t}$ and $\tilde{u}$ we solve the system of first order ordinary differential equations below.

$$\frac{d\tilde{r}}{d\epsilon} = \tilde{r}e^{-\sqrt{A}\tilde{t}},\ \frac{d\tilde{t}}{d\epsilon} = -\frac{1}{\sqrt{A}}e^{-\sqrt{A}\tilde{t}},\ \frac{d\tilde{u}}{d\epsilon} = \frac{1}{2\sigma}e^{-\sqrt{A}\tilde{t}}(\sqrt{A}\tilde{r} - f(\tilde{r}) + \frac{B}{\sqrt{A}})\tilde{u}$$

subject to the initial conditions $\tilde{r}(0) = r,\ \tilde{t}(0) = t,\ \tilde{u}(0) = u$.

The calculations are relatively straightforward and we see that if u is a solution of the CIR equation, then so is

$$\tilde{u}(r,t) = \exp\left\{\frac{\sqrt{A}r\epsilon}{2\sigma(e^{\sqrt{A}t}+\epsilon)} + \frac{1}{2\sigma}\left(F\left(\frac{re^{\sqrt{A}t}}{e^{\sqrt{A}t}+\epsilon}\right) - F(r)\right)\right\}$$
$$\times e^{-\frac{Bt}{2\sigma}}(e^{\sqrt{A}t}+\epsilon)^{\frac{B}{2\sigma\sqrt{A}}}\, u\left(\frac{re^{\sqrt{A}t}}{(e^{\sqrt{A}t}+\epsilon)}, \frac{1}{\sqrt{A}}\ln(e^{\sqrt{A}t}+\epsilon)\right),$$

where $F(r) = a\log r - br$.

5. Lie Symmetries and Integral Transforms

Study of the problem described in the previous section has recently led to a rather surprising connection between Lie group symmetries and integral transforms which allows us to obtain fundamental solutions of large classes of parabolic PDEs by inverting an integral transform that comes directly from the symmetry group. We outline the basic idea, then illustrate the process for the Black-Scholes equation of option pricing, before presenting a theorem which further clarifies the situation.

For simplicity we consider only a single *linear* equation

$$u_t = P(x, u^{(n)}), \quad x \in \Omega \subseteq \mathbb{R}, \tag{5.1}$$

We denote by $\tilde{u}_\epsilon = \rho(\exp\epsilon\mathbf{v})u(x,t)$ the action on solutions generated by $\mathbf{v}$. Typically we have

$$\rho(\exp\epsilon\mathbf{v})u(x,t) = \sigma(x,t;\epsilon)u\left(a_1(x,t;\epsilon), a_2(x,t;\epsilon)\right), \tag{5.2}$$

for some functions σ, a_1 and a_2. We call σ the *multiplier* and a_1 and a_2 the *change of variables* of the symmetry.

Now suppose that (5.1) has a fundamental solution $p(t,x,y)$. Then the function

$$u(x,t) = \int_\Omega f(y)p(t,x,y)dy, \tag{5.3}$$

solves the initial value problem for (5.1) with appropriate initial data $u(x,0) = f(x)$.

The idea is to connect the solutions (5.2) and (5.3). We take a stationary (time independent) solution $u = u_0(x)$. So in this case

$$\rho(\exp \epsilon \mathbf{v}) u_0(x) = \sigma(x,t;\epsilon) u_0\left(a_1(x,t;\epsilon)\right). \tag{5.4}$$

Setting $t = 0$ and using (5.3) suggests the relation

$$\int_\Omega \sigma(y,0;\epsilon) u_0(a_1(y,0;\epsilon) p(t,x,y) dy = \sigma(x,t;\epsilon) u_0\left(a_1(x,t;\epsilon)\right). \tag{5.5}$$

Since σ and a_1 are known, we have a family of integral equations for $p(t,x,y)$.

A priori, there is no reason to believe that this integral equation will be tractable. However, for large classes of PDEs, it turns out to be a standard integral transform. Thus we may recover the desired fundamental solution by inverting the transform. We will illustrate this procedure with the Black-Scholes equation from option pricing, before presenting a more general result which clarifies the situation further.

The *forward-propagating* Black-Scholes PDE, with the risk-free rate r and the volatility σ being taken as constants, can be written:

$$\frac{\partial u}{\partial t} = -ru + rx\frac{\partial u}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}.$$

This is subject to the initial condition $u(x,0) = f(x)$. We refer to this form of the Black-Scholes PDE as forward propagating, because we apply an *initial* condition for the option payoff, and we then solve the PDE *forward* in time from the initial condition. This should be contrasted to the normal way in which option-pricing problems are set up in finance, in which a *terminal value* corresponding to the option payoff at expiry is provided.

A straightforward application of the above techniques shows that the Lie algebra of symmetries of the Black-Scholes equation is spanned by the following vector fields:

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \quad \mathbf{v}_2 = x\frac{\partial}{\partial x},$$

$$\mathbf{v}_3 = 2t\frac{\partial}{\partial t} + \{\ln x - \mu t\} x\frac{\partial}{\partial x} - 2rtu\frac{\partial}{\partial u},$$

$$\mathbf{v}_4 = -\sigma^2 tx\frac{\partial}{\partial x} + \{\ln x + \mu t\} u\frac{\partial}{\partial u},$$

$$\mathbf{v}_5 = \sigma^2 tx \ln x^2 \frac{\partial}{\partial x} + 2\sigma^2 t^2 \frac{\partial}{\partial t} - \left\{(\ln x + \mu t)^2 + 2\sigma^2 r t^2 + \sigma^2 t\right\} u\frac{\partial}{\partial u},$$

$$\mathbf{v}_6 = u\frac{\partial}{\partial u}, \quad \mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u},$$

where $\mu = r - \frac{1}{2}\sigma^2$ and $\beta(x,t)$ is an arbitrary solution of the forward-propagating Black-Scholes PDE. Exponentiating the generators in turn, we obtain the following result.

Proposition 5.1. *For the infinitesimal symmetries of the backward propagating Black Scholes equation, we have*

$$\rho(\exp(\epsilon \mathbf{v}_1))u(x,t) = u(x, t+\epsilon). \tag{5.6}$$

$$\rho(\exp(\epsilon \mathbf{v}_2))u(x,t) = u(e^{\epsilon}x, t). \tag{5.7}$$

$$\rho(\exp(\epsilon \mathbf{v}_3))u(x,t) = e^{(c^2-1)rt}u(x^c e^{-c(c-1)(r-\frac{1}{2}\sigma^2)t}, c^2 t), \tag{5.8}$$

where $c = e^{\epsilon}$.

$$\rho(\exp(\epsilon \mathbf{v}_4))u(x,t) = x^{-\epsilon}e^{\left[-(r-\frac{1}{2}\sigma^2)\epsilon+\frac{1}{2}\sigma^2\epsilon^2\right]t}u(xe^{-\epsilon\sigma^2 t}, t). \tag{5.9}$$

$$\rho(\exp(\epsilon \mathbf{v}_5))u(x,t) = u\left(e^{\frac{\ln x}{1+2\sigma^2\epsilon t}}, \tfrac{t}{1+2\sigma^2\epsilon t}\right) \times \frac{1}{\sqrt{1+2\sigma^2\epsilon t}}\exp\left\{\frac{-\left(\left[\ln(x)+(r-\frac{1}{2}\sigma^2)t\right]^2+2r\sigma^2t^2\right)\epsilon}{1+2\sigma^2\epsilon t}\right\}. \tag{5.10}$$

$$\rho(\exp \epsilon \mathbf{v}_6)u(x,t) = e^{-\epsilon}u(x,t). \tag{5.11}$$

$$\rho(\exp \epsilon \mathbf{v}_\beta)u(x,t) = u(x,t)+\epsilon\beta(x,t). \tag{5.12}$$

5.0.3. *Calculating the fundamental solution of the Black-Scholes PDE.* We will use the symmetry

$$\mathbf{v}_4 = -\sigma^2 tx\frac{\partial}{\partial x} + \left[\ln(x)+(r-\tfrac{1}{2}\sigma^2)t\right]\frac{\partial}{\partial u}.$$

to obtain the fundamental solution, from the stationary solution

$$u_0(x) = x.$$

We identify:

$$u(x,t;\epsilon) = x^{-\epsilon}e^{\left[-(r-\frac{1}{2}\sigma^2)\epsilon+\frac{1}{2}\sigma^2\epsilon^2\right]t}$$

so that

$$u(y,0;\epsilon) = y^{-\epsilon}.$$

Substituting into the integral equation (5.5), we get after some simplification:

$$e^{\frac{1}{2}t\epsilon\left((\epsilon-1)\sigma^2-2r\right)}x^{1-\epsilon} = \int_0^\infty y^{1-\epsilon}p(t,x,y)\,dy.$$

We make the substitution of parameter $\epsilon = 2-s$ to obtain

$$e^{\frac{1}{2}(s-2)t\left((s-1)\sigma^2+2r\right)}x^{s-1} = \int_0^\infty y^{s-1}p(t,x,y)\,dy.$$

We recognise the right hand side as the Mellin Transform of the fundamental solution with respect to y:

$$\mathcal{M}\left\{p(t,x,y)\right\}(s) = e^{\frac{1}{2}(s-2)t\left((s-1)\sigma^2+2r\right)}x^{s-1}.$$

Hence we can recover $p(x,y,t)$ by performing a Mellin inversion from s to y:

$$\begin{aligned} p(t,x,y) &= \mathcal{M}^{-1}\left\{e^{\frac{1}{2}(s-2)t\left((s-1)\sigma^2+2r\right)}x^{s-1}\right\}(y) \\ &= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{e^{\frac{1}{2}(-iw-2)t\left((-iw-1)\sigma^2+2r\right)}x^{-iw-1}\right\}(-\ln(y)) \\ &= \frac{x}{y^2\sigma\sqrt{2\pi t}}\exp\left\{-\frac{\left[\ln(\frac{y}{x})-(r+\frac{1}{2}\sigma^2)t\right]^2}{2\sigma^2 t}\right\} \\ &= \frac{e^{-rt}}{\sigma\, y\sqrt{2\pi t}}\exp\left\{-\frac{\left[\ln(\frac{y}{x})-(r-\frac{1}{2}\sigma^2)t\right]^2}{2\sigma^2 t}\right\}. \end{aligned}$$

Note that we have performed the Mellin inversion by converting it to a corresponding Fourier inversion. We have used the well known result

$$\mathcal{M}\left\{f(x)\right\}(s) = \sqrt{2\pi}\,\mathcal{F}\left\{f(e^{-x})\right\}(is).$$

Here we have taken the Fourier transform $\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iyx}dx$.

Note also that the final expression is easily recognised as the *transition probability density function* for *Geometrical Brownian Motion* (GBM). This is precisely what we expect for the fundamental solution for the Black-Scholes PDE, as the underlying asset price dynamics is driven by GBM in the classic Black-Scholes case.

It turns out that this connection between fundamental solutions, symmetries and integral transforms is quite deep. Whenever a second order, linear, parabolic PDE in one spatial variable x has a nontrivial symmetry group, then for at least one of the vector fields in the Lie algebra, equation (5.5) can always be identified as a classical integral transform. There are a number of results along this line, in which we identify the multiplier of the symmetry and a stationary solution as an integral transform of a fundamental solution. The types of transforms that arise are all classical: Laplace transforms, Hankel transforms, Whittaker transforms and so on. We present one result, whose proof is in the recent paper by Craddock and Lennox, [4].

Theorem 5.2. *Let f be an analytic solution of the Riccati equation*

$$\sigma x f' - \sigma f + \frac{1}{2}f^2 + 2\mu\sigma x^{r+1} = Ax + B. \tag{5.13}$$

Let $u_0(x)$ be an analytic solution of the PDE

$$u_t = \sigma x u_{xx} + f(x)u_x - \mu x^r u, \tag{5.14}$$

which is independent of t and

$$U_\lambda(x,t) = \exp\left\{\frac{1}{2\sigma}\left(F\left(\frac{x}{(1+\sigma\lambda t)^2}\right) - F(x)\right) - \frac{\lambda(x+\frac{A}{2}t^2)}{(1+\sigma\lambda t)}\right\}\times$$
$$u_0\left(\frac{x}{(1+\sigma\lambda t)^2}\right), \tag{5.15}$$

where $F'(x) = f(x)/x$. Then

$$U_\lambda(x,t) = \int_0^\infty u_0(y)p_\mu(t,x,y)e^{-\lambda y}dy$$

where $p_\mu(t,x,y)$ is a fundamental solution of the PDE (5.14).

This theorem allows us to obtain the fundamental solution for PDEs with the given drift function by inverting a Laplace transform which arises directly from the symmetry group. This result has applications in the theory of bond pricing.

Example 5.1. We take $\sigma = 1, \mu = 0$. We may take $u_0 = 1$ in this case. For the PDE

$$u_t = xu_{xx} + \left(\frac{1}{2} + \sqrt{x}\coth(\sqrt{x})\right)u_x. \tag{5.16}$$

we have

$$U_\lambda(x,t) = \frac{\sinh\left(\frac{\sqrt{x}}{1+\lambda t}\right)}{\sinh(\sqrt{x})\sqrt{1+\lambda t}}\exp\left\{-\frac{\lambda(x+\frac{1}{4}t^2)}{1+\lambda t}\right\}.$$

Inverting the Laplace transform in λ is easily accomplished, to obtain the fundamental solution

$$p(t,x,y) = \frac{\sinh\left(\frac{2\sqrt{xy}}{t}\right)}{\sqrt{\pi y t}}\frac{\sinh(\sqrt{y})}{\sinh(\sqrt{x})}\exp\left\{-\frac{(x+y)}{t} - \frac{1}{4}t\right\}.$$

5.0.4. *An Application in Bond Pricing.* Let us consider a simple example. We don't propose this as a real model of interest rates, but simply present it as an illustration of the theory.

Consider the interest rate model $dr_t = f(r_t)dt + \sqrt{2\sigma r_t}dW_t$, where

$$f(r) = 2r\sqrt{\sigma}\,\cot(\frac{r}{\sqrt{\sigma}}).$$

We want to obtain the price of a zero coupon bond for this model. In order to do this, we obtain the fundamental solution of the PDE

$$V_t + \frac{1}{2}\sigma r V_{rr} + f(r)V_r - \mu r V = 0,$$

and set $\mu = 1$.

This is a straightforward application of Theorem 5.2. Omitting the calculations, we obtain the fundamental solution

$$p_\mu(t,r,y) = e^{\left\{\frac{-(r+y)}{t\,\sigma}\right\}}\left(\frac{\sqrt{r}}{t\,\sqrt{y}\,\sigma} I_1(\frac{2\sqrt{ry}}{t\,\sigma}) + \delta(y)\right)\frac{\sin(\frac{y\sqrt{\mu}}{\sqrt{\sigma}})}{\sin(\frac{r\sqrt{\mu}}{\sqrt{\sigma}})}.$$

Here δ is the Dirac delta function and I_1 is the usual Bessel function.

To obtain the bond price, we set $\mu = 1$ in our expression for $p_\mu(t,r,y)$, calculate the integral

$$B(r,t,T) = \int_0^\infty p_1(T-t,r,y)dy,$$

and replace t with $T-t$. This integral can be done exactly, giving the result

$$B(r,t,T) = e^{\left\{\frac{-r(T-t)}{1+\sigma(T-t)^2}\right\}}\frac{\sin\left(\frac{r}{\sqrt{\sigma}+\sigma^{\frac{3}{2}}(T-t)^2}\right)}{\sin\left(\frac{r}{\sqrt{\sigma}}\right)}.$$

Since we have the fundamental solution, we can do more than simply price bonds. We can also price options on bonds, swaps, caps and many other instruments. For example, the price of a call option C with strike E on this zero coupon bond is

$$C = \int_0^\infty \left(e^{\frac{-y(T-t)}{1+\sigma(T-t)^2}}\frac{\sin\left(\frac{y}{\sqrt{\sigma}+\sigma^{\frac{3}{2}}(T-t)^2}\right)}{\sin\left(\frac{y}{\sqrt{\sigma}}\right)} - E\right)^+ p_1(T-t,r,y)dy.$$

Here $(x-K)^+ = \max(x-K,0)$. It may be necessary to evaluate this integral numerically.

6. Group Invariant Solutions

One of the most powerful methods for constructing exact solutions of PDEs uses the notion of *group invariance*. A group invariant solution is one which is invariant under the action of a symmetry group.

We illustrate the idea of group invariance with an example. We know that if $u(x,t)$ is a solution of the heat equation, then so is

$$\tilde{u}_\epsilon(x,t) = \frac{1}{\sqrt{1+4\epsilon t}}\exp\left\{\frac{-\epsilon x^2}{1+4\epsilon t}\right\}u\left(\frac{x}{1+4\epsilon t},\frac{t}{1+4\epsilon t}\right). \tag{6.1}$$

Now we apply the symmetry (6.1) to the one dimensional heat kernel $k(x,t)$. The result is

$$\begin{aligned}\tilde{k}_\epsilon(x,t) &= \frac{1}{\sqrt{1+4\epsilon t}}\exp\left\{\frac{-\epsilon x^2}{1+4\epsilon t}\right\}\frac{1}{\sqrt{\frac{4\pi t}{1+4\epsilon t}}}\exp\left\{\frac{\frac{-x^2}{(1+4\epsilon t)^2}}{\frac{4t}{1+4\epsilon t}}\right\}\\ &= \frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{x^2}{4t}\frac{1+4\epsilon t}{1+4\epsilon t}\right\} = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t} = k(x,t).\end{aligned}$$

The point is that the application of the symmetry (6.1) has no effect on the fundamental solution. We say therefore that it is a *group invariant solution of the heat equation.*

This raises a number of questions. First, are there other solutions of the heat equation which are also invariant with respect to this group action? Secondly we can ask if there is a systematic method for computing invariant solutions? Thirdly, how representative of the PDEs solutions are group invariant solutions? For example, is the fundamental solution of a linear PDE always a group invariant solution?

The answer to the first two questions is yes. To demonstrate this, we will introduce the method of constructing group invariant solutions and go through this example for the heat equation in detail. The answer to the third question is not entirely clear, but it is certainly true that fundamental solutions for many PDEs may be obtained as group invariant solutions. In the next section we will obtain the heat kernel on the Heisenberg group as a group invariant solution.

The procedure for computing group invariant solutions is easy to describe and easy to use. It involves computing invariants of the group action by the method characteristics, and using these to reduce the number of variables in the problem.

The technical justification of this method is however quite involved and we refer the reader to Olver's book once more. Our aim of is to give a very quick introduction to the ideas, without being too concerned with technicalities.

Definition 6.1. Let G be a group which acts transversally on a manifold M. Let $x \in M$ and let the action of an element $g \in G$ on x be denoted $g.x$. The *orbit* of x under G is the set

$$\mathfrak{O}_x = \{y \in M | y = g.x, g \in G\}. \tag{6.2}$$

That is, the orbit is the set of all points that x is mapped to as g varies through the whole group.

Example 6.1. Consider the group $SO(2)$. If we take $(x,y) \in \mathbb{R}^2$ then the orbit of (x,y) under $SO(2)$ is the set of points of the form

$$(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta), \quad \theta \in [0, 2\pi).$$

It is not hard to identify this set of points. Elementary algebra shows that

$$(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 = x^2 + y^2.$$

This is clearly the equation of a circle of radius $r = \sqrt{x^2+y^2}$. So the orbits are circles. There is also a degenerate case, namely when $(x, y) = (0, 0)$. Here the orbit is the single point $(0, 0)$.

Notice that the orbit in the first case is a *submanifold* of $\mathbb{R}^2$. This is always the case, at least under suitable technical assumptions. In the case where the orbits are circles, the dimension of the orbit is one, since a circle is a one dimensional manifold. In general, if $\mathfrak{O}$ is an orbit, then the dimension of the orbit is the dimension of $\mathfrak{O}$ regarded as a submanifold.

Suppose that we have a PDE $P(x, D^\alpha u) = 0$, in n variables. Suppose also that there exists a symmetry group G of the PDE and that the orbits of G form a submanifold of dimension $p < n$. Then the PDE $P(x, D^\alpha u) = 0$ can always be reduced under a change of variables to a PDE in $n - p$ variables. In the literature it is common to write the reduced equation as $P/G(y, D^\alpha v) = 0$, where y and v are the new variables given by the change of variables.

The key to the method is finding invariants of the group action. Suppose that we have a one parameter group which is generated by a vector field of the form (2.1). To determine the action of $\mathbf{v}$ on a function f, we can form the Lie series

$$\exp(\epsilon\mathbf{v})f(x) = f(x) + \epsilon\mathbf{v}(f(x)) + \frac{1}{2}\epsilon^2\mathbf{v}^2(f(x)) + \cdots .$$

If f is invariant under the action of $\mathbf{v}$ then $f(x) = \exp(\epsilon\mathbf{v})f(x)$ which implies that $\mathbf{v}(f) = 0$. This means that the invariants are found by solving the first order PDE

$$\sum_{i=1}^{n} \xi_i(x, u)\frac{\partial f}{\partial x_i} + \phi(x, u)\frac{\partial f}{\partial u} = 0.$$

This can be done by the method of characteristics. We illustrate the procedure by finding invariants for some vector fields.

Example 6.2. Let

$$\mathbf{v} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

This generates rotations. To find the invariants we must solve $\mathbf{v}(f) = 0$. By the method of characteristics we see that the general solution of this PDE is $f = G(x^2 + y^2)$, for G an arbitrary differential function.

Example 6.3. The one dimensional heat equation $u_t = u_{xx}$ has the symmetry

$$\mathbf{v} = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}.$$

We find invariants of the group generated by $\mathbf{v}$ by solving

$$\frac{dx}{4xt} = \frac{dt}{4t^2} = -\frac{du}{(x^2 + 2t)u}. \tag{6.3}$$

To demonstrate the method, we will solve this in steps. First, solving

$$\frac{dx}{4xt} = \frac{dt}{4t^2} \tag{6.4}$$

gives $\ln x = \ln t + C$. Therefore $C = \ln(x/t)$. We could take $\eta = \ln(x/t)$, but since we may actually take any function of $\ln(x/t)$ for η, it makes sense just to write $\eta = e^{\ln(x/t)} = x/t$.

Notice that we then have $x = \eta t$. Returning to (6.3) we have to solve

$$\frac{dt}{4t^2} = -\frac{du}{(\eta^2 t^2 + 2t)u}. \tag{6.5}$$

Integration leads to

$$\ln u = -\frac{1}{4}\eta^2 t - \frac{1}{2}\ln t + D.$$

Where D is the result of combining the constants of integration from both sides of the equation. Since $\eta = x/t$ we have

$$D = \ln(\sqrt{t}u) + \frac{x^2}{4t}.$$

This gives us our second invariant. In fact, any function of D will be a second invariant. Let us take $v = e^D$ as our second invariant. That is, we set

$$v = \sqrt{t}e^{x^2/4t}u$$

to be the second invariant. Notice that the two sets of invariants we have obtained are *functionally independent*. Two invariants η and ξ are *functionally dependent* if there exists a continuous function F such that $\eta = F(\xi)$. If no such relationship exists, they are said to be functionally independent. In general, for a vector field with three variables, there will be two functionally independent sets of invariants.

We use the invariants to rewrite the PDE. We saw that $\eta = x^2 + y^2$ is an invariant of the rotation group $SO(2)$ in the plane. The Laplace equation $\Delta u = 0$ has $SO(2)$ as a group of symmetries.

Let $r = x^2 + y^2$ be our invariant. We look for a solution of the Laplace equation of the form $u(x, y) = U(x^2 + y^2) = U(r)$. Then by the chain rule Laplace's equation in the plane therefore becomes

$$\Delta u = 4\frac{dU}{dr} + 4r\frac{d^2U}{dr^2} = 0.$$

This ODE is called the *reduced equation*, because we have reduced a PDE in two variables to an ODE.

We may solve this to obtain $U(r) = A\ln r + D$, where D is a constant of integration. This is the family of solutions of the Laplace equation invariant under rotations. If we take $D = 0$ and $A = 1/4\pi$ we obtain the solution

$$U(r) = \frac{1}{4\pi}\ln r = \frac{1}{4\pi}\ln(x^2 + y^2),$$

which is the fundamental solution of the two dimensional Laplace equation.

What would happen if we chose a different invariant for the change of variables? We could equally have picked $r = \sqrt{x^2+y^2}$ for the Laplace equation. Doing this, we would obviously arrive at a different ODE. However the ODE which we arrived at would be equivalent to the previous one under the simple change of variables $r \to \sqrt{r}$. This is a special case of a more general situation. If η is an invariant and $\xi = f(\eta)$ is another invariant, then the reduced equations we obtain by using η and ξ respectively as changes of variables, will always be equivalent under the change of variables $\eta \to f(\eta)$.

Judicious choice of invariants can lead to easier forms of the reduced equation. For example, the so called heat equation on the $ax+b$ group can be written as

$$\frac{\partial u}{\partial t} = y^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + y\frac{\partial u}{\partial y}.$$

This has scaling symmetries $(x, y, t) \to (\lambda x, \lambda y, t), \lambda > 0$. If we make the obvious choice for an invariant $\xi = x/y$ then the reduced equation is

$$\frac{\partial u}{\partial t} = (1+\xi^2)\frac{\partial^2 u}{\partial \xi^2} + 2\xi\frac{\partial u}{\partial \xi}.$$

However, the less obvious choice $\xi = \sinh^{-1}(x/y)$ leads to the reduced PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2}.$$

This is just the one dimensional heat equation. So, some experimentation may be needed to obtain an optimal form of the reduced PDE.

Having illustrated the general procedure, let us turn to the heat equation.

Example 6.4. Let us find the group invariant solutions for the heat equation $u_t = u_{xx}$, where the group action is generated by the vector field $\mathbf{v} = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)u\frac{\partial}{\partial u}$.

We found two invariants, namely $\eta = x/t$ and $v = \sqrt{t}e^{x^2/4t}u$. Let our change of variables be $y = x/t,\ v = \sqrt{t}e^{x^2/4t}u$.

Applying the chain rule we have

$$u_t = \left(\frac{(-2\,t + x^2)\ v(y) - 4\,x\,v'(y)}{4\,t^{\frac{5}{2}}}\right) e^{-x^2/4t}.$$

Turning to the x derivatives gives

$$u_{xx} = \left(\frac{(-2\,t + x^2)\ v(y) - 4\,x\,v'(y) + 4\,v''(y)}{4\,t^{\frac{5}{2}}}\right) e^{-x^2/4t}.$$

So using our expressions for u_t and u_{xx}, the heat equation becomes $v''(y) = 0$. The general solution is just $v(y) = Ay + B$. Hence the group

invariant solutions are of the form
$$u(x,t) = \frac{1}{\sqrt{t}} e^{-x^2/4t}(A\frac{x}{t} + B). \tag{6.6}$$
Taking $A = 0, B = \frac{1}{\sqrt{4\pi}}$ will give the fundamental solution of the heat equation.

Example 6.5. Suppose we have the generalized bond pricing PDE
$$u_t = \sigma x u_{xx} + f(x)u_x - \mu x u.$$
If f satisfies a Riccati equation of the form
$$\sigma x f' - \sigma f + \frac{1}{2} f^2 + 2\mu\sigma x^2 = Ax + B$$
then there is symmetry of the form
$$\mathbf{v} = xt\partial_x + \frac{1}{2}t^2\partial_t - \frac{1}{2\sigma}(x + tf(x) + \frac{1}{2}At^2)u\partial_u.$$
To find the invariants of this group action, we have to solve
$$\frac{dx}{xt} = \frac{dt}{\frac{1}{2}t^2} = -\frac{du}{\frac{1}{2\sigma}(x + tf(x) + \frac{1}{2}At^2)u}. \tag{6.7}$$
This is straightforward and we take as our two invariants
$$y = x/t^2, \quad v = \exp\left(\frac{At}{2\sigma} + \frac{x}{\sigma t} + \frac{1}{2\sigma}F(x)\right)u.$$
We now wish to rewrite our PDE in terms of these invariants. That means we want u to be of the form
$$u(x,t) = \exp\left(\frac{-At}{2\sigma} - \frac{x}{\sigma t} - \frac{1}{2\sigma}F(x)\right)v(x/t^2).$$
We need to substitute this into our PDE and simplify to obtain an ODE for $v(y)$.

The reduced equation is simply
$$2\sigma^2 y^2 v''(y) = Bv(y).$$
This is the reduced equation for our generalised Bond pricing PDE and it is of Euler type. Substituting $v(y) = y^k$ gives the general solution for v as
$$v(y) = c_1 y^{\frac{\sigma + \sqrt{2B + \sigma^2}}{2\sigma}} + c_2 y^{-\frac{-\sigma + \sqrt{2B+\sigma^2}}{2\sigma}}.$$
Consequently the group invariant solutions of the generalised bond pricing PDE for this group action are
$$\begin{aligned} u(x,t) &= e^{\left(\frac{-At}{2\sigma} - \frac{x}{\sigma t} - \frac{1}{2\sigma}F(x)\right)} v(x/t^2) \\ &= e^{\left(\frac{-At}{2\sigma} - \frac{x}{\sigma t} - \frac{1}{2\sigma}F(x)\right)} \left(c_1 (\frac{x}{t^2})^{\frac{\sigma + \sqrt{2B+\sigma^2}}{2\sigma}} + c_2 (\frac{x}{t^2})^{-\frac{-\sigma+\sqrt{2B+\sigma^2}}{2\sigma}}\right). \end{aligned}$$
Other group invariant solutions can similarly be found.

6.1. **The Heat Equation on the Heisenberg Group.** In attempting to use Lie group symmetry methods to study PDEs, it is important for the researcher to be aware that very often the PDE of interest may not have many or possibly even any interesting symmetries. This does not mean that Lie symmetry analysis is of no use. In this section we will show that it may still be possible to analyse a PDE that has few useful symmetries, by a creative use of the ones available.

The key is the following somewhat surprising phenomenon. A PDE obtained by a group reduction can have more point symmetries than the original PDE. There are many examples of this known. Take the heat equation on the $ax + b$ group. This has a five dimensional Lie point symmetry algebra, excluding the infinite dimensional ideal which comes from superposition of solutions. However, as we saw in the previous section, if we reduce it under a scaling variable, then we obtain the heat equation, which has a six dimensional Lie point symmetry algebra. Using this, we can obtain considerable information about the heat kernel on the $ax + b$ group. See for example the analysis in [3].

The material in this section can be regarded as something of a case study. We will carry out a Lie symmetry analysis for the heat equation on the Heisenberg group. Even though the heat equation for the Heisenberg group has few point symmetries, we can still obtain the heat kernel by Lie symmetry methods. We also obtain an interesting new result on the invariance properties of the heat kernel. Other equations can be treated by the same type of analysis we make here.

A basis for the Heisenberg Lie algebra may be taken as

$$X = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ Z = \frac{\partial}{\partial z}.$$

Taking the subLaplacian $\Delta_H = X^2 + Y^2$ leads to the following heat equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2y\frac{\partial^2 u}{\partial x \partial z} - 2x\frac{\partial^2 u}{\partial y \partial z} + (x^2 + y^2)\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t}, \tag{6.8}$$

This does not have many useful symmetries, but is invariant under rotations. The fundamental solution should also be invariant under rotation. We therefore make the change of variables $r = \sqrt{x^2 + y^2}$. This reduces the heat equation to

$$U_t = U_{rr} + \frac{1}{r}U_r + r^2 z^2 U_{zz}. \tag{6.9}$$

It is then natural to remove the z by Fourier transform to obtain the reduced PDE

$$u_t = u_{rr} + \frac{1}{r}u_r - \lambda^2 r^2 u, \tag{6.10}$$

where $u(r, t; \lambda) = \int_{-\infty}^{\infty} U(r, z, t)e^{-i\lambda z} d\lambda$.

This PDE does have interesting symmetries. Our first task is to find a basis for the Lie algebra of symmetries of (6.10). The following is easy to prove.

Proposition 6.2. *A basis for the Lie algebra of symmetries of*

$$u_t = u_{rr} + \frac{1}{r}u_r - \lambda^2 r^2 u. \tag{6.11}$$

is given by

$$\begin{aligned}
\mathbf{v}_1 &= 2|\lambda| r e^{4|\lambda|t}\frac{\partial}{\partial r} + e^{4|\lambda|t}\frac{\partial}{\partial t} - 2e^{4|\lambda|t}(\lambda^2 r^2 + |\lambda|)u\frac{\partial}{\partial u}, \\
\mathbf{v}_2 &= -2|\lambda| r e^{-4|\lambda|t}\frac{\partial}{\partial r} + e^{-4|\lambda|t}\frac{\partial}{\partial t} - 2e^{-4|\lambda|t}(\lambda^2 r^2 - |\lambda|)u\frac{\partial}{\partial u}, \\
\mathbf{v}_3 &= \frac{\partial}{\partial t},\ \mathbf{v}_4 = u\frac{\partial}{\partial u},
\end{aligned}$$

and there is an infinite dimensional ideal spanned by vector fields of the form $\mathbf{v}_\beta = \beta(r,t)\frac{\partial}{\partial u}$*, where* β *is an arbitrary solution of (6.11).*

It is not difficult to see that the vector fields $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ span a copy of $\mathfrak{sl}_2$. Of more immediate interest is the question of what symmetries these vector fields produce. The proof of the following result is standard.

Proposition 6.3. *Let* u *be any solution of (6.11). Then the following are also solutions, at least for* ϵ *sufficiently small.*

$$\begin{aligned}
\rho(\exp(\epsilon\mathbf{v}_1))u(r,t) &= \frac{1}{\sqrt{1+4|\lambda|\epsilon e^{4|\lambda|t}}}\exp\left\{\frac{-2\lambda^2\epsilon r^2 e^{4|\lambda|t}}{1+4|\lambda|\epsilon e^{4|\lambda|t}}\right\}\times \\
&\qquad u\left(\frac{r}{\sqrt{1+4|\lambda|\epsilon e^{4|\lambda|t}}}, \frac{1}{4|\lambda|}\ln\left(\frac{e^{4|\lambda|t}}{1+4|\lambda|\epsilon e^{4|\lambda|t}}\right)\right)
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
\rho(\exp(\epsilon\mathbf{v}_2))u(r,t) &= \frac{e^{2|\lambda|t}}{\sqrt{e^{4|\lambda|t}-4|\lambda|\epsilon}}\exp\left\{-\frac{2\lambda^2\epsilon r^2}{e^{4|\lambda|t}-4|\lambda|\epsilon}\right\}\times \\
&\qquad u\left(\frac{re^{2|\lambda|t}}{\sqrt{e^{4|\lambda|t}-4|\lambda\epsilon}}, \frac{\ln(e^{4|\lambda|t}-4|\lambda|\epsilon)}{4|\lambda|}\right)
\end{aligned} \tag{6.13}$$

$$\rho(\exp(\epsilon\mathbf{v}_3))u(r,t) = u(r, t-\epsilon), \tag{6.14}$$

$$\rho(\exp(\epsilon\mathbf{v}_4))u(r,t) = e^{\epsilon}u(r,t). \tag{6.15}$$

It is now easy to obtain explicit solutions of the heat equation on the Heisenberg group from solutions of (6.11). Stationary solutions of (6.11) can be found by solving.

$$u_{rr} + \frac{1}{r}u_r - \lambda^2 r^2 u = 0 \tag{6.16}$$

The general solution of (6.16) is

$$u_0(r) = c_1 I_0(\frac{|\lambda| r^2}{2}) + c_2 K_0(\frac{|\lambda| r^2}{2}), \tag{6.17}$$

in which I_0 and K_0 are modified Bessel functions of the first and second kind.

Using the symmetries in (6.3), we can find families of solutions of the heat equation on H_3, such as

$$\begin{aligned} U_\epsilon^1(x,y,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+4|\lambda|\epsilon e^{4|\lambda|t}}} \exp\left\{ \frac{-2\epsilon\lambda^2(x^2+y^2)e^{4|\lambda|t}}{1+4|\lambda|\epsilon e^{2|\lambda|t}} \right\} \times \\ I_0\left(\frac{|\lambda|(x^2+y^2)}{2(1+4|\lambda|\epsilon e^{4|\lambda|t})} \right) e^{i\lambda z} d\lambda. \end{aligned} \tag{6.18}$$

Another family of solutions which comes from the symmetry given by $\mathbf{v}_1 + \mathbf{v}_2$ is

$$\begin{aligned} U_\epsilon(x,y,z,t) = \int_{-\infty}^{\infty} \frac{e^{-2\epsilon} \operatorname{sech}(2|\lambda|t)}{\sqrt{1-e^{-8\epsilon}\tanh^2(2|\lambda|t)}} e^{-\frac{|\lambda|(1-e^{-8\epsilon})\tanh(2|\lambda|t)(x^2+y^2)}{2(1-e^{-8\epsilon}\tanh^2(2|\lambda|t))}} \\ \times I_0\left(\frac{|\lambda| e^{-4\epsilon} \operatorname{sech}^2(2|\lambda|t)(x^2+y^2)}{2(1-e^{-8\epsilon}\tanh^2(2|\lambda|t))} \right) e^{i\lambda z} \frac{d\lambda}{2\pi}. \end{aligned} \tag{6.19}$$

There are more such families of solutions which can easily be written down. Our next task is to exhibit the fundamental solution for the heat equation.

6.1.1. *The Heat Kernel on the Heisenberg Group.* It is possible to obtain the heat kernel for the Heisenberg group using the integral transform approach as discussed in Section 5. This calculation is presented in [4] and a similar calculation is presented in the next section. There we deal with a form of the heat equation on the Heisenberg group that is associated with the classical harmonic oscillator. In this subsection we will show that the kernel may also be obtained as a group invariant solution. It is easy to verify that the vector field

$$\begin{aligned} \mathbf{v} = 2r\cosh(4|\lambda|t)\frac{\partial}{\partial r} + \frac{\sinh(4|\lambda|t)}{|\lambda|}\frac{\partial}{\partial t} \\ - \left(2|\lambda| r^2 \sinh(4|\lambda|t) + 2\cosh(4|\lambda|t) + 2\right) u\frac{\partial}{\partial u}. \end{aligned} \tag{6.20}$$

generates a local group of symmetries of (6.10). Invariants of the group action are

$$y = \frac{r}{\sqrt{\sinh(4|\lambda|t)}}, \tag{6.21}$$

and

$$w = \sqrt{\sinh(2|\lambda|t)} \exp\left\{ \frac{|\lambda| r^2}{2\tanh(4|\lambda|t)} \right\} u. \tag{6.22}$$

We now rewrite (6.11) in terms of the new variables y and w. Under the change of variables given by w and y, (6.11) becomes the ODE

$$w''(y) + \frac{1}{y}w'(y) - (\lambda^2 y^2 - 2|\lambda|)w(y) = 0. \tag{6.23}$$

We consider solutions of the form

$$w(y) = C(\lambda)e^{-\frac{|\lambda|y^2}{2}}, \tag{6.24}$$

in which $C(\lambda)$ is a constant of integration, which depends upon λ. If we reintroduce the original variables, we will get a solution of the original equation (6.10). We obtain

$$\begin{aligned} u(r,\lambda,t) &= \frac{C(\lambda)}{\sinh(2|\lambda|t)}\exp\left\{-\frac{|\lambda|r^2}{2\tanh(4|\lambda|t)} - \frac{|\lambda|r^2}{2\sinh(4|\lambda|t)}\right\} \\ &= \frac{C(\lambda)}{\sinh(2|\lambda|t)}\exp\left\{\frac{-|\lambda|r^2}{2\tanh(2|\lambda|t)}\right\}. \end{aligned} \tag{6.25}$$

In order to obtain a solution of (6.8), we require a choice of $C(\lambda)$, such that the following integral converges

$$U(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{C(\lambda)}{\sinh(2|\lambda|t)}\exp\left\{\frac{-|\lambda|(x^2+y^2)}{2\tanh(2|\lambda|t)} + i\lambda z\right\}d\lambda. \tag{6.26}$$

If we choose $C(\lambda) = \frac{|\lambda|}{2\pi}$, we obtain the solution

$$h(x,y,z,t) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{|\lambda|}{2\sinh(2|\lambda|t)}\exp\left\{\frac{-|\lambda|(x^2+y^2)}{2\tanh(2|\lambda|t)} + i\lambda z\right\}d\lambda. \tag{6.27}$$

In fact, h is the fundamental solution of the heat equation on the Heisenberg group, which is normally found by spherical Fourier transform. Here we obtain it through a simple calculation with symmetry groups. We further point out that generalisation to the $2n+1$ dimensional Heisenberg group is straightforward. We have also proved the following new result about the heat kernel on H_3.

Proposition 6.4. *Let $\mathcal{F}_3$ denote Fourier transform in the third variable. The heat kernel $h(x,y,z,t)$ on H_3 is invariant under the transformation $T_\epsilon u(x,y,z,t) = \mathcal{F}_3^{-1}\exp(\epsilon\mathbf{v})\mathcal{F}_3 u(x,y,z,t)$, where $\mathbf{v}$ is given by (6.20).*

The reader may note that here we have merely observed that the solution we have obtained is the known heat kernel for the Heisenberg group. However there are systematic procedures for searching for fundamental solutions by group invariant solution methods, such as those described in Bluman and Kumei's book. There are also approaches combining representation theory and symmetries, but we will not go into details here as this line of investigation will be the subject of a forthcoming publication.

7. The Heisenberg Group and Mehler's Formula

Let us now consider a different realisation of the subLaplacian for the Heisenberg group. This will lead us to an easy proof of Mehler's formula for the harmonic oscillator. The irreducible unitary representations of H_3 are indexed by nonzero real numbers λ and may be represented on $L^2(\mathbb{R})$ by

$$\pi_\lambda(a,b,c)f(x) = e^{i\lambda(c+a(x-\frac{1}{2}b))}f(x-b), \quad (a,b,c)\in H_3,\ f\in L^2(\mathbb{R}). \tag{7.1}$$

See for example [7]. We set

$$\begin{aligned} Xf &= \frac{d}{da}\pi_\lambda(a,0,0)f(x)\big|_{a=0} = \lambda i x f(\xi) \\ Yf &= \frac{d}{db}\pi_\lambda(0,b,0)f(x)\big|_{b=0} = -\frac{d}{dx}f(x), \\ Zf &= \frac{d}{dc}\pi_\lambda(0,0,c)f(x)\big|_{c=0} = i\lambda f(x). \end{aligned}$$

Plainly $[X,Y]=-Z$. We are interested in solving the heat equation associated with the sublaplacian $\mathcal{H} = X^2+Y^2$. In other words, we would like the heat kernel for

$$u_t = u_{xx} - \lambda^2 x^2 u. \tag{7.2}$$

We will find it more convenient to study the equation

$$u_t = u_{xx} - (\lambda^2x^2 - |\lambda|)u. \tag{7.3}$$

We can obtain a solution of (7.2) from a solution u of (7.3) by multiplying u by $e^{-|\lambda|t}$.

We determine the symmetries for (7.3). Leaving the details to the reader, we state the necessary result.

Proposition 7.1. *A basis for the Lie algebra of symmetries of (7.3) is given by*

$$\begin{aligned} \mathbf{v}_1 &= 2|\lambda|xe^{4|\lambda|t}\frac{\partial}{\partial r} + e^{4|\lambda|t}\frac{\partial}{\partial t} - 2\lambda^2x^2e^{4|\lambda|t}u\frac{\partial}{\partial u}, \\ \mathbf{v}_2 &= -2|\lambda|xe^{-4|\lambda|t}\frac{\partial}{\partial r} + e^{-4|\lambda|t}\frac{\partial}{\partial t} - 2(\lambda^2x^2-|\lambda|)e^{-4|\lambda|t}u\frac{\partial}{\partial u}, \\ \mathbf{v}_3 &= \frac{\partial}{\partial t}, \quad \mathbf{v}_4 = e^{2|\lambda|t}\frac{\partial}{\partial x} - |\lambda|xe^{2|\lambda|t}u\frac{\partial}{\partial u}, \\ \mathbf{v}_5 &= e^{-2|\lambda|t}\frac{\partial}{\partial x} + |\lambda|xe^{-2|\lambda|t}u\frac{\partial}{\partial u}, \quad \mathbf{v}_6 = u\frac{\partial}{\partial u}, \end{aligned}$$

and there is an infinite dimensional ideal spanned by vector fields of the form $\mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u}$, where β is an arbitrary solution of (7.3).

We are concerned with $\mathbf{v}_2$. We have

Lemma 7.2. *If $u(x,t)$ is a solution of (7.3), then so is*

$$\rho(\exp\epsilon\mathbf{v}_2)u(x,t) = \frac{e^{2|\lambda|t}}{\sqrt{e^{4|\lambda|t}-4|\lambda|\epsilon}}\exp\left\{\frac{-2\lambda^2\epsilon x^2}{e^{4|\lambda|t}-4|\lambda|\epsilon}\right\}\times$$
$$u\left(\frac{xe^{2|\lambda|t}}{\sqrt{e^{4|\lambda|t}-4|\lambda|\epsilon}}, \frac{\ln(e^{4|\lambda|t}-4|\lambda|\epsilon)}{4|\lambda|}\right). \tag{7.4}$$

We make use of the fact that $u_0(x) = e^{\frac{-|\lambda|x^2}{2}}$ is a solution. Applying the symmetry (7.4) to u_0 gives

$$\rho(\exp\epsilon\mathbf{v}_2)u_0(x) = \frac{e^{2|\lambda|t}}{\sqrt{e^{4|\lambda|t}-4|\lambda|\epsilon}}\exp\left\{\frac{-|\lambda|x^2}{2}\left(\frac{e^{4|\lambda|t}+4|\lambda|\epsilon}{e^{4|\lambda|t}-4|\lambda|\epsilon}\right)\right\}. \tag{7.5}$$

It is possible to obtain the heat kernel for (7.3) from this function.

Theorem 7.3. *The heat kernel for (7.3) is given by*

$$p(t,x,y) = \frac{\sqrt{|\lambda|}e^{|\lambda|t}}{\sqrt{2\pi\sinh(2|\lambda|t)}}\exp\left\{\frac{-|\lambda|(x^2+y^2)}{2\tanh(2|\lambda|t)}\right\}\cosh\left(\frac{|\lambda|xy}{\sinh(2|\lambda|t)}\right). \tag{7.6}$$

Proof. We set

$$u_\epsilon(x,t) = \frac{e^{2|\lambda|t}}{\sqrt{e^{4|\lambda|t}-4|\lambda|\epsilon}}\exp\left\{\frac{-|\lambda|x^2}{2}\left(\frac{e^{4|\lambda|t}+4|\lambda|\epsilon}{e^{4|\lambda|t}-4|\lambda|\epsilon}\right)\right\}. \tag{7.7}$$

The integral equation (5.5), implies that we must have

$$\int_{-\infty}^{\infty} u_\epsilon(y,0)p(t,x,y)dy = \frac{e^{2|\lambda|t}}{\sqrt{e^{4|\lambda|t}-4|\lambda|\epsilon}}\exp\left\{\frac{-|\lambda|x^2}{2}\left(\frac{e^{4|\lambda|t}+4|\lambda|\epsilon}{e^{4|\lambda|t}-4|\lambda|\epsilon}\right)\right\} \tag{7.8}$$

Introduce the change of parameters $s = \frac{|\lambda|}{2}\left(\frac{1+4|\lambda|\epsilon}{1-4|\lambda|\epsilon}\right)$. Then $u_\epsilon(y,0) = \frac{e^{-sy^2}}{\sqrt{1-4|\lambda|\epsilon}}$. Using the symmetry of $p(t,x,y)$ in x and y, this reduces the problem of obtaining p from (7.8) to the inversion of a Laplace transform when we let $y^2 = \xi$. Inversion of the transform and replacement of the original variables gives the result. □

Example 7.1. We use this result to obtain a solution of the heat equation. Integrating the kernel against $\phi(y) = 1$ gives

$$\int_{-\infty}^{\infty}\frac{\sqrt{|\lambda|}e^{|\lambda|t}}{\sqrt{2\pi\sinh(2|\lambda|t)}}\exp\left\{\frac{-|\lambda|(x^2+y^2)}{2\tanh(2|\lambda|t)}\right\}\cosh\left(\frac{|\lambda|xy}{\sinh(2|\lambda|t)}\right)d\xi$$
$$= \frac{e^{|\lambda|t}}{\sqrt{\cosh(2|\lambda|t)}}\exp\left\{-\frac{|\lambda|x^2}{2}\tanh(2|\lambda|t)\right\}. \tag{7.9}$$

Differentiation shows that $u(x,t)$ is a solution of (7.3) and it is clear that $u(x,0)=1$.

We obtain the heat kernel for (7.2) in the obvious way.

Corollary 7.4. *The heat kernel for (7.2) is*

$$p_1(t,x,y)=\frac{\sqrt{|\lambda|}}{\sqrt{2\pi\sinh(2|\lambda|t)}}\exp\left\{\frac{-|\lambda|(x^2+y^2)}{2\tanh(2|\lambda|t)}\right\}\cosh\left(\frac{|\lambda|xy}{\sinh(2|\lambda|t)}\right). \tag{7.10}$$

If we let $\lambda=1$ in expression (7.10) then we obtain what is essentially Mehler's formula for the harmonic oscillator, as given on page 113 of Davies book, [9]. Previously Craddock and Dooley have shown that Mehler's formula can be obtained directly by transforming the initial solution $u_0(x)=e^{-|\lambda|x^2/2}$ by the appropriate symmetries.

Now let us take the inverse Fourier transform in λ. We see that if u is a solution of (7.2), then $U(x,z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x,t;\lambda)e^{i\lambda z}$ is a solution of

$$U_t=U_{xx}+x^2U_{zz}. \tag{7.11}$$

This is the heat equation (6.8) restricted to the $y=0$ plane. It is easy to see that solutions of (7.11) can be obtained by

$$U(x,z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{|\lambda|}e^{i\lambda z}}{\sqrt{2\pi\sinh(2|\lambda|t)}}\exp\left\{\frac{-|\lambda|x^2}{2\tanh(2|\lambda|t)}\right\}G_\lambda(x,z,t)d\lambda \tag{7.12}$$

with

$$G_\lambda(x,z,t)=\int_{-\infty}^{\infty}\varphi(\xi)\exp\left\{\frac{-|\lambda|(x^2+\xi^2)}{2\tanh(2|\lambda|t)}\right\}\cosh\left(\frac{|\lambda|x\xi}{\sinh(2|\lambda|t)}\right)d\xi.$$

for suitable initial data. Taking $\varphi(y)=\delta(y)$ yields the heat kernel for the restricted heat equation (7.11). It is

$$h_R(x,z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{|\lambda|}e^{i\lambda z}}{\sqrt{2\pi\sinh(2|\lambda|t)}}\exp\left\{\frac{-|\lambda|x^2}{2\tanh(2|\lambda|t)}\right\}d\lambda. \tag{7.13}$$

7.0.2. *An Orthogonal Family of Solutions.* As remarked previously, every linear PDE has symmetries which reflect the fact that superposition of solutions leads to new solutions. Here we will show how this fact can sometimes be exploited to obtain interesting results, by combining it with other symmetries. The basic idea is that we use the Lie algebra property of the infinitesimal symmetries.

Consider the equation (7.3). Since the vector fields in Proposition 7.1 form a Lie algebra, we can generate a family of solutions in the following manner. We know that if $\mathbf{v}_i$ and $\mathbf{v}_j$ are infinitesimal symmetries, then so is the $\mathbf{v}=[\mathbf{v}_i,\mathbf{v}_j]$. We will use two vector fields, $\mathbf{v}_5=e^{-2|\lambda|t}\frac{\partial}{\partial x}+$

$|\lambda|xe^{-2|\lambda|t}u\frac{\partial}{\partial u}$and $\mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u}$, where β is an arbitrary solution of 7.1. Then the Lie bracket

$$[\mathbf{v}_5, \mathbf{v}_\beta] = \left(e^{-2|\lambda|t}\beta_x - |\lambda|xe^{-2|\lambda|t}\beta\right)\frac{\partial}{\partial u}$$

is also an infinitesimal symmetry. This means that if β is a solution of (7.3), then so $u(x,t) = e^{-2|\lambda|t}\beta_x - |\lambda|xe^{-2|\lambda|t}\beta$.

We can thus generate a family of solutions of (7.3) by setting $\beta_0 = e^{-|\lambda|x^2/2}$ and

$$\beta_{n+1} = e^{-2|\lambda|t}\frac{\partial \beta_n}{\partial x} - |\lambda|xe^{-2|\lambda|t}\beta_n.$$

It is possible to produce an explicit formula for the solutions β_n. We will see that the family of solutions generated in this manner is orthogonal on $L^2(\mathbb{R}\times[0,\infty))$. To prove this, we observe that if we exponentiate $\mathbf{v}_5$ we find that

$$\rho(\exp(\epsilon\mathbf{v}_5))u(x,t) = \exp\left\{\epsilon|\lambda|xe^{-2|\lambda|t} - \frac{|\lambda|}{2}\epsilon^2e^{-4|\lambda|t}\right\}u(x-\epsilon e^{-2|\lambda|t},t). \tag{7.14}$$

Applying this symmetry to the solution $u_0 = e^{-|\lambda|x^2/2}$, we see that

$$\tilde{u}_\epsilon(x,t) = e^{-\frac{|\lambda|}{2}x^2}e^{2\epsilon|\lambda|xe^{-2|\lambda|t}-|\lambda|\epsilon^2e^{-4|\lambda|t}}$$

is also a solution of (7.3). It is well known that if we expand $\tilde{u}_\epsilon$ in a Taylor series we obtain

$$\tilde{u}_\epsilon(x,t) = \sum_{n=0}^\infty \frac{\epsilon^n}{n!}\beta_n(x,t). \tag{7.15}$$

This may be checked by direct calculation. However, $\tilde{u}_\epsilon$ is the generating function for the Hermite polynomials, H_n. So we have

$$\begin{aligned} e^{-\frac{|\lambda|}{2}x^2}e^{2\epsilon|\lambda|xe^{-2|\lambda|t}-|\lambda|\epsilon^2e^{-4|\lambda|t}} &= e^{-|\lambda|x^2/2}\sum_{n=0}^\infty\frac{(\sqrt{|\lambda|}\epsilon e^{-2|\lambda|t})^n}{n!}H_n(\sqrt{|\lambda|}x) \\ &= e^{-|\lambda|x^2/2}\sum_{n=0}^\infty\frac{(-\epsilon)^n}{n!}\beta_n(x,t). \end{aligned} \tag{7.16}$$

We thus have an explicit expression for the solutions β_n :

$$\beta_n(x,t) = (-1)^ne^{-|\lambda|x^2/2}(\sqrt{|\lambda|})^ne^{-2n|\lambda|t}H_n(\sqrt{|\lambda|}x).$$

Now observe that if $(\beta_n,\beta_m) = \int_0^\infty\int_{-\infty}^\infty \beta_n(x,t)\beta_m(x,t)dxdt$ then

$$\begin{aligned} (\beta_n,\beta_m) &= \int_0^\infty\int_{-\infty}^\infty e^{-|\lambda|x^2}|\lambda|^{\frac{n+m}{2}}e^{-2(n+m)|\lambda|t}H_n(\sqrt{|\lambda|}x)H_m(\sqrt{|\lambda|}x)dxdt \\ &= \frac{2^nn!|\lambda|^n}{4|\lambda|}\delta_{nm} = 2^{n-2}n!|\lambda|^{n-1}\delta_{nm}, \end{aligned}$$

where δ_{nm} is the Kronecker delta. Thus the family $\{\beta_n\}_{n\in\mathbb{N}}$ is an orthogonal family of solutions. We can thus study the expansion of arbitary solutions of the (7.3) in terms of the functions β_n.

In principle, a similar analysis should be possible for any second order parabolic PDE with non trivial symmetries. For example, the reader can check that using this method, it is possible to construct the heat polynomials for the heat equation from the solution $u = 1$. Theorems for the expansion of arbitrary solutions of the heat equation in terms of heat polynomials and associated functions have been established by Widder [19].

8. Symmetries and Group Representations

There is a close connection between Lie symmetry groups and group representation theory. This connection has been explored in [6], [7] and [8]. The basic idea is that Lie group symmetries for linear PDEs can often be constructed from standard representations of the underlying group. One has to find a suitably restricted space of solutions and an intertwining operator and this can often be done. This approach to symmetries has many potential applications. We will present one such here.

We illustrate the basic connection with an example involving the heat equation. Consider the symmetry of the one dimensional heat equation given by

$$\tilde{u}_\epsilon(x,t) = e^{-\epsilon x+\epsilon^2 t}u(x-2\epsilon t,t).$$

The idea is to apply this to the solution of the heat equation with initial data $u(x,0) = f(x)$ given by

$$u(x,t) = \int_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{(x-y)^2}{4t}\right\}dy.$$

We have

$$\begin{aligned}\tilde{u}_\epsilon(x,t) &= e^{-\epsilon x+\epsilon^2 t}\int_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{(x-2\epsilon t-y)^2}{4t}\right\}dy \\ &= \int_{-\infty}^{\infty} e^{-\epsilon y}f(y)\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{(x-y)^2}{4t}\right\}dy.\end{aligned}$$

Notice that the complicated symmetry $\tilde{u}_\epsilon(x,t) = e^{-\epsilon x+\epsilon^2 t}u(x-2\epsilon t,t)$ is revealed here to be nothing more than the resulting of transforming the initial data $f(x)$ to $e^{-\epsilon x}f(x)$. This is clear if we substitute $t = 0$ into $\tilde{u}_\epsilon(x,t)$. We have

$$\tilde{u}_\epsilon(x,0) = e^{-\epsilon x}u(x,0).$$

So the symmetry solution $\tilde{u}_\epsilon(x,t)$ is the solution of the heat equation whose initial value is $e^{-\epsilon x}$ times the initial value of the original solution $u(x,t)$.

In fact all the point symmetries of the heat equation arise in a similar fashion. However this is not so easy to prove. The difficulty lies in the fact that the representations which the construction relies upon are not unitary and cannot define an action on any Banach space. This makes it necessary to construct a topological vector space of distributions upon which the representations act.

The unitary case is much easier to handle. For example, consider the complex heat equation $iu_t = u_{xx}$. This possesses a six dimensional Lie symmetry algebra spanned by

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial x}, \ \mathbf{v}_2 = \frac{\partial}{\partial t}, \ \mathbf{v}_3 = iu\frac{\partial}{\partial u}, \ \mathbf{v}_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \frac{1}{2}iu\frac{\partial}{\partial u} \\
\mathbf{v}_5 &= 2t\frac{\partial}{\partial x} - ixu\frac{\partial}{\partial u}, \ \mathbf{v}_6 = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (ix^2+2t)u\frac{\partial}{\partial u}.
\end{aligned}$$

It is a straightforward exercise to show that $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\}$ generate the three dimensional Heisenberg group. The remaining three vector fields generate $SL(2, \mathbb{R})$. We denote the action of the symmetry group on a solution u by $\sigma(g)u(x,t)$. We then have

$$\sigma(\exp \epsilon \mathbf{v}_1)u(x,t) = u(x-\epsilon, t) \tag{8.1}$$

$$\sigma(\exp \epsilon \mathbf{v}_2)u(x,t) = u(x, t-\epsilon) \tag{8.2}$$

$$\sigma(\exp \epsilon \mathbf{v}_3)u(x,t) = e^{i\epsilon}u(x,t) \tag{8.3}$$

$$\sigma(\exp \epsilon \mathbf{v}_4)u(x,t) = e^{-\frac{1}{2}i\epsilon}u(e^{\epsilon}x, e^{2\epsilon}t) \tag{8.4}$$

$$\sigma(\exp \epsilon \mathbf{v}_5)u(x,t) = e^{-i\epsilon x + i\epsilon^2 t}u(x-2\epsilon t, t) \tag{8.5}$$

$$\sigma(\exp \epsilon \mathbf{v}_6)u(x,t) = \frac{1}{\sqrt{1+4\epsilon t}} \exp\left\{\frac{-i\epsilon x^2}{1+4\epsilon t}\right\} u\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right). \tag{8.6}$$

We define the operator

$$Af(x,t) = \lim \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{iy^2t+2iyx}dy,$$

where lim denotes the limit in the L^2 mean. This defines solutions of the complex heat equation for all $f \in L^2(\mathbb{R})$. (See Kato [13] for a proof of this claim).

We work with the three dimensional Heisenberg group $H_3 = \mathbb{R}^2 \times \mathbb{R}$. We take the group product to be

$$(a,b,c)\cdot(a',b',c') = (a+a', b+b', c+c' + \frac{1}{2}(ab'-a'b)).$$

We will use the representation of the Heisenberg group given by (7.1). For each λ this is an irreducible representation of the Heisenberg group. See for example Folland's book, [11] for a proof of irreducibility. For

$SL(2,\mathbb{R})$ we have the projective representation $(R_\lambda(g), L^2(\mathbb{R}))$ given by

$$R_\lambda\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right) f(z) = e^{i\lambda b z^2} f(z) \tag{8.7}$$

$$R_\lambda\left(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right) f(z) = |a|^{1/2} f(az) \tag{8.8}$$

$$R_\lambda\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) f(z) = |\lambda|^{1/2} \widehat{f}(\lambda z). \tag{8.9}$$

where $\widehat{f}(z)$ is the Fourier transform of f. This is not a genuine representation of $SL(2,\mathbb{R})$, since it has a nontrivial cocycle. That is,

$$R_\lambda(g)R_\lambda(h) = \alpha(gh)R_\lambda(gh)$$

where α is a nontrivial cocyle and $|\alpha(gh)| = 1$ for all $g, h \in SL(2,\mathbb{R})$. Of course we can extend R_λ to a genuine representation, but this is not necessary for our purposes. (It is in fact a genuine representation of $SL(2,\mathbb{C})$).

We consider the projective representation of $G = H_3 \rtimes SL(2,\mathbb{R})$ given by $\rho_\lambda = \pi_\lambda \otimes R_\lambda$ with $L^2(\mathbb{R})$ as the representation space. Then the following is true.

Theorem 8.1. *Let σ denote the Lie symmetry operator given for basis vectors by (8.1) and let $(\rho_\lambda, L^2(\mathbb{R}))$ be as above. Then for all $g \in H_3 \rtimes SL(2,\mathbb{R})$ and $f \in L^2(\mathbb{R})$ we have*

$$\sigma(g)Af(x,t) = (AT_1(g)f)(x,t).$$

A proof of this result is in [7], with slightly different notation. Equivalent results are true for at least the following list of PDEs:

$$iu_t = u_{xx} + kx^2 u,\ \ k \neq 0 \tag{8.10}$$

$$iu_t = u_{xx} + axu,\ \ a \neq 0 \tag{8.11}$$

$$iu_t = u_{xx} + \frac{a}{x^2} u \tag{8.12}$$

$$iu_t = u_{xx} + \left(\frac{a}{x^2} + kx^2\right) u,\ \ a \neq 0, k \neq 0. \tag{8.13}$$

In other words, for these equations, the Lie point symmetries are actually global projective representations of the underlying Lie group, intertwined with a standard representation via a unitary intertwining operator.

The technical details in the non unitary case are rather difficult. The problem is that in order to prove a result for the heat equation, which is equivalent to Theorem 8.1, we have to take the parameter λ in the representation to be purely imaginary. This leads to a non unitary representation which cannot act on any Banach space. We thus have to construct a locally convex topological vector space of distributions upon which the non unitary representation acts. Call this space $\mathcal{H}^\lambda$. The construction is rather involved and we refer to [7] for the details.

It turns out that there exists a family of representations T_λ of $H_3 \rtimes SL(2,\mathbb{R})$, which has a nonunitary action on $\mathcal{H}^\lambda$. The following is true.

Theorem 8.2. *Let the symmetries of the heat equation $u_t = u_{xx}$ be denoted by*

$$\tilde{u}_\epsilon(x,t) = \sigma(\exp(\epsilon\mathbf{v})u(x,t),$$

where $\mathbf{v}$ is an infinitesimal symmetry. Let $G = H_3 \rtimes SL(2,\mathbb{R})$. Then there exists a locally convex topological vector space of distributions $\mathcal{H}^\lambda$, a non unitary representation T_λ of G acting on $\mathcal{H}^\lambda$ for each $\lambda \in \mathbb{C}$ such that for all $f \in \mathcal{H}^\lambda$ we have

$$(\sigma(g)Af)(x,t) = (AT_i(g)f)(x,t), \tag{8.14}$$

where $Af(x,t) = \frac{1}{2\pi}\int f(y)e^{-y^2t+iyx}dy$.

Similar results can be proven for many other equations, such as certain Fokker-Planck equations and the axially symmetric wave equation. Notice that viewing symmetries as representations immediately reveals information that the standard approach does not. To see why, suppose that we have a linear PDE

$$P(x, D^\alpha)u = 0 \quad (1)$$

for $x \in \Omega$. Suppose also that there is a kernel function $K(x,y)$ defined on $\Omega \times \Omega$ such that for functions f in some suitable function space, $V(\Omega)$. (eg. for $f \in L^1(\Omega)$) the integral

$$u(x) = \int_\Omega f(y)K(x,y)dy \tag{8.15}$$

defines solutions of (1). This requires the kernel $K(x,y)$ to be a solution of (1) for each fixed y. Now suppose that G is a group (usually a Lie group in our context) and T is a representation of G on $V(\Omega)$. Then for all $g \in G$, the mapping $T(g) : V(\Omega) \to V(\Omega)$. This means that if $f \in V(\Omega)$ then $T(g)f \in V(\Omega)$.

We have assumed that the integral (8.15) converges for all $f \in V$. So if it converges for f it also converges for $T(g)f$. Thus we may define a symmetry of (1) by setting

$$\sigma(g)u(x) = \int_\Omega (T(g)f)(y)K(x,y)dy. \tag{8.16}$$

There are interesting questions about this set up. First, is it the case that the symmetries we have just defined are always the same as the Lie point symmetries provided by Lie's method? The answer is that in many cases they are, such as the examples given above. However there are also cases for which they are not. For the cases in which the symmetries defined by (8.16) are point symmetries, it is clear that the integrand of the function can be rewritten

$$(T(g)f)(y)K(x,y) = \rho(x)f(y)K(a(x),y). \tag{8.17}$$

In other words

$$\sigma(g)u(x) = \int_\Omega (T(g)f)(y)K(x,y)dy$$
$$= \rho(x)\int_\Omega f(y)K(a(x),y)dy$$
$$= \rho(x)u(a(x)), \tag{8.18}$$

for some functions ρ and a which depend on the individual symmetry. In the example of the heat equation above, (8.17) reads

$$e^{-\epsilon y}f(y)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}} = e^{-\epsilon x+\epsilon^2 t}f(y)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-2\epsilon t-y)^2}{4t}}.$$

Here $\rho(x,t) = e^{-\epsilon x+\epsilon^2 t}, a(x,t) = x - 2\epsilon t$.

A useful feature of the representation theory approach to symmetries is that it can be applied even when a PDE has no non trivial point symmetries. It can also produce group symmetries which are not Lie point symmetries.

To illustrate this claim, suppose that we consider the third order PDE $u_t = u_{xxx}$. This is the linearised KdV equation. We may obtain solutions of this PDE by means of the Fourier transform. If f is integrable, then

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{-iy^3t+iyx}dy$$

is a solution of the linearised KdV equation.

Now let us consider the group $\mathbb{R}$ and let it act on $L^1(\mathbb{R})$ by translation. That is, if $\epsilon \in \mathbb{R}$ then $T(\epsilon)f(y) = f(y-\epsilon)$. It is clear that $f \in L^1(\mathbb{R})$ implies that $f(y-\epsilon) \in L^1(\mathbb{R})$. So the integral

$$u_\epsilon(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y-\epsilon)e^{-iy^3t+iyx}dy \tag{8.19}$$

also converges. Hence it too defines a solution of the linearized KdV equation. We have

$$\begin{aligned}(\sigma(\epsilon)u)(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(\pi(\epsilon))f(y)\exp\{-iy^3t+iyx\}dy \\ &= \frac{1}{2\pi}e^{i\epsilon x-i\epsilon^3 t}\int_{-\infty}^{\infty}(f(s)e^{3it\epsilon s^2-is^3t})e^{isz}ds,\end{aligned}$$

where $z = x - 3\epsilon^2 t$. Thus

$$(\sigma(\epsilon)u)(x,t) = e^{i\epsilon x-i\epsilon^3 t}\int_{-\infty}^{\infty} u(x-3\epsilon^2 t - y, t)K_{3i\epsilon t}(y)dy,$$

where $K_{3\epsilon it}(x) = \frac{1}{\sqrt{12\epsilon it}}\exp\{\frac{ix^2}{12\epsilon t}\}$

This is certainly not a Lie point symmetry. The only Lie point symmetries of the linearized KdV equation are

$$(\sigma(\epsilon)u)(x,t) = u(x-\epsilon,t), \tag{8.20}$$
$$(\sigma(\epsilon)u)(x,t) = u(x,t-\epsilon), \tag{8.21}$$
$$(\sigma(\epsilon)u)(x,t) = u(e^{\epsilon}, e^{3\epsilon}t) \tag{8.22}$$
$$(\sigma(\epsilon)u)(x,t) = e^{\epsilon}u(x,t). \tag{8.23}$$

However, (8.19) is a symmetry which arises from a Lie group. Other non point symmetries may be identified by choosing different pairs of groups and representations. The actions of many of these symmetries are very hard to explicitly describe in terms of the original solutions u.

Thus the representation theory approach to symmetries reveals more symmetries than does the standard Lie approach. The drawback is that we require a mapping which sends some initial data f to a solution. In contrast, Lie's method provides an algorithm for computing symmetries, which does not require us to know such a mapping. However exploiting the connection between symmetries and representations can be very fruitful.

To connect symmetries and representations, we need an intertwining operator. It is actually possibly to find such an operator by identifying the kernel of the operator as a group invariant solution. However, study of this problem is beyond the scope of this article.

To conclude the discussion on symmetries and representations, we will see how the representation theory approach leads to a new class of operators which map solutions to solutions.

9. A Heisenberg Group Fourier Transform for the Complex Heat Equation

We work with the three dimensional Heisenberg group $H_3 = \mathbb{R}^2 \times \mathbb{R}$. For $f \in L^2(\mathbb{R})$ we set

$$(\pi_\lambda(a,b,c)f)(\xi) = e^{i\lambda(c+a(\xi-\frac{1}{2}b))}f(\xi-b), \quad \lambda \in \mathbb{R}-\{0\}, \tag{9.1}$$

as above. The Fourier transform on the Heisenberg group is defined as follows.

Definition 9.1. Let $f \in \mathcal{S}(H_3)$, where $\mathcal{S}(H_3)$ represents the Schwartz space. The Fourier transform on the Heisenberg group is the operator

$$\left(\pi_\lambda(f)\phi\right)(\xi) = \int_{H_3} f(a,b,c)\pi_\lambda(a,b,c)\phi(\xi)dadbdc, \tag{9.2}$$

for $\phi \in L^2(\mathbb{R})$.

The operator $\pi_\lambda(f)$ is a Hilbert-Schmidt operator of trace class. The following Fourier inversion theorem is easy to prove. See for example the paper by Fabec, [10].

Theorem 9.2. *Let $f \in \mathcal{S}(H_3)$. Then*

(1)
$$f(e) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{tr}(\pi_\lambda(f))|\lambda| d\lambda. \tag{9.3}$$

(2)
$$\|f\|_2^2 = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \|\pi_\lambda(f)\|_2^2|\lambda| d\lambda. \tag{9.4}$$

(3)
$$f(h) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{tr}(\pi_\lambda(h^{-1})\pi_\lambda(f))|\lambda| d\lambda. \tag{9.5}$$

Here $\|T\| = \mathrm{tr}(T^*T)^{1/2}$ is the Hilbert-Schmidt norm.

We now turn to the complex heat equation $iu_t = u_{xx}$. We consider the Heisenberg group symmetries

$$\sigma(\exp \epsilon \mathbf{v}_1)u(x,t) = u(x-\epsilon, t) \tag{9.6}$$

$$\sigma(\exp \epsilon \mathbf{v}_3)u(x,t) = e^{i\epsilon}u(x,t) \tag{9.7}$$

$$\sigma(\exp \epsilon \mathbf{v}_5)u(x,t) = e^{-i\epsilon x + i\epsilon^2 t}u(x-2\epsilon t, t). \tag{9.8}$$

This symmetry is unitarily equivalent to $(\pi_\lambda, L^2(\mathbb{R}))$ when $\lambda = 1$. We wish to extend σ in such a way that we obtain a family of unitary operators depending on λ. For any $\phi \in L^2(\mathbb{R})$ we know that

$$u(x,t) = A\phi(x,t) = \int_{-\infty}^{\infty} \phi(y)K_{it}(x-y)dy, \tag{9.9}$$

$K(x,t) = 1/\sqrt{4\pi t}e^{-x^2/4t}$, is a solution of $iu_t = u_{xx}$ with initial data $u(x,0) = \phi(x)$. We define σ_λ by the rule

$$\sigma_\lambda(h)A\phi = A\pi_\lambda(h)\phi \tag{9.10}$$

for all $\phi \in H_3$. Then for all $h = (a,b,c) \in H_3$ we have:

$$\sigma_\lambda(a,0,0)u(x,t) = u(x-\lambda a, t) \tag{9.11}$$

$$\sigma_\lambda(0,b,0)u(x,t) = e^{-ibx+ib^2t}u(x-2bt, t) \tag{9.12}$$

$$\sigma_\lambda(0,0,c)u(x,t) = e^{i\lambda c}u(x,t). \tag{9.13}$$

Using the decomposition $(a,b,c) = (a,0,0)(0,b,0)(0,0,c-\frac{1}{2}ab)$ we easily obtain

$$\sigma_\lambda(a,b,c)u(x,t) = e^{-ibx+ib^2t+i\lambda(c+\frac{1}{2}ab)}u(x-\lambda a-2bt, t). \tag{9.14}$$

Thus we have a family of unitary representations of H_3 unitarily equivalent to $(\pi_\lambda, L^2(\mathbb{R}))$, which act on solutions of the complex heat equation. The Hilbert space $\mathcal{H}$ on which σ_λ acts is the image Hilbert space of $L^2(\mathbb{R})$ under the unitary mapping A. We can therefore use σ_λ to define a Fourier transform. This will produce operators which map solutions to solutions. In essence, we can realise $L^1(H_3)$ as a Banach algebra of symmetries of the complex heat equation.

We have the following easy result.

Corollary 9.3. *For $f \in \mathcal{S}(H_3)$ we have the Heisenberg group Fourier transform*

$$\sigma_\lambda(f)u = \int_{H_3} f(a,b,c)\sigma_\lambda(a,b,c)u(x,t)dadbdc \tag{9.15}$$

which acts on solutions of $iu_t = u_{xx}$. The following properties hold. Let $f \in \mathcal{S}(H_3)$. Then

(1)
$$f(e) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{tr}(\sigma_\lambda(f))|\lambda|d\lambda. \tag{9.16}$$

(2)
$$\|f\|_2^2 = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \|\sigma_\lambda(f)\|_2^2|\lambda|d\lambda. \tag{9.17}$$

(3)
$$f(h) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{tr}(\sigma_\lambda(h^{-1})\sigma_\lambda(f))|\lambda|d\lambda. \tag{9.18}$$

Here $\|T\| = \mathrm{tr}(T^*T)^{1/2}$ is the Hilbert-Schmidt norm.

The result for $\{\sigma_\lambda, \mathcal{H}\}$ follows from the corresponding result for $\{\pi_\lambda, L^2(\mathbb{R})\}$ by unitary equivalence. This Fourier transform has some very interesting properties. Not only does it map solutions to solutions, we also have the useful fact that the eigenfunctions of the Fourier transform form an orthonormal basis for $\mathcal{H}$.

The following lemma is easy to prove, involving nothing more than writing out the definition of the Fourier transform and evaluating the integrals.

Lemma 9.4. *For each $f \in \mathcal{S}(H_3)$, $\sigma_\lambda(f)$ is a mapping from $\mathcal{H}$ into itself. Thus $L^1(H_3)$ is a Banach algebra of symmetries of $iu_t = u_{xx}$. Further we may write*

$$\sigma_\lambda(f)u(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}^3(a,b,c-\frac{1}{2}ab)e^{-ibx+ib^2t}u(x-\lambda a-2bt,t)dadb. \tag{9.19}$$

Here $\widehat{f}^3$ denotes the classical Fourier transform of f in the third variable. We take the classical Fourier transform to be
$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{iyx}dx$. If $u = A\phi$ then we also have

$$\sigma_\lambda(f)u(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}^{1,3}(\lambda\frac{(\xi+b)}{2},\xi-b,\lambda)\phi(\xi)e^{i(\xi+b)^2t-i(\xi+b)x}\frac{d\xi}{2\pi}db. \tag{9.20}$$

Here $\widehat{f}^{1,3}$ denotes the classical Fourier transform in the first and third variables.

Example 9.1. Let us take $f(a,b,c) = e^{-\frac{1}{2}(a^2+b^2+c^2)}$ and the solution $u = 1$. The integration is straightforward and we have

$$\begin{aligned}\sigma_\lambda(f)u(x,t) &= \int_{H_3} e^{-\frac{1}{2}(a^2+b^2+c^2)} e^{-ibx+ib^2+i\lambda(c+\frac{1}{2}ab)} dadbdc \\ &= \frac{(2\pi)^{3/2} e^{-\frac{1}{2}\lambda^2}}{\sqrt{1+\frac{\lambda^2}{4}-2it}} \exp\left\{\frac{-x^2}{2(1+\frac{\lambda^2}{4}-2it)}\right\}. \end{aligned} \tag{9.21}$$

From the solution $u = 1$ we have obtained a family of solutions of the complex heat equation, indexed by a parameter λ. We have thus a new notion of symmetry, which may potentially be fruitful in the analysis of solutions of PDEs. It is possible to construct Fourier transforms on solution spaces of other PDEs, but that is a subject for later work.

10. Conclusion

Lie symmetry group methods provide one of the most powerful tools available for the construction of exact solutions for PDEs. They can be used to derive complex solutions from trivial solutions and construct the fundamental solution of a PDE from a trivial (stationary) solution. They have applications in a diverse range of fields, from physics to mathematical finance. The theory of Lie group symmetries is a highly active field of research, with new results appearing regularly. It is possible that the deep connections between symmetries, transform theory and representation theory, can be exploited to give new insights into the problem of obtaining exact solutions of PDEs.

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