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Local Analysis for a Mutual Inhibition in Presence of Two Viruses in a Chemostat

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Abstract: A competition with mutual inhibition is a form of direct competition between the populations of two species where each actively inhibits the other. In this paper, we consider a mathematical system of ordinary differential equations describing two species, with mutual inhibition, competing for a limiting substrate in the presence of two viruses. A detailed local qualitative analysis of the restriction of the system to the attractor set is carried out. We prove that for general nonlinear response functions, the Competitive Exclusion Principle is still fulfilled so that at most one species can survive. Initial species concentrations are important in determining which is the winning species. The results obtained were validated by numerical simulations using Matlab software.

Keywords: *chemostat; competition; reversible inhibition; virus; local analysis; competitive exclusion principle.*

Mathematics Subject Classification (2010): 34D20, 37C75, 65L07, 65L20, 92B05, 92B10, 93B18, 93D20.

1 Introduction

A chemostat is a laboratory device (bioreactor) in which organisms grow on the available nutrient in a controlled manner. In many applications, it is simply a vessel used as a wastewater treatment process [18]. In ecology, it refers to an artificial lake for the continuous culture of bacteria which allows us to analyse inter-specific interactions between bacteria. A large number of mathematical studies have been published [18]. The most used mathematical system modelling the bacterial competition for a single obligate limiting substrate predicts competitive exclusion [12], that is, at least one competitor bacteria loses the competition [18]. Hsu et al. [15] in 1977, were among the first to study the problem of competition in a chemostat. They considered n populations in competition for the same nutrient and showed that competitive exclusion was verified, namely, the competitor which is better at using the substrate in small quantities survives and the others are extinguished. In the case of nonmonotonic growth functions, Butler and Wolkowicz [2] in 1985, also verified the competitive exclusion principle. In 1992, Wolkowicz and Lu [19] used Lyapunov functions to also verify the competitive exclusion principle in the case of general shape-growth functions, but with different mortality rates. For each species, the competitive exclusion principle was further checked (the resulting equilibrium being globally stable). Li [16] recently extended this result to

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an even wider class of growth functions. In 1994, Smith and Waltman [17] verified this principle for the Droop model. Wolkowicz and Xia [20] and Wolkowicz et al. [21] studied competition in a chemostat with the recycling of dead organisms for different types of delays (discrete, distributed). This theoretical result (Competitive Exclusion Principle) was confirmed experimentally by Hansen and Hubbell [11].

In many cases, the competing bacteria can produce a plethora of secondary metabolites to increase their competitiveness against other bacteria. For example, the production of *Nisin* by a number of strains of *Lactococcus lactis*, which exert a high antibacterial activity against Gram-positive bacteria, has been widely studied [13,14]. This inter-specific interaction is classified as an inhibition relationship. Viruses are the most abundant and diverse form of life on the Earth. They can infect all types of organisms (*Vertebrates, Invertebrates, Plants, Fungi, Bacteria, Archaea*). Viruses that infect bacteria are called *bacteriophages* or *phages*.

In this work, we extend the chemostat model [18] to general growth rates taking into account the reversible inhibition between species as in [3,4,6], but in the presence of two viruses. As our study is qualitative, we assume that the two species are feeding on a nonreproducing limiting substrate that is essential for both species. We also assume that the chemostat is well-mixed so that environmental conditions are homogeneous. We neglect the natural mortality of the species and the viruses, compared to the removal rate D. We prove that with general nonlinear response functions, the mutual inhibitory relationship between two competing species confirms the competitive exclusion principle (CEP). We have shown that at least one of the species becomes extinct and that initial species concentrations are important in determining which is the winning species.

The rest of the paper is structured as follows. In Section 2, we propose a mathematical model for this association and we recall some useful results of the chemostat theory. In Section 3, we restrict the model to four dimensions since the conservation of the total biomass is fulfilled. In Sections 4, 5 and 6, three cases are considered, where the main results of the local stability are presented. Finally, in Section 7, some numerical examples are presented to illustrate the obtained results confirming the competitive exclusion principle.

2 Mathematical Model and Properties

The proposed normalised mathematical model is given by

$$\begin{cases} \dot{s} &= Ds^{in} - f_1(s,x_2)\,x_1 - f_2(s,x_1)\,x_2 - Ds, \\ \dot{x}_1 &= f_1(s,x_2)x_1 - \alpha_1\,x_1 v_1 - Dx_1, \\ \dot{x}_2 &= f_2(s,x_1)x_2 - \alpha_2\,x_2 v_2 - Dx_2\ , \\ \dot{v}_1 &= \alpha_1\,x_1 v_1 - Dv_1, \\ \dot{v}_2 &= \alpha_2\,x_2 v_2 - Dv_2, \end{cases} \tag{1}$$

where $s^{in} > 0$ is the input concentration of substrate into the chemostat, $D > 0$ is the dilution rate. $\alpha_i > 0$ is the rate of infection, $s(t)$ is the concentration of substrate in the chemostat at time t. $x_i(t)$ is the i^{th} species concentration in the chemostat at time t, $v_i(t)$ is the i^{th} virus concentration in the chemostat at time t, $f_i(s,x_j)$ is the species growth rate depending on substrate and the concentration of the other species. The functions $f_i : \mathbb{R}_+^2 \to \mathbb{R}_+,\ i = 1,2$, are of class $\mathcal{C}^1$, and satisfy

A1 $f_1(0,x_2) = f_2(0,x_1) = 0, \quad \forall\, x_1, x_2 \in \mathbb{R}_+$.

A2 $\dfrac{\partial f_1}{\partial s}(s,x_2) > 0, \quad \forall\,(s,x_2) \in \mathbb{R}_+^2 \quad \dfrac{\partial f_2}{\partial s}(s,x_1) > 0, \qquad \forall\,(s,x_1) \in \mathbb{R}_+^2$.

A3 $\dfrac{\partial f_1}{\partial x_2}(s,x_2) < -\alpha_1 < 0, \quad \forall\,(s,x_2) \in \mathbb{R}_+^2, \quad \dfrac{\partial f_2}{\partial x_1}(s,x_1) < -\alpha_2 < 0, \qquad \forall\,(s,x_1) \in \mathbb{R}_+^2$.

Hypothesis **A1** states that the substrate is essential for the bacteria growth; hypothesis **A2** states that the growth rate increases with substrate. Hypothesis **A3** states that species inhibit each other and that each species is more sensitive to the other species than to the virus.

The system (1) plus **A1-A3** is not a realistic model for the biological system under consideration. To be more realistic, we should introduce two other variables describing intermediate proteins. Each protein produced by species x_i inhibits the growth of species j, where $i, j = 1, 2$ and $i \neq j$. In this case, the model will be huge ($\mathbb{R}^7$) and then difficult to study.

El Hajji [3] considered two species feeding on limiting substrate in a chemostat assuming a mutual inhibitory relationship between both species. The proposed model is the same as the one we have proposed here, but with $\alpha_1 = \alpha_2 = 0$ (no viruses associated with both species). The author proved that at most one species can survive, which confirms the competitive exclusion principle. The author also proved that, in the case where there are two locally stable equilibrium points, the initial concentrations of species are of great importance in determining which species is the winner.

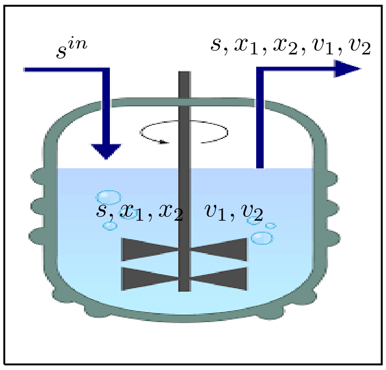

Figure 1: A simple chemostat shematic [3]: a continuous stirring mechanism at equal inflow and outflow rates (D), where two species (x_1, x_2) are competing for a limiting substrate (s) in the presence of two viruses (v_1, v_2), with an input concentration of substrate (s^{in}) and an output concentration of substrate (s), species concentrations (x_1, x_2) and virus concentrations (v_1, v_2).

Proposition 2.1 *1. Let the initial condition $\Big(s(0), x_1(0), x_2(0), v_1(0), v_2(0)\Big) \in \mathbb{R}^5_+$, the solution of model (1) admit positive bounded components and then be definite for all $t \geq 0$.*

2. $\Omega = \Big\{(s, x_1, x_2, v_1, v_2) \in \mathbb{R}^5_+ \ / s + x_1 + x_2 + v_1 + v_2 = s^{in}\Big\}$ is an invariant attractor set of all solutions of model (1).

Proof. The solutions' positivity can be proved as follows. If $s = 0$, then $\dot{s} = Ds^{in} > 0$, and if $x_i = 0$, then $\dot{x}_i = 0$ for $i = 1, 2$. If $v_i = 0$, then $\dot{v}_i = 0$ for $i = 1, 2$.

Next we prove the boundedness of solutions of model (1). Let $B(t) = s(t) + x_1(t) + x_2(t) + v_1(t) + v_2(t) - s^{in}$, then one obtains a single equation given by

$$\dot{B}(t) = \dot{s}(t) + \dot{x}_1(t) + \dot{x}_2(t) + \dot{v}_1(t) + \dot{v}_2(t) = D\Big(s^{in} - s(t) - x_1(t) - x_2(t) - v_1(t) - v_2(t)\Big) = -DB(t),$$

then $B(t) = B(0)e^{-Dt}$, which means that

$$s(t) + x_1(t) + x_2(t) + v_1(t) + v_2(t) = s^{in} + (s(0) + x_1(0) + x_2(0) + v_1(0) + v_2(0) - s^{in})e^{-Dt}. \tag{2}$$

Since s, x_1, x_2, v_1 and v_2 are positive, the solution of model (1) is bounded.
The invariance of the attractor Ω is a consequence of equation (2).

3 Restriction of System (1) to the Invariant Attractor Set Ω

The solutions of model (1) converge exponentially into Ω. Since we are studying the asymptotic behavior of (1), it is sufficient to restrict the study of model (1) to Ω. The projection of the restriction of model (1) to Ω on the plane (x_1, x_2, v_1, v_2) is given as follows:

$$\begin{cases} \dot{x}_1 &= f_1\big(s^{in} - (x_1 + x_2 + v_1 + v_2), x_2\big)x_1 - \alpha_1 x_1 v_1 - Dx_1, \\ \dot{x}_2 &= f_2\big(s^{in} - (x_1 + x_2 + v_1 + v_2), x_1\big)x_2 - \alpha_2 x_2 v_2 - Dx_2, \\ \dot{v}_1 &= \alpha_1 x_1 v_1 - Dv_1, \\ \dot{v}_2 &= \alpha_2 x_2 v_2 - Dv_2, \end{cases} \tag{3}$$

where the state vector (x_1, x_2, v_1, v_2) inside the sub-set is defined by

$$\mathcal{S} = \left\{(x_1, x_2, v_1, v_2) \in \mathbb{R}_+^4 :\ x_1 + x_2 + v_1 + v_2 \leq s^{in}\right\}.$$

In this section, the equilibria of system (3) are determined and their local stability properties are established. Define the parameters $\bar{x}_1$, $\bar{x}_2$, $\bar{v}_1$, $\bar{v}_2$, $\bar{\bar{x}}_1$, $\bar{\bar{x}}_2$, $\bar{\bar{\bar{x}}}_1$, $\bar{\bar{\bar{x}}}_2$, $\bar{\bar{\bar{v}}}_1$, $\bar{\bar{\bar{v}}}_2$, as follows:

- $\bar{x}_1$ is the solution of the equation $f_1(s^{in} - \bar{x}_1, 0) = D$.
- $\bar{x}_2$ is the solution of the equation $f_2(s^{in} - \bar{x}_2, 0) = D$.
- $\bar{v}_1$ is the solution of the equation $f_1(s^{in} - \dfrac{D}{\alpha_1} - \bar{v}_1, 0) = D + \alpha_1 \bar{v}_1$.
- $\bar{v}_2$ is the solution of the equation $f_2(s^{in} - \dfrac{D}{\alpha_2} - \bar{v}_2, 0) = D + \alpha_2 \bar{v}_2$.
- $(\bar{\bar{x}}_1, \bar{\bar{x}}_2)$ is the solution of the equations $f_1(s^{in} - \bar{\bar{x}}_1 - \bar{\bar{x}}_2, \bar{\bar{x}}_2) = f_2(s^{in} - \bar{\bar{x}}_1 - \bar{\bar{x}}_2, \bar{\bar{x}}_1) = D$.
- $(\bar{\bar{\bar{x}}}_1, \bar{\bar{\bar{v}}}_2)$ is the solution of the equations $f_1(s^{in} - \bar{\bar{\bar{x}}}_1 - \dfrac{D}{\alpha_2} - \bar{\bar{\bar{v}}}_2, \dfrac{D}{\alpha_2}) = D$ and $f_2(s^{in} - \bar{\bar{\bar{x}}}_1 - \dfrac{D}{\alpha_2} - \bar{\bar{\bar{v}}}_2, \bar{\bar{\bar{x}}}_1) - \alpha_2 \bar{\bar{\bar{v}}}_2 = D$.
- $(\bar{\bar{\bar{x}}}_2, \bar{\bar{\bar{v}}}_1)$ is the solution of the equations $f_1(s^{in} - \dfrac{D}{\alpha_1} - \bar{\bar{\bar{x}}}_2 - \bar{\bar{\bar{v}}}_1, \bar{\bar{\bar{x}}}_2) - \alpha_1 \bar{\bar{\bar{v}}}_1 = D$ and $f_2(s^{in} - \dfrac{D}{\alpha_1} - \bar{\bar{\bar{x}}}_2 - \bar{\bar{\bar{v}}}_1, \dfrac{D}{\alpha_1}) = D$.

Then the system (3) admits $F_0 = (0,0,0,0), F_1 = (\bar{x}_1, 0, 0, 0)$, $F_2 = (0, \bar{x}_2, 0, 0), F_3 = (\dfrac{D}{\alpha_1}, 0, \bar{v}_1, 0)$, $F_4 = (0, \dfrac{D}{\alpha_2}, 0, \bar{v}_2)$, $F_5 = (\bar{\bar{x}}_1, \bar{\bar{x}}_2, 0, 0), F_6 = (\bar{\bar{\bar{x}}}_1, \dfrac{D}{\alpha_2}, 0, \bar{\bar{\bar{v}}}_2)$ and $F_7 = (\dfrac{D}{\alpha_1}, \bar{\bar{\bar{x}}}_2, \bar{\bar{\bar{v}}}_1, 0)$ as equilibrium points.

Let $D_1 = f_1(s^{in}, 0)$, $D_2 = f_2(s^{in}, 0), D_3 = f_1(s^{in} - \dfrac{D}{\alpha_1}, 0), D_4 = f_2(s^{in} - \dfrac{D}{\alpha_2}, 0), D_5 = f_1(s^{in} - \dfrac{D}{\alpha_2} - \bar{v}_2, \dfrac{D}{\alpha_2}), D_6 = f_2(s^{in} - \dfrac{D}{\alpha_1} - \bar{v}_1, \dfrac{D}{\alpha_1}), D_7 = f_1(s^{in} - \bar{x}_2, \bar{x}_2), D_8 = f_2(s^{in} - \bar{x}_1, \bar{x}_1)$, $D_9 = f_1(s^{in} - \bar{v}_1 - \dfrac{D}{\alpha_1}, \bar{v}_1)$ and $D_{10} = f_2(s^{in} - \bar{v}_2 - \dfrac{D}{\alpha_2}, \bar{v}_2)$. Note that $D_9 < D_3 < D_1, D_5 < D_1, D_7 < D_1$, $D_{10} < D_4 < D_2$, $D_6 < D_2$ and $D_8 < D_2$.

In the rest of the paper, for simplicity and without any loss of generality, we will assume that $\alpha_1 > \alpha_2$, then $\dfrac{D}{\alpha_1} < \dfrac{D}{\alpha_2}$ and we will consider only three situations, where $s^{in} < \dfrac{D}{\alpha_1}$, $\dfrac{D}{\alpha_1} < s^{in} < \dfrac{D}{\alpha_2}$ and $\dfrac{D}{\alpha_2} < s^{in} < \dfrac{D}{\alpha_1} + \dfrac{D}{\alpha_2}$.

4 First Case : $s^{in} < \dfrac{D}{\alpha_1}$

The system (3) admits F_0, F_1, F_2 and F_5 as equilibria with $\bar{x}_1, \bar{x}_2, \bar{\bar{x}}_1, \bar{\bar{x}}_2 < \dfrac{D}{\alpha_1} < \dfrac{D}{\alpha_2}$. The conditions of existence of the equilibria are given in the lemmas hereafter.

Lemma 4.1 *The trivial equilibrium point F_0 exists always. If $D < \max(D_1, D_2)$, then F_0 is a saddle point, however, if $D > \max(D_1, D_2)$, then F_0 is a stable node.*

Proof. The Jacobian matrix J_0 of system (3) on F_0 is then given by

$$J_0 = \begin{bmatrix} D_1 - D & 0 & 0 & 0 \\ 0 & D_2 - D & 0 & 0 \\ 0 & 0 & -D & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}.$$

Its eigenvalues are given by $\lambda_1 = \lambda_2 = -D < 0, \lambda_3 = D_1 - D$ and $\lambda_4 = D_2 - D$. Therefore, if $D < \max(D_1, D_2)$, then F_0 is a saddle point, and if $D > \max(D_1, D_2)$, then F_0 is a stable node.

Lemma 4.2 *The equilibrium point F_1 exists if and only if $D < D_1$. If $D > D_8$, then F_1 is a stable node, however, if $D < D_8$, then F_1 is a saddle point.*

Proof. An equilibrium F_1 exists if and only if $\bar{x}_1 \in]0, s^{in}[$ is a solution of

$$f_1(s^{in} - \bar{x}_1, 0) = D. \tag{4}$$

Let $\psi_1(x_1) = f_1(s^{in} - x_1, 0) - D$. Since $\psi_1'(x_1) = -\frac{\partial f_1}{\partial s}(s^{in} - x_1, 0) < 0$, $\psi_1(0) = D_1 - D$ and $\psi_1(s^{in}) = -D < 0$, equation (4) admits a unique positive solution $\bar{x}_1 \in]0, s^{in}[$ if and only if $D < D_1$.

Assume that F_1 exists ($D < D_1$). The Jacobian matrix J_1 of model (3) at F_1 is given by

$$J_1 = \begin{bmatrix} -\bar{x}_1\frac{\partial f_1}{\partial s} & \bar{x}_1\frac{\partial f_1}{\partial x_2} - \bar{x}_1\frac{\partial f_1}{\partial s} & -\alpha_1\,\bar{x}_1 - \bar{x}_1\frac{\partial f_1}{\partial s} & -\bar{x}_1\frac{\partial f_1}{\partial s} \\ 0 & D_8 - D & 0 & 0 \\ 0 & 0 & \alpha_1\bar{x}_1 - D & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}.$$

J_1 admits four eigenvalues given by $\lambda_1 = -\bar{x}_1\frac{\partial f_1}{\partial s}(s^{in} - \bar{x}_1, 0) < 0, \lambda_2 = -(D - D_8), \lambda_3 = \alpha_1(\bar{x}_1 - \frac{D}{\alpha_1}) < 0$ and $\lambda_4 = -D < 0$. It follows that if $D > D_8$, then F_1 is a stable node, and if $D < D_8$, then F_1 is a saddle point.

Lemma 4.3 *The equilibrium point F_2 exists if and only if $D < D_2$. If $D > D_7$, then F_2 is a stable node, and if $D < D_7$, then F_2 is a saddle point.*

Proof. An equilibrium F_2 exists if and only if $\bar{x}_2 \in]0, s^{in}[$ is a solution of

$$f_2(s^{in} - \bar{x}_2, 0) = D. \tag{5}$$

Let $\psi_2(x_2) = f_2(s^{in} - x_2, 0) - D$. Since $\psi_2'(x_2) = -\frac{\partial f_2}{\partial s}(s^{in} - \bar{x}_2, 0) < 0$, $\psi_2(0) = D_2 - D$ and $\psi_2(s^{in}) = -D < 0$, equation (5) admits a unique positive solution $\bar{x}_2 \in]0, s^{in}[$ if and only if $D < D_2$.

Assume that F_2 exists ($D < D_2$). The Jacobian matrix J_2 of system (3) at F_2 is given by

$$J_2 = \begin{bmatrix} D_7 - D & 0 & 0 & 0 \\ x_2\frac{\partial f_2}{\partial x_1} - x_2\frac{\partial f_2}{\partial s} & -\bar{x}_2\frac{\partial f_2}{\partial s} & -\bar{x}_2\frac{\partial f_2}{\partial s} & -\alpha_2\,\bar{x}_2 - \bar{x}_2\frac{\partial f_2}{\partial s} \\ 0 & 0 & -D & 0 \\ 0 & 0 & 0 & \alpha_2\bar{x}_2 - D \end{bmatrix}.$$

J_2 admits four eigenvalues given by $\lambda_1 = -\bar{x}_2\frac{\partial f_2}{\partial s}(s^{in} - \bar{x}_2, 0) < 0, \lambda_2 = -(D - D_7), \lambda_3 = \alpha_2(\bar{x}_2 - \frac{D}{\alpha_2}) < 0$ and $\lambda_4 = -D < 0$. It follows that if $D > D_7$, then F_2 is a stable node, however, if $D < D_7$, then F_2 is a saddle point.

Lemma 4.4 *The situation $D < \min(D_7, D_8)$ is impossible.*

Proof. Assume that $0 < D < \min(D_7, D_8)$. From Lemmas 4.2 and 4.3, F_1 and F_2 exist.

1. If $\bar{x}_1 \geq \bar{x}_2$, then $D = f_2(s^{in} - \bar{x}_2, 0) \geq f_2(s^{in} - \bar{x}_1, 0) > f_2(s^{in} - \bar{x}_1, \bar{x}_1) = D_8 > D$, which is impossible.
2. If $\bar{x}_1 \leq \bar{x}_2$, then $D = f_1(s^{in} - \bar{x}_1, 0) \geq f_1(s^{in} - \bar{x}_2, 0) > f_1(s^{in} - \bar{x}_2, \bar{x}_2) = D_7 > D$, which is impossible.

Lemma 4.5 *An equilibrium F_5 exists if and only if $\max(D_7, D_8) < D < \min(D_1, D_2)$. If it exists, then F_1 and F_2 exist and satisfy $\bar{\bar{x}}_1 < \bar{x}_1$ and $\bar{\bar{x}}_2 < \bar{x}_2$. F_5 is always a saddle point.*

Proof. Since the functions $x_2 \to f_1(s^{in} - x_1 - x_2, x_2)$ and $x_2 \to f_2(s^{in} - x_1 - x_2, x_1)$ are noncreasing, one deduces that the isoclines are the graphs of two functions $x_2 = \varphi_1(x_1)$ and $x_2 = \varphi_2(x_1)$ and then $0 = \varphi_1(\bar{x}_1)$ and $\bar{x}_2 = \varphi_2(0)$. $\bar{\bar{x}}_1$ is a solution of $\psi_5(\bar{\bar{x}}_1) = 0$, where $\psi_5(x_1) = \varphi_2(x_1) - \varphi_1(x_1)$. The derivatives of φ_1 and φ_2 are given by $\varphi_2'(x_1) = -1 + \frac{\partial f_2}{\partial x_1}/\frac{\partial f_2}{\partial s} < -1 < \varphi_1'(x_1) = -1 + \frac{\partial f_1}{\partial x_2}/(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_1}{\partial s}) < 0$. One deduces that $\psi_5'(x_1) = \varphi_2'(x_1) - \varphi_1'(x_1) < 0$. $\psi_5(0) = \varphi_2(0) - \varphi_1(0) = \bar{x}_2 - \varphi_1(0)$ and $\psi_5(\bar{x}_1) = \varphi_2(\bar{x}_1)$, then $\bar{\bar{x}}_1$ exists and is unique if and only if $\bar{x}_2 > \varphi_1(0)$ and $\varphi_2(\bar{x}_1) < 0$, and these are satisfied only if $D = f_1(s^{in} - \varphi_1(0), \varphi_1(0)) > f_1(s^{in} - \bar{x}_2, \bar{x}_2) = D_7$ and $D = f_2(s^{in} - \bar{x}_1 - \varphi_2(\bar{x}_1), \bar{x}_1) > f_2(s^{in} - \bar{x}_1, \bar{x}_1) = D_8$. The existence and the uniqueness of $\bar{\bar{x}}_2 = \varphi_1(\bar{\bar{x}}_1) = \varphi_2(\bar{\bar{x}}_1)$ are easily deduced since the two functions $\varphi_1(.)$ and $\varphi_2(.)$ are decreasing.

Assume that F_5 exists. One has

$$\psi_3(\bar{\bar{x}}_1) = 0 = f_1(s^{in} - \bar{\bar{x}}_1, 0) - D > f_1(s^{in} - \bar{\bar{x}}_1 - \bar{\bar{x}}_2, \bar{\bar{x}}_2) - D = 0 = \psi_3(\bar{x}_1),$$

then $\psi_3(\bar{\bar{x}}_1) > \psi_3(\bar{x}_1)$ since the function $\psi_3(.)$ is decreasing, $\bar{x}_1 > \bar{\bar{x}}_1$.

$$\psi_4(\bar{\bar{x}}_2) = f_2(s^{in} - \bar{\bar{x}}_2, 0) - D > f_2(s^{in} - \bar{\bar{x}}_1 - \bar{\bar{x}}_2, \bar{\bar{x}}_1) - D = 0 = \psi_4(\bar{x}_2),$$

then $\psi_4(\bar{x}_2) < \psi_4(\bar{\bar{x}}_2)$ since the function $\psi_4(.)$ is decreasing, $\bar{x}_2 > \bar{\bar{x}}_2$.

Assume that F_5 exists. The Jacobian matrix J_5 of system (3) at $F_5 = (\bar{\bar{x}}_1, \bar{\bar{x}}_2, 0, 0)$ is given by

$$J_5 = \begin{bmatrix} -\bar{\bar{x}}_1 \frac{\partial f_1}{\partial s} & \bar{\bar{x}}_1 \frac{\partial f_1}{\partial x_2} - \bar{\bar{x}}_1 \frac{\partial f_1}{\partial s} & -\alpha_1 \bar{\bar{x}}_1 - \bar{\bar{x}}_1 \frac{\partial f_1}{\partial s} & -\bar{\bar{x}}_1 \frac{\partial f_1}{\partial s} \\ \bar{\bar{x}}_2 \frac{\partial f_2}{\partial x_1} - \bar{\bar{x}}_2 \frac{\partial f_2}{\partial s} & -\bar{\bar{x}}_2 \frac{\partial f_2}{\partial s} & -\bar{\bar{x}}_2 \frac{\partial f_2}{\partial s} & -\alpha_2 \bar{\bar{x}}_2 - \bar{\bar{x}}_2 \frac{\partial f_2}{\partial s} \\ 0 & 0 & \alpha_1 \bar{\bar{x}}_1 - D & 0 \\ 0 & 0 & 0 & \alpha_2 \bar{\bar{x}}_2 - D \end{bmatrix}.$$

J_5 admits four eigenvalues given by $\lambda_1 = \alpha_1(\bar{\bar{x}}_1 - \frac{D}{\alpha_1}) < 0$, $\lambda_2 = \alpha_2(\bar{\bar{x}}_2 - \frac{D}{\alpha_2}) < 0$ and two other eigenvalues of the solutions of

$$\lambda^2 + a\lambda + b = 0,$$

where

$$a = \bar{\bar{x}}_1 \frac{\partial f_1}{\partial s} + \bar{\bar{x}}_2 \frac{\partial f_2}{\partial s} > 0$$

and

$$b = \bar{\bar{x}}_1 \bar{\bar{x}}_2 \Big[- \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} + \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial s} + \frac{\partial f_1}{\partial s} \frac{\partial f_2}{\partial x_1} \Big] < 0.$$

It follows that F_5 is a saddle point.

The number and the nature of equilibria of system (3) are summarized in the theorem below.

Theorem 4.1

A) If $\min(D_7, D_8) < D < \max(D_7, D_8)$, *then*

- *(i) if* $D_8 < D_7$ *and* $D_8 < D < \min(D_2, D_7)$, *then system (3) admits three equilibria* F_0, F_1 *and* F_2. F_1 *is a stable node, however,* F_0 *and* F_2 *are two saddle points.*
- *(ii) if* $D_8 < D_7$ *and* $D_2 < D < D_7$, *then system (3) admits two equilibria* F_0 *and* F_1. F_1 *is a stable node and* F_0 *is a saddle point.*
- *(iii) if* $D_7 < D_8$ *and* $D_7 < D < \min(D_8, D_1)$, *then system (3) admits three equilibria* F_0, F_1 *and* F_2. F_2 *is a stable node, however,* F_0 *and* F_1 *are two saddle points.*
- *(iv) if* $D_7 < D_8$ *and* $D_1 < D < D_8$, *then system (3) admits two equilibria* F_0 *and* F_2. F_2 *is a stable node, however,* F_0 *is a saddle point.*

B) If $\max(D_7, D_8) < D < \min(D_1, D_2)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_5*.* F_1 *and* F_2 *are two stable nodes, however,* F_0 *and* F_5 *are two saddle points.*

C) If $\min(D_1, D_2) < D < \max(D_1, D_2)$*, then*

(i) if $D_1 < D_2$*, then system (3) admits two equilibria* F_0 *and* F_2*.* F_2 *is a stable node, however,* F_0 *is a saddle point.*

(ii) if $D_2 < D_1$*, then system (3) admits two equilibria* F_0 *and* F_1*.* F_1 *is a stable node, however,* F_0 *is a saddle point.*

D) If $\max(D_1, D_2) < D$*, then system (3) admits one stationary point* F_0*.* F_0 *is a stable node.*

5 Second Case : $\frac{D}{\alpha_1} < s^{in} < \frac{D}{\alpha_2}$

The system (3) admits F_0, F_1, F_2, F_3, F_5 and F_7 as equilibrium points with $\bar{x}_1, \bar{x}_2, \bar{\bar{x}}_1, \bar{\bar{x}}_2, \bar{v}_1 < \frac{D}{\alpha_2}$. The conditions of existence of the equilibria are stated in the lemmas hereafter.

Lemma 5.1 *F_0 exists always. If $D < \max(D_1, D_2)$, then F_0 is a saddle point. If $D > \max(D_1, D_2)$, then F_0 is a stable node.*

Proof. See the proof of Lemma 4.1.

Lemma 5.2 *The equilibrium point F_2 exists if and only if $D < D_2$. If $D > D_7$, then F_2 is a stable node, however, if $D < D_7$, then F_2 is a saddle point.*

Proof. See the proof of Lemma 4.3.

Lemma 5.3 *The situation $D < \min(D_7, D_8)$ is impossible.*

Proof. See the proof of Lemma 4.4.

Lemma 5.4 *An equilibrium F_5 exists if and only if $\max(D_7, D_8) < D < \min(D_1, D_2)$. If it exists, then F_1 and F_2 exist and satisfy $\bar{\bar{x}}_1 < \bar{x}_1$ and $\bar{\bar{x}}_2 < \bar{x}_2$. F_5 is always a saddle point.*

Proof. See the proof of Lemma 4.5.

Lemma 5.5 *F_1 exists if and only if $D < D_1$. If $D > \max(D_3, D_8)$, then F_1 is a stable node, however, if $D < D_3$ or $D_3 < D < D_8$, then F_1 is a saddle point.*

Proof. The proof of existence and uniqueness of F_1 is given in the proof of Lemma 4.2. Assume that F_1 exists ($D < D_1$). One has

- If $D < D_3$, then $f_1(s^{in} - \bar{x}_1, 0) = D < D_3 = f_1(s^{in} - \frac{D}{\alpha_1}, 0)$ and then $\bar{x}_1 > \frac{D}{\alpha_1}$.
- If $D > D_3$, then $f_1(s^{in} - \bar{x}_1, 0) = D > D_3 = f_1(s^{in} - \frac{D}{\alpha_1}, 0)$ and then $\bar{x}_1 < \frac{D}{\alpha_1}$.

The Jacobian matrix J_1 of system (3) at F_1 is given by

$$J_1 = \begin{bmatrix} -\bar{x}_1 \frac{\partial f_1}{\partial s} & \bar{x}_1 \frac{\partial f_1}{\partial x_2} - \bar{x}_1 \frac{\partial f_1}{\partial s} & -\alpha_1 \bar{x}_1 - \bar{x}_1 \frac{\partial f_1}{\partial s} & -\bar{x}_1 \frac{\partial f_1}{\partial s} \\ \\ 0 & D_8 - D & 0 & 0 \\ 0 & 0 & \alpha_1 \bar{x}_1 - D & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}.$$

J_1 admits four eigenvalues given by $\lambda_1 = -\bar{x}_1 \frac{\partial f_1}{\partial s}(s^{in} - \bar{x}_1, 0) < 0, \lambda_2 = -(D - D_8), \lambda_3 = \alpha_1(\bar{x}_1 - \frac{D}{\alpha_1})$ and $\lambda_4 = -D < 0$. It follows that

- F_1 is a saddle point if $D < D_3$.
- F_1 is a stable node if $D > D_3$ and $D > D_8$.
- F_1 is a saddle point if $D > D_3$ and $D < D_8$.

Lemma 5.6 *F_3 exists if and only if $D < D_3$. If $D_6 < D < D_3$, then F_3 is locally asymptotically stable. If $D < \min(D_3, D_6)$, then F_3 is unstable.*

Proof. An equilibrium F_3 exists if and only if $\bar{v}_1 \in]0, s^{in} - \frac{D}{\alpha_1}[$ is a solution of

$$f_1(s^{in} - \frac{D}{\alpha_1} - \bar{v}_1, 0) = D + \alpha_1 \bar{v}_1. \tag{6}$$

Let $\psi_3(v_1) = f_1(s^{in} - \frac{D}{\alpha_1} - v_1, 0) - D - \alpha_1 v_1$. Since $\psi_3'(v_1) = -\frac{\partial f_1}{\partial s}(s^{in} - \frac{D}{\alpha_1} - v_1, 0) - \alpha_1 < 0$, $\psi_3(0) = D_3 - D$ and $\psi_3(s^{in} - \frac{D}{\alpha_1}) = -D - \alpha_1(s^{in} - \frac{D}{\alpha_1}) < 0$, equation (6) admits a unique positive solution $\bar{v}_1 \in]0, s^{in} - \frac{D}{\alpha_1}[$ if and only if $D < D_3$.

If F_3 exists, the Jacobian matrix J_1 of model (3) at F_3 is stated as follows:

$$J_3 = \begin{bmatrix} -\frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & \frac{D}{\alpha_1}\frac{\partial f_1}{\partial x_2} - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & -D - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & -\frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} \\ 0 & D_6 - D & 0 & 0 \\ \alpha_1 \bar{v}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}.$$

J_3 admits four eigenvalues given by $\lambda_1 = -D < 0$ and $\lambda_2 = -(D - D_6)$ and two other eigenvalues of the solution of the equation

$$\lambda^2 + a\lambda + b = 0,$$

where $a = \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s}(s^{in} - \frac{D}{\alpha_1} - \bar{v}_1, 0) > 0$ and $b = \alpha_1 \bar{v}_1 \Big(D + \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s}(s^{in} - \frac{D}{\alpha_1} - \bar{v}_1, 0)\Big) > 0$. It follows that

- If $D_6 < D < D_3$, then $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0, \lambda_4 < 0$ and F_3 is then locally asymptotically stable.
- If $D < \min(D_3, D_6)$, then F_3 is a saddle point.

Lemma 5.7 *An equilibrium F_7 exists if and only if $\max(D_6, D_9) < D < D_3$. If F_7 exists, it follows that $\bar{\bar{\bar{v}}}_1 < \bar{v}_1$ and F_7 is always unstable.*

Proof. Since the functions $x_2 \to f_1(s^{in} - x_2 - \frac{D}{\alpha_1} - v_1, x_2) - \alpha_1 v_1$ and $x_2 \to f_2(s^{in} - x_2 - \frac{D}{\alpha_1} - v_1, \frac{D}{\alpha_1})$ are decreasing, one deduces that the isoclines are the graphs of two functions $x_2 = \varphi_5(v_1)$ and $x_2 = \varphi_6(v_1)$. $\bar{\bar{\bar{v}}}_1$ is a solution of $\psi_7(\bar{\bar{\bar{v}}}_1) = 0$, where $\psi_7(v_1) = \varphi_6(v_1) - \varphi_5(v_1)$. The derivatives of φ_5 and φ_6 are given by $\varphi_6'(v_1) = -1 < \varphi_5'(v_1) = -1 + \left(\frac{\partial f_1}{\partial x_2} + \alpha_1\right) / \left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_1}{\partial s}\right) < 0$. One deduces that $\psi_7'(v_1) = \varphi_6'(v_1) - \varphi_5'(v_1) < 0$. $\psi_7(0) = \varphi_6(0) - \varphi_5(0)$ and $\psi_7(\bar{v}_1) = \varphi_6(\bar{v}_1) - \varphi_5(\bar{v}_1)$, then $\bar{\bar{\bar{v}}}_1$ exists and is unique if and only if $\varphi_5(0) < \varphi_6(0)$ and $\varphi_6(\bar{v}_1) < \varphi_5(\bar{v}_1)$. Note that $\varphi_5(\bar{v}_1) = 0$ and $\varphi_6(0) < \bar{v}_1$. Then the existence is satisfied only if $D = f_1(s^{in} - \varphi_5(0) - \frac{D}{\alpha_1}, \varphi_5(0)) > f_1(s^{in} - \bar{v}_1 - \frac{D}{\alpha_1}, \bar{v}_1) = D_9$ and $D = f_2(s^{in} - \varphi_6(\bar{v}_1) - \bar{v}_1 - \frac{D}{\alpha_1}, \frac{D}{\alpha_1}) > f_2(s^{in} - \bar{v}_1 - \frac{D}{\alpha_1}, \frac{D}{\alpha_1}) = D_6$.

Assume that F_7 exists. One has

$$\psi_4(\bar{\bar{\bar{x}}}_2) = f_2(s^{in} - \bar{\bar{\bar{x}}}_2, 0) - D \geq f_2(s^{in} - \bar{\bar{\bar{x}}}_2 - \frac{D}{\alpha_1} - \bar{\bar{\bar{v}}}_1, \frac{D}{\alpha_1}) - D = 0 = \psi_4(\bar{x}_2),$$

then $\psi_4(\bar{x}_2) < \psi_4(\bar{\bar{\bar{x}}}_2)$ since the function $\psi_4(.)$ is decreasing, $\bar{x}_2 > \bar{\bar{\bar{x}}}_2$. The Jacobian matrix J_7 of system (3) at $F_7 = (\frac{D}{\alpha_1}, \bar{\bar{\bar{x}}}_2, \bar{\bar{\bar{v}}}_1, 0)$ is given by

$$J_7 = \begin{bmatrix} -\frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & \frac{D}{\alpha_1}\frac{\partial f_1}{\partial x_2} - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & -D - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & -\frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} \\ \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial x_1} - \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} & -\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} & -\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} & -\alpha_2\,\bar{\bar{\bar{x}}}_2 - \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} \\ \alpha_1\bar{\bar{\bar{v}}}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2\bar{\bar{\bar{x}}}_2 - D \end{bmatrix}.$$

J_7 admits four eigenvalues given by $\lambda_1 = \alpha_2\bar{\bar{\bar{x}}}_2 - D$ and three other eigenvalues of the roots of the following characteristic polynomial:

$$P_7(X) = \begin{vmatrix} -X - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & \frac{D}{\alpha_1}\frac{\partial f_1}{\partial x_2} - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & -D - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} \\ \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial x_1} - \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} & -X - \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} & -\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} \\ \alpha_1\bar{\bar{\bar{v}}}_1 & 0 & -X \end{vmatrix},$$

$$P_7(X) = -X\begin{vmatrix} -X - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & \frac{D}{\alpha_1}\frac{\partial f_1}{\partial x_2} - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} \\ \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial x_1} - \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} & -X - \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} \end{vmatrix} + \alpha_1\bar{\bar{\bar{v}}}_1\begin{vmatrix} \frac{D}{\alpha_1}\frac{\partial f_1}{\partial x_2} - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} & -D - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} \\ -X - \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} & -\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} \end{vmatrix},$$

$$\begin{aligned} P_7(X) = -X\Big| &(X + \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s})(X + \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}) - (\frac{D}{\alpha_1}\frac{\partial f_1}{\partial x_2} - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s})(\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial x_1} - \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}) \Big| \\ &+\alpha_1\bar{\bar{\bar{v}}}_1\Big| -\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}(\frac{D}{\alpha_1}\frac{\partial f_1}{\partial x_2} - \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s}) - (D + \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s})(X + \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}) \Big|, \end{aligned}$$

$$\begin{aligned} P_7(X) = -X\Big| &X^2 + X(\frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} + \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}) - \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} + \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial s} + \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_1}{\partial s}\frac{\partial f_2}{\partial x_1} \Big| \\ &-\alpha_1\bar{\bar{\bar{v}}}_1\Big| \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}\frac{\partial f_1}{\partial x_2} + (D + \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s})X + D\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s} \Big|, \end{aligned}$$

$$\begin{aligned} P_7(X) \;=\; & -X^3 - X^2\Big(\frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} + \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}\Big) \\ & -X\Big(- \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} + \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial s} + \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_1}{\partial s}\frac{\partial f_2}{\partial x_1} + \alpha_1\bar{\bar{\bar{v}}}_1(D + \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s})\Big) \\ & -\alpha_1\bar{\bar{\bar{v}}}_1\left(\frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}\frac{\partial f_1}{\partial x_2} + D\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}\right). \end{aligned}$$

Then

$$P_7(X) \;= -(X^3 + b_1X^2 + b_2X + b_3)$$

with

$$\begin{aligned} b_1 \;&= \Big(\frac{D}{\alpha_1}\frac{\partial f_1}{\partial s} + \bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}\Big) > 0, \\ b_2 \;&= \Big(- \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} + \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial s} + \frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_1}{\partial s}\frac{\partial f_2}{\partial x_1} + \alpha_1\bar{\bar{\bar{v}}}_1(D + \frac{D}{\alpha_1}\frac{\partial f_1}{\partial s})\Big), \\ b_3 \;&= \alpha_1\bar{\bar{\bar{v}}}_1\left(\frac{D}{\alpha_1}\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}\frac{\partial f_1}{\partial x_2} + D\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}\right) = D\bar{\bar{\bar{v}}}_1\bar{\bar{\bar{x}}}_2\frac{\partial f_2}{\partial s}\left(\frac{\partial f_1}{\partial x_2} + \alpha_1\right) < 0. \end{aligned}$$

So, the conditions for the stability of F_7 are not satisfied, then F_7 is unstable.

The number and the nature of equilibria of model (3) are given in the theorem hereafter.

Theorem 5.1 *A) If* $\min(D_7, D_8) < D < \max(D_7, D_8)$*, then*

(i) if $D_8 < D_7$*, then*

1. *if* $\max(D_2, D_3) < D < D_7$*, then system (3) admits two equilibria* F_0 *and* F_1*.* F_1 *is a stable node, however,* F_0 *is a saddle point.*
2. *if* $\max(D_2, D_9) < D < \min(D_3, D_7)$*, then system (3) admits four equilibria* F_0, F_1, F_3 *and* F_7*.* F_0, F_1 *and* F_7 *are three saddle points, however,* F_3 *is a stable node.*
3. *if* $D_2 < D < \min(D_9, D_7)$*, then system (3) admits three equilibria* F_0, F_1 *and* F_3*.* F_0 *and* F_1 *are two saddle points, however,* F_3 *is a stable node.*
4. *if* $\max(D_3, D_6, D_8) < D < \min(D_2, D_7)$*, then system (3) admits three equilibria* F_0, F_1 *and* F_2*.* F_0 *and* F_2 *are two saddle points, however,* F_1 *is a stable node.*
5. *if* $\max(D_6, D_8, D_9) < D < \min(D_2, D_3, D_7)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_7*.* F_0, F_1, F_2 *and* F_7 *are four saddle points, however,* F_3 *is a stable node.*
6. *if* $\max(D_6, D_8) < D < \min(D_2, D_9, D_7)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*.* F_0, F_1 *and* F_2 *are three saddle points, however,* F_3 *is a stable node.*
7. *if* $\max(D_3, D_8) < D < \min(D_6, D_7)$*, then system (3) admits three equilibria* F_0, F_1 *and* F_2*.* F_0 *and* F_2 *are two saddle points, however,* F_1 *is a stable node.*
8. *if* $D_8 < D < \min(D_3, D_6, D_7)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*, all of them are saddle points.*

(ii) if $D_7 < D_8$*, then*

1. *if* $D_7 < D < \min(D_9, D_6, D_8)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*.* F_0, F_1 *and* F_3 *are three saddle points, however,* F_2 *is a stable node.*
2. *if* $\max(D_9, D_7) < D < \min(D_3, D_6, D_8)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*.* F_0, F_1 *and* F_3 *are three saddle points, however,* F_2 *is a stable node.*
3. *if* $\max(D_3, D_7) < D < \min(D_1, D_6, D_8)$*, then system (3) admits three equilibria* F_0, F_1 *and* F_2*.* F_0 *and* F_1 *are two saddle points, however,* F_2 *is a stable node.*
4. *if* $D_1 < D < \min(D_6, D_8)$*, then system (3) admits two equilibria* F_0 *and* F_2*.* F_0 *is a saddle point, however,* F_2 *is a stable node.*
5. *if* $\max(D_6, D_7) < D < \min(D_8, D_9)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*.* F_0 *and* F_1 *are two saddle points, however,* F_2 *and* F_3 *are two stable nodes.*
6. *if* $\max(D_6, D_7, D_9) < D < \min(D_3, D_8)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_7*.* F_0, F_1 *and* F_7 *are three saddle points, however,* F_2 *and* F_3 *are two stable nodes.*
7. *if* $\max(D_6, D_7, D_3) < D < \min(D_1, D_8)$*, then system (3) admits three equilibria* F_0, F_1 *and* F_2*.* F_0 *and* F_1 *are two saddle points, however,* F_2 *is a stable node.*
8. *if* $\max(D_6, D_1) < D < D_8$*, then system (3) admits two equilibria* F_0 *and* F_2*.* F_0 *is a saddle point, however,* F_2 *is a stable node.*

B) If $\max(D_7, D_8) < D < \min(D_1, D_2)$*, then*

(i) If $\max(D_3, D_6, D_7, D_8) < D < \min(D_1, D_2)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_5*.* F_0 *and* F_5 *are saddle points,* F_1 *and* F_2 *are stable nodes.*

(ii) If $\max(D_6, D_7, D_8, D_9) < D < \min(D_2, D_3)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_5 *and* F_7*.* F_0, F_1, F_5 *and* F_7 *are saddle points,* F_2 *and* F_3 *are stable nodes.*

(iii) If $\max(D_3, D_7, D_8) < D < \min(D_1, D_6)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_5*.* F_0 *and* F_5 *are saddle points,* F_1 *and* F_2 *are stable nodes.*

(iv) If $\max(D_7, D_8, D_9) < D < \min(D_3, D_6)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_5*.* F_0, F_1, F_3 *and* F_5 *are saddle points,* F_2 *is a stable node.*

(v) *If* $\max(D_7, D_8) < D < \min(D_6, D_9)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_5*.* F_0, F_1, F_3 *and* F_5 *are saddle points,* F_2 *is a stable node.*

(vi) *If* $\max(D_6, D_7, D_8) < D < \min(D_2, D_9)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_5*.* F_0, F_1 *and* F_5 *are saddle points,* F_2 *and* F_3 *are stable nodes.*

C) *If* $\min(D_1, D_2) < D < \max(D_1, D_2)$*, then*

(i) *If* $D_1 < D < D_2$*, then system (3) admits two equilibria* F_0 *and* F_2*.* F_0 *is a saddle point, however,* F_2 *is a stable node.*

(ii) *If* $D_2 < D < D_1$*, then*

1. *if* $D_2 < D < D_9$*, then system (3) admits three equilibria* F_0, F_1 *and* F_3*.* F_0 *and* F_1 *are two saddle points, however,* F_3 *is a stable node.*
2. *if* $\max(D_2, D_9) < D < D_3$*, then system (3) admits four equilibria* F_0, F_1, F_3 *and* F_7*.* F_0, F_1 *and* F_7 *are three saddle points, however,* F_3 *is a stable node.*
3. *if* $\max(D_2, D_3) < D < D_1$*, then system (3) admits two equilibria* F_0 *and* F_1*.* F_0 *is a saddle point, however,* F_1 *is a stable node.*

D) *If* $\max(D_1, D_2) < D$*, then model (3) admits only* F_0 *as an equilibrium point.* F_0 *is a stable node.*

6 Third Case : $\dfrac{D}{\alpha_2} < s^{in} < \dfrac{D}{\alpha_1} + \dfrac{D}{\alpha_2}$

The system (3) admits $F_0, F_1, F_2, F_3, F_4, F_5, F_6$ and F_7 as equilibrium points with

$$\bar{v}_1 < \min(s^{in} - \frac{D}{\alpha_1}, \frac{D}{\alpha_2}) \text{ and } \bar{v}_2 < \min(s^{in} - \frac{D}{\alpha_2}, \frac{D}{\alpha_1}).$$

The conditions of existence of the equilibria are stated in the lemmas hereafter.

Lemma 6.1 F_0 *exists always. If* $D < \max(D_1, D_2)$*, then* F_0 *is a saddle point, however, if* $D > \max(D_1, D_2)$*, then* F_0 *is a stable node.*

Proof. See the proof of Lemma 4.1.

Lemma 6.2 *The equilibrium point* F_1 *exists if and only if* $D < D_1$*. If* $D > \max(D_3, D_8)$*, then* F_1 *is a stable node, however, if* $D < D_3$ *or* $D_3 < D < D_8$*, then* F_1 *is a saddle point.*

Proof. See the proof of Lemma 5.5.

Lemma 6.3 *The equilibrium point* F_3 *exists if and only if* $D < D_3$*. If* $D_6 < D < D_3$*, then* F_3 *is locally asymptotically stable. If* $D < \min(D_3, D_6)$*, then* F_3 *is unstable.*

Proof. See the proof of Lemma 5.6.

Lemma 6.4 *The situation* $D < \min(D_7, D_8)$ *is impossible.*

Proof. See the proof of Lemma 4.4.

Lemma 6.5 *An equilibrium* F_5 *exists if and only if* $\max(D_7, D_8) < D < \min(D_1, D_2)$*. If it exists, then* F_1 *and* F_2 *exist and satisfy* $\bar{\bar{x}}_1 < \bar{x}_1$ *and* $\bar{\bar{x}}_2 < \bar{x}_2$*.* F_5 *is always a saddle point.*

Proof. See the proof of Lemma 4.5.

Lemma 6.6 *An equilibrium* F_7 *exists if and only if* $\max(D_6, D_9) < D < D_3$*. Therefore,* $\bar{\bar{\bar{v}}}_1 < \bar{v}_1$ *and* F_7 *is always unstable.*

Proof. See the proof of Lemma 5.7.

Lemma 6.7 *The equilibrium point* F_2 *exists if and only if* $D < D_2$*. If* $D > \max(D_4, D_7)$*, then* F_2 *is a stable node, however, if* $D < D_4$ *or* $D_4 < D < D_7$*, then* F_2 *is a saddle point.*

Proof. Existence and uniqueness of F_2 are given in the proof of Lemma 4.3. Assume that F_2 exists ($D < D_2$). One has

- If $D < D_4$, then $f_2(s^{in} - \bar{x}_2, 0) = D < D_4 = f_2(s^{in} - \frac{D}{\alpha_2}, 0)$ and then $\bar{x}_2 > \frac{D}{\alpha_2}$.
- If $D > D_4$, then $f_2(s^{in} - \bar{x}_2, 0) = D > D_4 = f_2(s^{in} - \frac{D}{\alpha_2}, 0)$ and then $\bar{x}_2 < \frac{D}{\alpha_2}$.

The Jacobian matrix J_2 of model (3) at F_2 is given as follows:

$$J_2 = \begin{bmatrix} D_7 - D & 0 & 0 & 0 \\ x_2\frac{\partial f_2}{\partial x_1} - x_2\frac{\partial f_2}{\partial s} & -\bar{x}_2\frac{\partial f_2}{\partial s} & -\bar{x}_2\frac{\partial f_2}{\partial s} & -\alpha_2\,\bar{x}_2 - \bar{x}_2\frac{\partial f_2}{\partial s} \\ 0 & 0 & -D & 0 \\ 0 & 0 & 0 & \alpha_2\bar{x}_2 - D \end{bmatrix}.$$

J_2 admits four eigenvalues given by $\lambda_1 = -\bar{x}_2\frac{\partial f_2}{\partial s}(s^{in} - \bar{x}_2, 0) < 0, \lambda_2 = -(D - D_7), \lambda_3 = \alpha_2(\bar{x}_2 - \frac{D}{\alpha_2})$ and $\lambda_4 = -D < 0$. It follows that

- If $D < D_4$, then F_2 is a saddle point.
- If $D > D_4$ and $D > D_7$, then F_2 is a stable node.
- If $D > D_4$ and $D < D_7$, then F_2 is a saddle point.

Lemma 6.8 *F_4 exists if and only if $D < D_4$. If $D_5 < D < D_4$, then F_4 is locally asymptotically stable. If $D < \min(D_4, D_5)$, then F_4 is unstable (saddle point).*

Proof. An equilibrium F_4 exists if and only if $\bar{v}_2 \in]0, s^{in} - \frac{D}{\alpha_2}[$ is a solution of

$$f_2(s^{in} - \frac{D}{\alpha_2} - \bar{v}_2, 0) = D + \alpha_2\bar{v}_2. \tag{7}$$

Let $\psi_4(v_2) = f_2(s^{in} - \frac{D}{\alpha_2} - v_2, 0) - D - \alpha_2 v_2$. Since $\psi_4'(v_2) = -\frac{\partial f_2}{\partial s}(s^{in} - \frac{D}{\alpha_2} - v_2, 0) - \alpha_2 < 0$, $\psi_4(0) = D_4 - D$, $\psi_4(s^{in} - \frac{D}{\alpha_2}) = -D - \alpha_2(s^{in} - \frac{D}{\alpha_2}) < 0$, equation (7) admits a unique positive solution $\bar{v}_2 \in]0, s^{in} - \frac{D}{\alpha_2}[$ if and only if $D < D_4$.

If F_4 exists, the Jacobian matrix J_4 of system (3) at F_4 is given by

$$J_4 = \begin{bmatrix} D_5 - D & 0 & 0 & 0 \\ \frac{D}{\alpha_2}\frac{\partial f_2}{\partial x_1} - \frac{D}{\alpha_2}\frac{\partial f_2}{\partial s} & -\frac{D}{\alpha_2}\frac{\partial f_2}{\partial s} & -\frac{D}{\alpha_2}\frac{\partial f_2}{\partial s} & -D - \frac{D}{\alpha_2}\frac{\partial f_2}{\partial s} \\ 0 & 0 & -D & 0 \\ 0 & \alpha_2\bar{v}_2 & 0 & 0 \end{bmatrix}.$$

J_4 admits four eigenvalues given by $\lambda_1 = -D < 0$ and $\lambda_2 = -(D - D_5)$ and two other eigenvalues of the solution of the equation

$$\lambda^2 + a\lambda + b = 0,$$

where $a = \frac{D}{\alpha_2}\frac{\partial f_2}{\partial s}(s^{in} - \frac{D}{\alpha_2} - \bar{v}_2, 0) > 0$ and $b = \alpha_2\bar{v}_2\Big(D + \frac{D}{\alpha_2}\frac{\partial f_2}{\partial s}(s^{in} - \frac{D}{\alpha_2} - \bar{v}_2, 0)\Big) > 0$. It follows that

- If $D_5 < D < D_4$, then $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0, \lambda_4 < 0$ and F_4 is then locally asymptotically stable.
- If $D < \min(D_4, D_5)$, then F_4 is a saddle point.

Lemma 6.9 *An equilibrium F_6 exists if and only if* $\max(D_5, D_{10}) < D < D_4$. *Therefore,* $\bar{\bar{\bar{v}}}_2 < \bar{v}_2$ *and F_6 is always unstable.*

Proof. Since the functions $x_1 \to f_1(s^{in} - x_1 - \frac{D}{\alpha_2} - v_2, \frac{D}{\alpha_2})$ and $x_1 \to f_2(s^{in} - x_1 - \frac{D}{\alpha_2} - v_2, x_1) - \alpha_2 v_2$ are nonincreasing, one deduces that the isoclines are the graphs of two functions $x_1 = \varphi_3(v_2)$ and $x_1 = \varphi_4(v_2)$. $\bar{\bar{\bar{v}}}_2$ is the solution of $\psi_6(\bar{\bar{\bar{v}}}_2) = 0$, where $\psi_6(v_2) = \varphi_4(v_2) - \varphi_3(v_2)$. The derivatives of φ_3 and φ_4 are given by $\varphi_3'(v_2) = -1 < \varphi_4'(v_2) = -1 + \left(\frac{\partial f_2}{\partial x_1} + \alpha_2\right) / \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_2}{\partial s}\right) < 0$. One deduces that $\psi_6'(v_2) = \varphi_4'(v_2) - \varphi_3'(v_2) > 0$. $\psi_6(0) = \varphi_4(0) - \varphi_3(0)$ and $\psi_6(\bar{v}_2) = \varphi_4(\bar{v}_2) - \varphi_3(\bar{v}_2)$, then $\bar{\bar{\bar{v}}}_2$ exists and is unique if and only if $\varphi_4(0) < \varphi_3(0)$ and $\varphi_3(\bar{v}_2) < \varphi_4(\bar{v}_2)$. Note that $\varphi_4(\bar{v}_2) = 0$ and $\varphi_3(0) < \bar{v}_2$. The existence is satisfied only if

$$D = f_2(s^{in} - \varphi_4(0) - \frac{D}{\alpha_2}, \varphi_4(0)) > f_2(s^{in} - \bar{v}_2 - \frac{D}{\alpha_2}, \bar{v}_2) = D_{10}$$

and

$$D = f_1(s^{in} - \varphi_3(\bar{v}_2) - \bar{v}_2 - \frac{D}{\alpha_2}, \frac{D}{\alpha_2}) > f_1(s^{in} - \bar{v}_2 - \frac{D}{\alpha_2}, \frac{D}{\alpha_2}) = D_5.$$

Assume that F_6 exists. One has

$$\psi_3(\bar{\bar{\bar{x}}}_1) = f_1(s^{in} - \bar{\bar{\bar{x}}}_1, 0) - D \geq f_1(s^{in} - \bar{\bar{\bar{x}}}_1 - \frac{D}{\alpha_2} - \bar{\bar{\bar{v}}}_2, \frac{D}{\alpha_2}) - D = 0 = \psi_3(\bar{x}_1),$$

then $\psi_3(\bar{x}_1) < \psi_3(\bar{\bar{\bar{x}}}_1)$ since the function $\psi_3(.)$ is decreasing, $\bar{x}_1 > \bar{\bar{\bar{x}}}_1$. The Jacobian matrix J_6 of system (3) at $F_6 = (\bar{\bar{\bar{x}}}_1, \frac{D}{\alpha_2}, 0, \bar{\bar{\bar{v}}}_2)$ is given by

$$J_6 = \begin{bmatrix} -\bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} & \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial x_2} - \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} & -\alpha_1 \bar{\bar{\bar{x}}}_1 - \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} & -\bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} \\ \frac{D}{\alpha_2} \frac{\partial f_2}{\partial x_1} - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} & -\frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} & -\frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} & -D - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} \\ 0 & 0 & \alpha_1 \bar{\bar{\bar{x}}}_1 - D & 0 \\ 0 & \alpha_2 \bar{\bar{\bar{v}}}_2 & 0 & 0 \end{bmatrix}.$$

J_6 admits four eigenvalues given by $\lambda_1 = \alpha_1 \bar{\bar{\bar{x}}}_1 - D$ and three other eigenvalues of the roots of the following characteristic polynomial:

$$P_6(X) = \begin{vmatrix} -X - \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} & \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial x_2} - \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} & -\bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} \\ \frac{D}{\alpha_2} \frac{\partial f_2}{\partial x_1} - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} & -X - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} & -D - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} \\ 0 & \alpha_2 \bar{\bar{\bar{v}}}_2 & -X \end{vmatrix},$$

$$P_6(X) = -X \begin{vmatrix} -X - \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} & \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial x_2} - \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} \\ \frac{D}{\alpha_2} \frac{\partial f_2}{\partial x_1} - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} & -X - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} \end{vmatrix} - \alpha_2 \bar{\bar{\bar{v}}}_2 \begin{vmatrix} -X - \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} & -\bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} \\ \frac{D}{\alpha_2} \frac{\partial f_2}{\partial x_1} - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} & -D - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s} \end{vmatrix},$$

$$\begin{aligned} P_6(X) &= -X \left[\left(X + \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s}\right)\left(X + \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s}\right) - \left(\bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial x_2} - \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s}\right)\left(\frac{D}{\alpha_2} \frac{\partial f_2}{\partial x_1} - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s}\right)\right] \\ &\quad -\alpha_2 \bar{\bar{\bar{v}}}_2 \left[\left(X + \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s}\right)\left(D + \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s}\right) + \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s}\left(\frac{D}{\alpha_2} \frac{\partial f_2}{\partial x_1} - \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s}\right)\right] \\ &= -X \left[X^2 + X\left(\bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} + \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s}\right) - \bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2} \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} + \bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2} \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial s} + \bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2} \frac{\partial f_1}{\partial s} \frac{\partial f_2}{\partial x_1}\right] \\ &\quad -\alpha_2 \bar{\bar{\bar{v}}}_2 \left[X\left(D + \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s}\right) + D \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} + \bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2} \frac{\partial f_1}{\partial s} \frac{\partial f_2}{\partial x_1}\right] \\ &= -X^3 - X^2\left(\bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} + \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s}\right) \\ &\quad -X\left(-\bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2} \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} + \bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2} \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial s} + \bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2} \frac{\partial f_1}{\partial s} \frac{\partial f_2}{\partial x_1} + \alpha_2 \bar{\bar{\bar{v}}}_2 \left(D + \frac{D}{\alpha_2} \frac{\partial f_2}{\partial s}\right)\right) \\ &\quad -D \bar{\bar{\bar{v}}}_2 \bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} \left[\alpha_2 + \frac{\partial f_2}{\partial x_1}\right]. \end{aligned}$$

Then $P_6(X) = -(X^3 + b_1X^2 + b_2X + b_3)$ with

$$\begin{aligned}
b_1 &= \Big(\bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s} + \frac{D}{\alpha_2}\frac{\partial f_2}{\partial s}\Big) > 0,\\
b_2 &= \Big(-\bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2}\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} + \bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2}\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial s} + \bar{\bar{\bar{x}}}_1 \frac{D}{\alpha_2}\frac{\partial f_1}{\partial s}\frac{\partial f_2}{\partial x_1} + \alpha_2 \bar{\bar{\bar{v}}}_2\Big(D + \frac{D}{\alpha_2}\frac{\partial f_2}{\partial s}\Big)\Big),\\
b_3 &= D\bar{\bar{\bar{v}}}_2\bar{\bar{\bar{x}}}_1 \frac{\partial f_1}{\partial s}\left[\alpha_2 + \frac{\partial f_2}{\partial x_1}\right] < 0.
\end{aligned}$$

So, the conditions for the stability of F_6 are not satisfied, then F_6 is unstable.

The number and the nature of equilibrium points of model (3) are stated in the following theorem.

Theorem 6.1 *A) If* $\min(D_7, D_8) < D < \max(D_7, D_8)$, *then*

(i) if $D_8 < D_7$, *then*

1. *if* $\max(D_2, D_3) < D < D_7$, *then system (3) admits two equilibria* F_0 *and* F_1. F_0 *is a saddle point, however,* F_1 *is a stable node.*
2. *if* $\max(D_2, D_9) < D < \min(D_3, D_7)$, *then system (3) admits four equilibria* F_0, F_1, F_3 *and* F_7. F_0, F_1 *and* F_7 *are three saddle points, however,* F_3 *is a stable node.*
3. *if* $D_2 < D < \min(D_7, D_9)$, *then system (3) admits three equilibria* F_0, F_1 *and* F_3. F_0 *and* F_1 *are two saddle points, however,* F_3 *is a stable node.*
4. *if* $\max(D_3, D_4, D_6, D_8) < D < \min(D_2, D_7)$, *then system (3) admits three equilibria* F_0, F_1 *and* F_2. F_0 *and* F_2 *are two saddle points, however,* F_1 *is a stable node.*
5. *if* $\max(D_3, D_5, D_6, D_8, D_{10}) < D < \min(D_4, D_7)$, *then system (3) admits five equilibria* F_0, F_1, F_2, F_4 *and* F_6. F_0, F_2 *and* F_6 *are three saddle points, however,* F_1 *and* F_4 *are two stable nodes.*
6. *if* $\max(D_3, D_5, D_6, D_8) < D < \min(D_7, D_{10})$, *then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_4. F_0 *and* F_2 *are saddle points, however,* F_1 *and* F_4 *are two stable nodes.*
7. *if* $\max(D_3, D_6, D_8) < D < \min(D_4, D_5, D_7)$, *then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_4. F_0, F_2 *and* F_4 *are three saddle points, however,* F_1 *is a stable node.*
8. *if* $\max(D_4, D_6, D_8, D_9) < D < \min(D_2, D_3, D_7)$, *then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_7. F_0, F_1, F_2 *and* F_7 *are four saddle points, however,* F_3 *is a stable node.*
9. *if* $\max(D_5, D_6, D_8, D_9, D_{10}) < D < \min(D_3, D_4, D_7)$, *then system (3) admits seven equilibria* $F_0, F_1, F_2, F_3, F_4, F_6$ *and* F_7. F_0, F_1, F_2, F_6 *and* F_7 *are five saddle points, however,* F_3 *and* F_4 *are two stable nodes.*
10. *if* $\max(D_5, D_6, D_8, D_9) < D < \min(D_3, D_7, D_{10})$, *then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_7. F_0, F_1, F_2 *and* F_7 *are four saddle points, however,* F_3 *and* F_4 *are two stable nodes.*
11. *if* $\max(D_6, D_8, D_9) < D < \min(D_3, D_4, D_5, D_7)$, *then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_7. F_0, F_1, F_2, F_4 *and* F_7 *are five saddle points, however,* F_3 *is a stable node.*
12. *if* $\max(D_4, D_6, D_8) < D < \min(D_2, D_7, D_9)$, *then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3. F_0, F_1 *and* F_2 *are three saddle points, however,* F_3 *is a stable node.*
13. *if* $\max(D_5, D_6, D_8, D_{10}) < D < \min(D_4, D_7, D_9)$, *then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_6. F_0, F_1, F_2 *and* F_6 *are four saddle points, however,* F_3 *and* F_4 *are stable nodes.*
14. *if* $\max(D_5, D_6, D_8) < D < \min(D_7, D_9, D_{10})$, *then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4. F_0, F_1 *and* F_2 *are three saddle points, however,* F_3 *and* F_4 *are stable nodes.*

15. *if* $\max(D_6, D_8) < D < \min(D_4, D_5, D_7, D_9)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*.* F_0, F_1, F_2 *and* F_4 *are four saddle points, however,* F_3 *is a stable node.*
16. *if* $\max(D_3, D_4, D_8) < D < \min(D_6, D_7)$*, then system (3) admits three equilibria* F_0, F_1 *and* F_2*.* F_0 *and* F_2 *are two saddle points, however,* F_1 *is a stable node.*
17. *if* $\max(D_3, D_5, D_8, D_{10}) < D < \min(D_4, D_6, D_7)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_4 *and* F_6*.* F_0, F_1, F_2 *and* F_6 *are three saddle points, however,* F_1 *and* F_4 *are stable nodes.*
18. *if* $\max(D_3, D_5, D_8) < D < \min(D_6, D_7, D_{10})$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_4*.* F_0 *and* F_2 *are two saddle points, however,* F_1 *and* F_4 *are stable nodes.*
19. *if* $\max(D_3, D_8) < D < \min(D_4, D_5, D_6, D_7)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_4*.* F_0, F_2 *and* F_4 *are three saddle points, however,* F_1 *is a stable node.*
20. *if* $\max(D_4, D_8, D_9) < D < \min(D_3, D_6, D_7)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*, all of them are saddle points.*
21. *if* $\max(D_5, D_8, D_9, D_{10}) < D < \min(D_3, D_4, D_6, D_7)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_6*.* F_0, F_1, F_2, F_3 *and* F_6 *are five saddle points, however,* F_4 *is a stable node.*
22. *if* $\max(D_5, D_8, D_9) < D < \min(D_3, D_6, D_7, D_{10})$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*.* F_0, F_1, F_2 *and* F_3 *are four saddle points, however,* F_4 *is a stable node.*
23. *if* $\max(D_8, D_9) < D < \min(D_3, D_4, D_5, D_6, D_7)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*, all of them are saddle points.*
24. *if* $\max(D_4, D_8) < D < \min(D_6, D_7, D_9)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*, all of them are saddle points.*
25. *if* $\max(D_5, D_8, D_{10}) < D < \min(D_4, D_6, D_7, D_9)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_6*.* F_0, F_1, F_2, F_3 *and* F_6 *are five saddle points, however,* F_4 *is a stable node.*
26. *if* $\max(D_5, D_8) < D < \min(D_6, D_7, D_9, D_{10})$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*.* F_0, F_1, F_2 *and* F_3 *are four saddle points, however,* F_4 *is a stable node.*
27. *if* $D_8 < D < \min(D_4, D_5, D_6, D_7, D_9)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*, all of them are saddle points.*

(ii) if $D_7 < D_8$*, then*

1. *if* $\max(D_4, D_7) < D < \min(D_6, D_8, D_9)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*.* F_0, F_1 *and* F_3 *are three saddle points, however,* F_2 *is a stable node.*
2. *if* $\max, D_5, D_7, D_{10}) < D < \min(D_4, D_6, D_8, D_9)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_6*.* F_0, F_1, F_2, F_3 *and* F_6 *are five saddle points, however,* F_4 *is a stable node.*
3. *if* $\max(D_5, D_7) < D < \min(D_6, D_8, D_9, D_{10})$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*.* F_0, F_1, F_2 *and* F_3 *are four saddle points, however,* F_4 *is a stable node.*
4. *if* $D_7 < D < \min(D_4, D_5, D_6, D_8, D_9)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*, all of them are saddle points.*
5. *if* $\max(D_4, D_7, D_9) < D < \min(D_3, D_6, D_8)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*.* F_0, F_1 *and* F_3 *are three saddle points, however,* F_2 *is a stable node.*
6. *if* $\max(D_5, D_7, D_9, D_{10}) < D < \min(D_3, D_4, D_6, D_8)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_6*.* F_0, F_1, F_2, F_3 *and* F_6 *are five saddle points, however,* F_4 *is a stable node.*

7. *if* $\max(D_5, D_7, D_9) < D < \min(D_3, D_6, D_8, D_{10})$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*.* F_0, F_1, F_2 *and* F_3 *are four saddle points, however,* F_4 *is a stable node.*
8. *if* $\max(D_7, D_9) < D < \min(D_3, D_4, D_5, D_6, D_8)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*, all of them are saddle points.*
9. *if* $\max(D_3, D_4, D_7) < D < \min(D_1, D_6, D_8)$*, then system (3) admits three equilibria* F_0, F_1 *and* F_2*.* F_0 *and* F_1 *are two saddle points, however,* F_2 *is a stable node.*
10. *if* $\max(D_3, D_5, D_7, D_{10}) < D < \min(D_1, D_4, D_6, D_8)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_4 *and* F_6*.* F_0, F_1, F_2 *and* F_6 *are four saddle points, however,* F_4 *is a stable node.*
11. *if* $\max(D_3, D_5, D_7) < D < \min(D_1, D_6, D_8, D_{10})$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_4*.* F_0, F_1 *and* F_2 *are three saddle points, however,* F_4 *is a stable node.*
12. *if* $\max(D_3, D_7) < D < \min(D_4, D_5, D_6, D_8)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_4*, all of them are saddle points.*
13. *if* $\max(D_4, D_1) < D < \min(D_6, D_8)$*, then system (3) admits two equilibria* F_0 *and* F_2*.* F_0 *is a saddle point, however,* F_2 *is a stable node.*
14. *if* $\max(D_{10}, D_1) < D < \min(D_4, D_6, D_8)$*, then system (3) admits four equilibria* F_0, F_2, F_4 *and* F_6*.* F_0, F_2 *and* F_6 *are three saddle points, however,* F_4 *is a stable node.*
15. *if* $D_1 < D < \min(D_6, D_8, D_{10})$*, then system (3) admits three equilibria* F_0, F_2 *and* F_4*.* F_0 *and* F_2 *are two saddle points, however,* F_4 *is a stable node.*
16. *if* $\max(D_4, D_6, D_7) < D < \min(D_8, D_9)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_3*.* F_0 *and* F_1 *are two saddle points, however,* F_2 *and* F_3 *are two stable nodes.*
17. *if* $\max(D_5, D_6, D_7, D_{10}) < D < \min(D_4, D_8, D_9)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_6*.* F_0, F_1, F_2 *and* F_6 *are four saddle points, however,* F_3 *and* F_4 *are two stable nodes.*
18. *if* $\max(D_5, D_6, D_7) < D < \min(D_8, D_9, D_{10})$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*.* F_0, F_1 *and* F_2 *are three saddle points, however,* F_3 *and* F_4 *are stable nodes.*
19. *if* $\max(D_6, D_7) < D < \min(D_4, D_5, D_8, D_9)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_4*.* F_0, F_1, F_2 *and* F_4 *are four saddle points, however,* F_3 *is a stable node.*
20. *if* $\max(D_4, D_6, D_7, D_9) < D < \min(D_3, D_8)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_7*.* F_0, F_1 *and* F_7 *are three saddle points, however,* F_2 *and* F_3 *are two stable nodes.*
21. *if* $\max(D_5, D_6, D_7, D_9, D_{10}) < D < \min(D_3, D_4, D_8)$ *, then system (3) admits seven equilibria* $F_0, F_1, F_2, F_3, F_4, F_6$ *and* F_7*.* F_0, F_1, F_2, F_6 *and* F_7 *are five saddle points, however,* F_3 *and* F_4 *are two stable nodes.*
22. *if* $\max(D_5, D_6, D_7, D_9) < D < \min(D_3, D_8, D_{10})$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_7*.* F_0, F_1, F_2 *and* F_7 *are four saddle points, however,* F_3 *and* F_4 *are two stable nodes.*
23. *if* $\max(D_6, D_7, D_9) < D < \min(D_3, D_4, D_5, D_8)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_7*.* F_0, F_1, F_2, F_4 *and* F_7 *are five saddle points, however,* F_3 *is a stable node.*
24. *if* $\max(D_3, D_4, D_6, D_7) < D < \min(D_1, D_8)$*, then system (3) admits three equilibria* F_0, F_1 *and* F_2*.* F_0 *and* F_1 *are two saddle points, however,* F_2 *is a stable node.*
25. *if* $\max(D_3, D_5, D_6, D_7, D_{10}) < D < \min(D_1, D_4, D_8)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_4 *and* F_6*.* F_0, F_1, F_2 *and* F_6 *are four saddle points, however,* F_4 *is a stable node.*
26. *if* $\max(D_3, D_5, D_6, D_7) < D < \min(D_1, D_8, D_{10})$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_4*.* F_0, F_1 *and* F_2 *are three saddle points, however,* F_4 *is a stable node.*

27. *if* $\max(D_3, D_6, D_7) < D < \min(D_4, D_5, D_8)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_4*, all of them are saddle points.*
28. *if* $\max(D_6, D_4, D_1) < D < D_8$*, then system (3) admits two equilibria* F_0 *and* F_2*.* F_0 *is a saddle point, however,* F_2 *is a stable node.*
29. *if* $\max(D_{10}, D_6, D_1) < D < \min(D_4, D_8)$*, then system (3) admits four equilibria* F_0, F_2, F_4 *and* F_6*.* F_0, F_2 *and* F_6 *are three saddle points, however,* F_4 *is a stable node.*
30. *if* $\max(D_6, D_1) < D < \min(D_{10}, D_8)$*, then system (3) admits three equilibria* F_0, F_2 *and* F_4*.* F_0 *and* F_2 *are two saddle points, however,* F_4 *is a stable node.*

B) If $\max(D_7, D_8) < D < \min(D_1, D_2)$*, then*

1. *If* $\max(D_3, D_6, D_7, D_8, D_4) < D < \min(D_1, D_2)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_5*.* F_0 *and* F_5 *are saddle points,* F_1 *and* F_2 *are stable nodes.*
2. *If* $\max(D_3, D_5, D_6, D_7, D_8, D_{10}) < D < \min(D_1, D_4)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_4, F_5 *and* F_6*.* F_0, F_2, F_5 *and* F_6 *are saddle points,* F_1 *and* F_4 *are stable nodes.*
3. *If* $\max(D_3, D_5, D_6, D_7, D_8) < D < \min(D_1, D_{10})$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_4 *and* F_5*.* F_0, F_2 *and* F_5 *are saddle points,* F_1 *and* F_4 *are stable nodes.*
4. *If* $\max(D_3, D_6, D_7, D_8) < D < \min(D_4, D_5)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_4 *and* F_5*.* F_0, F_2, F_4 *and* F_5 *are saddle points,* F_1 *is a stable node.*
5. *If* $\max(D_4, D_6, D_7, D_8, D_9) < D < \min(D_2, D_3)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_5 *and* F_7*.* F_0, F_1, F_5 *and* F_7 *are saddle points,* F_2 *and* F_3 *are stable nodes.*
6. *If* $\max(D_5, D_6, D_7, D_8, D_9, D_{10}) < D < \min(D_3, D_4)$*, then system (3) admits eight equilibria* $F_0, F_1, F_2, F_3, F_4, F_5, F_6$ *and* F_7*.* F_0, F_1, F_2, F_5, F_6 *and* F_7 *are saddle points,* F_3 *and* F_4 *are stable nodes.*
7. *If* $\max(D_5, D_6, D_7, D_8, D_9) < D < \min(D_3, D_{10})$*, then system (3) admits seven equilibria* $F_0, F_1, F_2, F_3, F_4, F_5$ *and* F_7*.* F_0, F_1, F_2, F_5 *and* F_7 *are saddle points,* F_3 *and* F_4 *are stable nodes.*
8. *If* $\max(D_6, D_7, D_8, D_9) < D < \min(D_3, D_4, D_5)$*, then system (3) admits seven equilibria* $F_0, F_1, F_2, F_3, F_4, F_5$ *and* F_7*.* F_0, F_1, F_2, F_4, F_5 *and* F_7 *are saddle points,* F_3 *is a stable node.*
9. *If* $\max(D_3, D_4, D_7, D_8) < D < \min(D_1, D_6)$*, then system (3) admits four equilibria* F_0, F_1, F_2 *and* F_5*.* F_0 *and* F_5 *are saddle points,* F_1 *and* F_2 *are stable nodes.*
10. *If* $\max(D_3, D_5, D_7, D_8, D_{10}) < D < \min(D_1, D_4, D_6)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_4, F_5 *and* F_6*.* F_0, F_2, F_5 *and* F_6 *are saddle points,* F_1 *and* F_4 *are stable nodes.*
11. *If* $\max(D_3, D_5, D_7, D_8) < D < \min(D_1, D_6, D_{10})$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_4 *and* F_5*.* F_0, F_2 *and* F_5 *are saddle points,* F_1 *and* F_4 *are stable nodes.*
12. *If* $\max(D_3, D_7, D_8) < D < \min(D_4, D_5, D_6)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_4 *and* F_5*.* F_0, F_2, F_4 *and* F_5 *are saddle points,* F_1 *is a stable node.*
13. *If* $\max(D_4, D_7, D_8, D_9) < D < \min(D_3, D_6)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_5*.* F_0, F_1, F_3 *and* F_5 *are saddle points,* F_2 *is a stable node.*
14. *If* $\max(D_5, D_7, D_8, D_9, D_{10}) < D < \min(D_3, D_4, D_6)$*, then system (3) admits seven equilibria* $F_0, F_1, F_2, F_3, F_4, F_5$ *and* F_6*.* F_0, F_1, F_2, F_3, F_5 *and* F_6 *are saddle points,* F_4 *is a stable node.*
15. *If* $\max(D_5, D_7, D_8, D_9) < D < \min(D_3, D_6, D_{10})$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_5*.* F_0, F_1, F_2, F_3 *and* F_5 *are saddle points,* F_4 *is a stable node.*
16. *If* $\max(D_7, D_8, D_9) < D < \min(D_3, D_4, D_5, D_6)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_5*, all of them are saddle points.*

17. *If* $\max(D_4, D_7, D_8) < D < \min(D_6, D_9)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_5*.* F_0, F_1, F_3 *and* F_5 *are saddle points,* F_2 *is a stable node.*
18. *If* $\max(D_5, D_7, D_8, D_{10}) < D < \min(D_4, D_6, D_9)$*, then system (3) admits seven equilibria* $F_0, F_1, F_2, F_3, F_4, F_5$ *and* F_6*.* F_0, F_1, F_2, F_3, F_5 *and* F_6 *are saddle points,* F_4 *is a stable node.*
19. *If* $\max(D_5, D_7, D_8) < D < \min(D_6, D_9, D_{10})$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_5*.* F_0, F_1, F_2, F_3 *and* F_5 *are saddle points,* F_4 *is a stable node.*
20. *If* $\max(D_7, D_8) < D < \min(D_4, D_5, D_6, D_9)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_5*, all of them are saddle points.*
21. *If* $\max(D_4, D_6, D_7, D_8) < D < \min(D_2, D_9)$*, then system (3) admits five equilibria* F_0, F_1, F_2, F_3 *and* F_5*.* F_0, F_1 *and* F_5 *are saddle points,* F_2 *and* F_3 *are stable nodes.*
22. *If* $\max(D_5, D_6, D_7, D_8, D_{10}) < D < \min(D_4, D_9)$*, then system (3) admits seven equilibria* $F_0, F_1, F_2, F_3, F_4, F_5$ *and* F_6*.* F_0, F_1, F_2, F_5 *and* F_6 *are saddle points,* F_3 *and* F_4 *are stable nodes.*
23. *If* $\max(D_5, D_6, D_7, D_8) < D < \min(D_9, D_{10})$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_5*.* F_0, F_1, F_2 *and* F_5 *are saddle points,* F_3 *and* F_4 *are stable nodes.*
24. *If* $\max(D_6, D_7, D_8) < D < \min(D_4, D_5, D_9)$*, then system (3) admits six equilibria* F_0, F_1, F_2, F_3, F_4 *and* F_5*.* F_0, F_1, F_2, F_4 *and* F_5 *are saddle points,* F_3 *is a stable node.*

C) If $\min(D_1, D_2) < D < \max(D_1, D_2)$*, then*

(i) If $D_1 < D < D_2$*, then*

1. *if* $D_1 < D < D_{10}$*, then system (3) admits three equilibria* F_0, F_2 *and* F_4*.* F_0 *and* F_2 *are two saddle points, however,* F_4 *is a stable node.*
2. *if* $\max(D_1, D_{10}) < D < D_4$*, then system (3) admits four equilibria* F_0, F_2, F_4 *and* F_6*.* F_0, F_2 *and* F_6 *are three saddle points, however,* F_4 *is a stable node.*
3. *if* $\max(D_1, D_4) < D < D_2$*, then system (3) admits two equilibria* F_0 *and* F_2*.* F_0 *is a saddle point, however,* F_2 *is a stable node.*

(ii) If $D_2 < D < D_1$*, then*

1. *if* $D_2 < D < D_9$*, then system (3) admits three equilibria* F_0, F_1 *and* F_3*.* F_0 *and* F_1 *are two saddle points, however,* F_3 *is a stable node.*
2. *if* $\max(D_2, D_9) < D < D_3$*, then system (3) admits four equilibria* F_0, F_1, F_3 *and* F_7*.* F_0, F_1 *and* F_7 *are three saddle points, however,* F_3 *is a stable node.*
3. *if* $\max(D_2, D_3) < D < D_1$*, then system (3) admits two equilibria* F_0 *and* F_1*.* F_0 *is a saddle point, however,* F_1 *is a stable node.*

D) If $\max(D_1, D_2) < D$*, then model (3) admits only* F_0 *as an equilibrium point.* F_0 *is a stable node.*

7 Numerical Simulations

We validated the obtained results by some numerical simulations on a system that uses Monod growth rates and takes into account the reversible inhibition between species:

$$\begin{cases} \dot{s} = D\,(s^{in} - s) - \dfrac{4\,s\,x_1}{(1+s)(1+x_2)} - \dfrac{4\,s\,x_2}{(2+s)(1.5+x_1)}\,, \\ \dot{x}_1 = \left(\dfrac{4\,s}{(1+s)(1+x_2)} - D - 0.2v_1\right) x_1, \\ \dot{x}_2 = \left(\dfrac{4\,s}{(2+s)(1.5+x_1)} - D - 0.1v_2\right) x_2, \\ \dot{v}_1 = (0.2\,x_1 - D)v_1, \\ \dot{v}_2 = (0.1\,x_2 - D)v_2. \end{cases} \tag{8}$$

One can readily check that the functional responses satisfy Assumptions **A1** to **A3**.

7.1 First case

In Fig. 2, if the dilution rate $D = 4$ satisfying $D_2 = 2.42 < D_1 = 3.8 < D = 4$, each solution with the initial condition inside the whole domain converges to the equilibrium F_0, from where the extinction of the two species. However, in Fig. 3, for $D = 2.5$ satisfying $D_2 = 2.3 < D < D_1 = 3.7$,

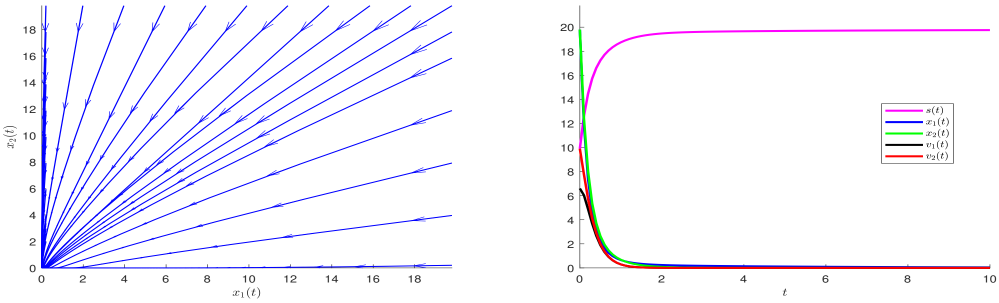

Figure 2: $x_1 - x_2$ behaviour for $D = 4, s^{in} = 19.8$.

each solution with the initial condition inside the whole domain is converging to the equilibrium F_1, from where only species 1 can survive.

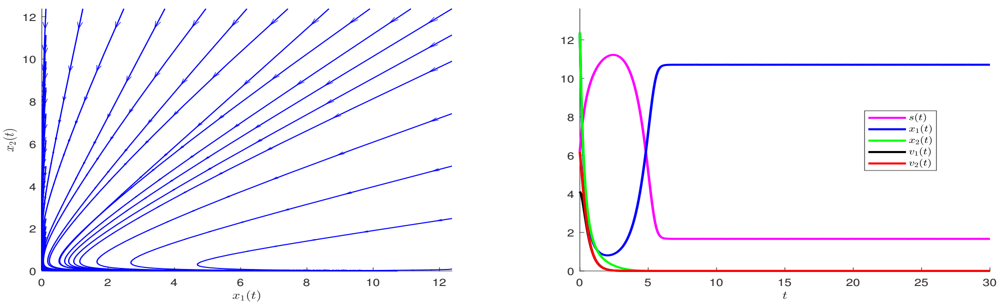

Figure 3: $x_1 - x_2$ behaviour for $D = 2.5, s^{in} = 12.38$.

In Fig. 4, for $D = 1.2$ satisfying then $D = 1.2 < D_2 = 2 < D_1 = 3.42$, each solution with the initial condition inside the red domain converges to the equilibrium F_2 and each solution with the initial condition inside the blue domain converges to the equilibrium F_1. The competitive exclusion principle is fulfilled here since at least one species goes extinct. As seen in Fig. 4, initial species concentrations are important in determining which is the winning species.

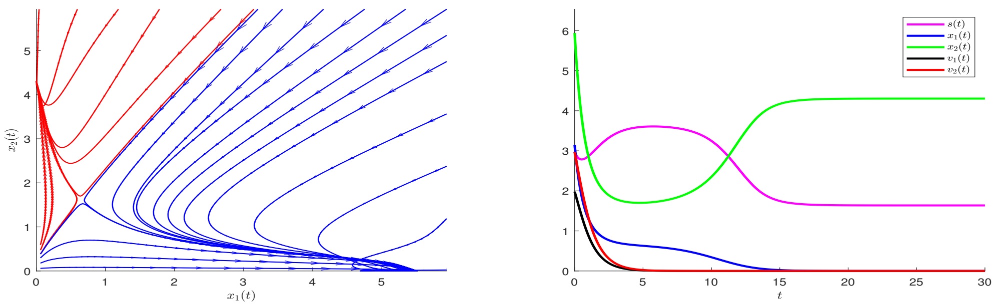

Figure 4: $x_1 - x_2$ behaviour for $D = 1.2, s^{in} = 5.94$.

7.2 Second case

In Fig. 5, if $D = 4$, which satisfies $D_2 = 2.5 < D_1 = 3.87 < D = 4$, each solution with the initial condition inside the whole domain is converging to the equilibrium F_0, from where the extinction of the two species.

However, in Fig. 6, if $D = 2.5$, which satisfies $D_9 = 0.16 < D_7 = 0.36 < D_2 = 2.41 < D < D_1 = 3.8$, each solution with the initial condition inside the whole domain is converging to the equilibrium F_1, from where only species 1 can survive.

In Fig. 7, if $D = 2$, which satisfies $D_8 = 0.09 < D_9 = 0.2 < D_7 = 0.34 < D = 2 < D_3 = 3.33 < D_2 = 2.35 < D_1 = 3.75$, each solution with the initial condition inside the red domain converges to the equilibrium F_2 and each solution with the initial condition inside the blue domain converges to the equilibrium F_3. The competitive exclusion principle is fulfilled here since at least one species goes extinct.

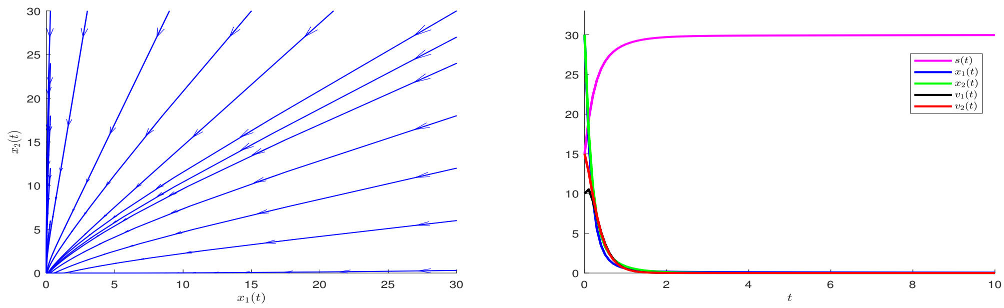

Figure 5: $x_1 - x_2$ behaviour for $D = 4, s^{in} = 30$.

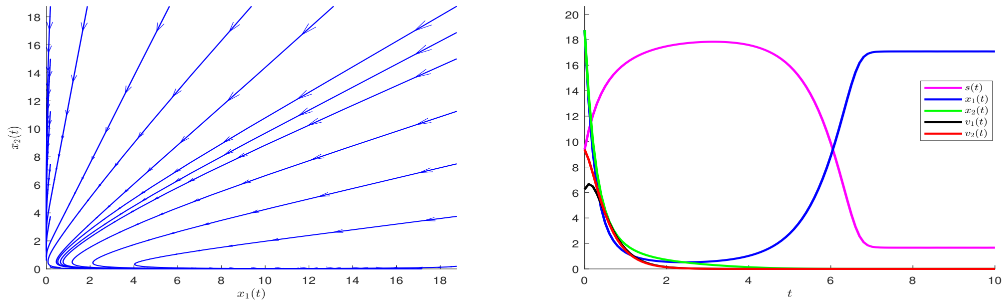

Figure 6: $x_1 - x_2$ behaviour for $D = 2.5, s^{in} = 18.75$.

7.3 Third case

In Fig. 8, if $D = 4$, which satisfies $D_2 = 2.55 < D_1 = 3.91 < D = 4$, each solution with the initial condition inside the whole domain is converging to the equilibrium F_0, from where the extinction of the two species.

However, in Fig. 9, if $D = 2.5$, which satisfies $D_2 = 2.49 < D < D_1 = 3.86$, each solution with the initial condition inside the whole domain is converging to the equilibrium F_3, from where only species 1 can survive.

In Fig. 10, if $D = 1.2$, which satisfies $D_6 = 0.23 < D = 1.2 < D_2 = 2.32 < D_3 = 3.53 < D_1 = 3.72$, each solution with the initial condition inside the red domain converges to the equilibrium F_2 and each solution with the initial condition inside the blue domain converges to the equilibrium F_3. The competitive exclusion principle is fulfilled here since at least one species goes extinct.

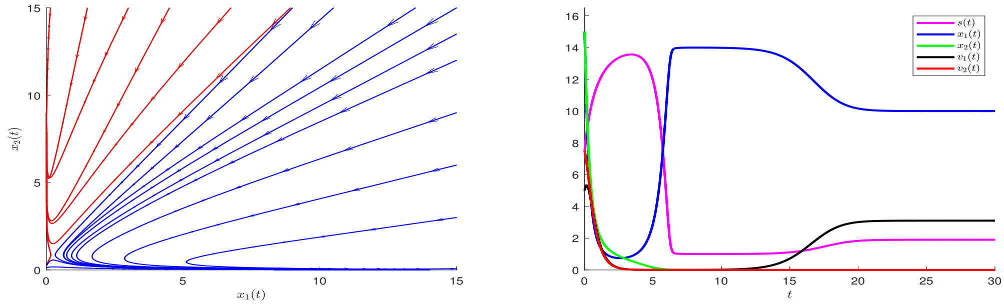

Figure 7: $x_1 - x_2$ behaviour for $D = 2, s^{in} = 15$.

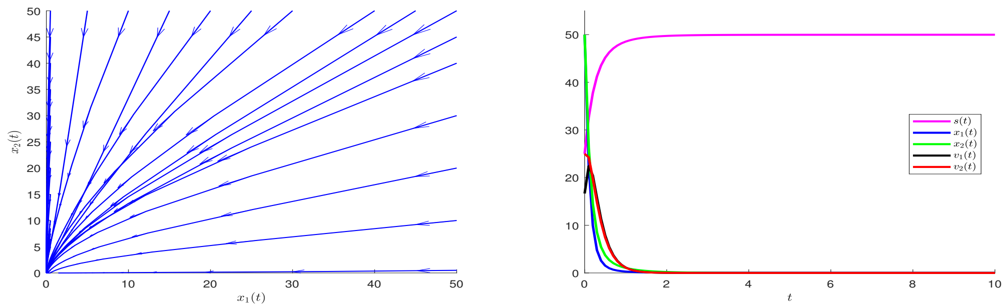

Figure 8: $x_1 - x_2$ behaviour for $D = 4, s^{in} = 45$.

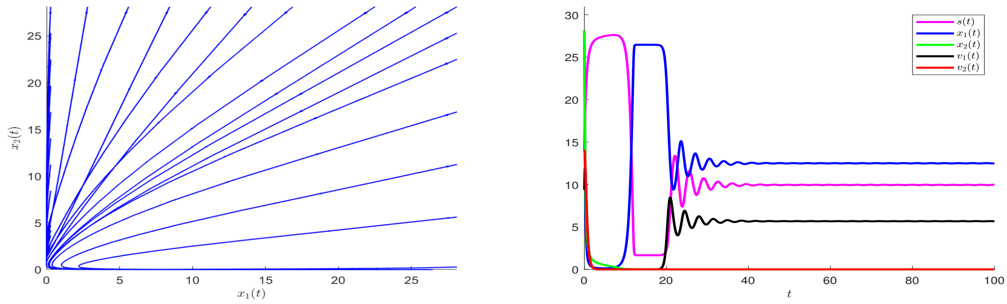

Figure 9: $x_1 - x_2$ behaviour for $D = 2.5, s^{in} = 31.25$.

In the case where we have two equilibrium points which are locally stable (Figures 4,7 and 10), the initial concentrations of species are important in determining which species is the winner. If the initial concentration is inside the attraction domain of the equilibrium point corresponding to the persistence of species 1, then species 2 becomes extinct, and if the initial concentration is inside the attraction domain of the equilibrium point corresponding to the persistence of species 2, then species 1 becomes extinct.

8 Conclusion

The CEP has been widely studied in the scientific literature. In 1932, Gause conducted experiments on the growth of yeasts and paramecia [10]. He deduced that the most competitive species consistently wins the competition. In 1960, this principle became quite popular in ecology. In fact, the CEP is still valid for many kinds of ecosystems [12]. Hsu et al. [15] in 1977, were among the first to study the problem of competition in a chemostat. They considered n populations in com-

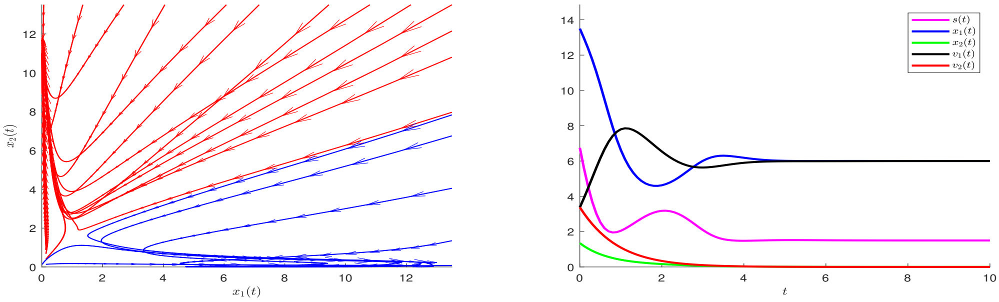

Figure 10: $x_1 - x_2$ behaviour for $D = 1.2, s^{in} = 13.5$.

petition for the same nutrient and verified the competitive exclusion, namely, that the competitor which better uses the substrate in small quantities survives, whereas the others are extinguished.

In this paper, we proposed a mathematical model (1) describing a reversible inhibition relationship between two competing bacteria for one resource in the presence of two viruses. We locally analysed the restriction of system (1) to the attractor set Ω. We proved that in a continuous reactor and under nonlinear general functional responses f_1 and f_2, the competitive exclusion principle is still fulfilled with at least one species becoming extinct. Initial species concentrations are important in determining which is the winning species.

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A Note on Explicit Solutions of FitzHugh-Rinzel System

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Abstract: The numerous scientific feedbacks that the FitzHugh-Rinzel system (FHR) is having in various scientific fields, lead to further studies on the determination of its explicit solutions. Indeed, such a study can help to get a better understanding of several behaviours in the complex dynamics of biological systems. In this note, a class of travelling wave solutions is determined and specific solutions are achieved to explicitly show the contribution due to a diffusion term considered in the FHR model.

Keywords: *FitzHugh-Rinzel model; exact solutions; travelling wave solution.*

Mathematics Subject Classification (2010): 44A10, 35K57, 35E05.

1 Introduction

One of the most commonly known models in biomathematics is the FitzHugh-Rinzel (FHR) system [1–3]. It derives from the FitzHugh-Nagumo (FHN) model [4–12] and unlike the latter, it has an additional variable suitable for evaluating and studying nerve cell bursting phenomena.

In general, bursting oscillations can be described by a system variable that changes periodically from a rapid spike oscillation to a silent phase during which the membrane potential changes slowly [13].

Studies concerning bursting phenomena are increasingly present in various scientific fields (see, for instance, [14] and references therein), and in particular, some applications concern the restoration of synaptic connections. In fact, it seems that certain nanoscale memristor devices have the potential to reproduce the behaviour of a biological synapse,

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suggesting that in the future electronic synapses may be introduced to directly connect neurons [15, 16].

The interest aroused the FHR system applications also leads to the research of explicit solutions. Indeed, in an attempt to understand the various phenomena that the FitzHugh-Rinzel system is able to describe, knowing the expression of the solution can lead to a more complete analysis of the phenomenon itself. In view of this, in this paper, the exact solutions are determined by pointing the research to travelling wave solutions.

The paper is organized as follows. In Section 2, the mathematical problem is defined. In Section 3, taking into account travelling waves, a class of explicit solutions is determined, and in Section 4, a solution has been developed to show the incidence of the diffusive term inserted in the FHR system. Finally, in Section 5, some concluding remarks have been underlined.

2 Mathematical Considerations

Generally, the FitzHugh-Rinzel model under consideration is the following:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u - u^3/3 + I - w + y, \\ \dfrac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u), \\ \dfrac{\partial y}{\partial t} = \delta(-u + h - dy), \end{cases} \tag{1}$$

where $I, \varepsilon, \beta, c, d, h, \delta$ indicate arbitrary constants.

In this paper, in order to evaluate also the contribution due to a diffusion term, the following FHR system is considered:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} - au + ku^2\,(a + 1 - u) - w + y + I, \\ \dfrac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u), \\ \dfrac{\partial y}{\partial t} = \delta(-u + h - dy). \end{cases} \tag{2}$$

Indeed, the term with $D > 0$ represents just the diffusion contribution and it derives from the Hodgkin-Huxley (HH) theory for nerve membranes when the spatial variation in the potential $V$is considered [11].

When $D = 0$, $k = 1/3$ and $a = -1$, system (2) turns into (1).

After indicating by means of

$$u(x,0) = u_0\,, \qquad w(x,0) = w_0, \qquad y(x,0) = y_0, \qquad (\,x \in \Re\,) \tag{3}$$

the initial values, from (2) it follows that

$$\begin{cases} w = w_0\, e^{-\varepsilon\beta t} + \dfrac{c}{\beta}(1 - e^{-\varepsilon\beta t}) + \varepsilon\displaystyle\int_0^t e^{-\varepsilon\beta(t-\tau)}\,u(x,\tau)\,d\tau, \\ y = y_0\, e^{-\delta d t} + \dfrac{h}{d}(1 - e^{-\delta d t}) - \delta\displaystyle\int_0^t e^{-\delta d(t-\tau)}\,u(x,\tau)\,d\tau. \end{cases} \tag{4}$$

So, when $k = 1$, problem (2) turns into

$$\begin{cases} u_t - Du_{xx} + au + \int_0^t [\varepsilon e^{-\varepsilon\beta(t-\tau)} + \delta e^{-\delta d(t-\tau)}] u(x,\tau) d\tau = F(x,t,u), \\ u(x,0) = u_0(x), \quad x \in \Re, \end{cases} \tag{5}$$

where

$$F = u^2(a+1-u) + I - w_0 e^{-\varepsilon\beta t} + y_0 e^{-\delta dt} - \frac{c}{\beta}(1-e^{-\varepsilon\beta t}) + \frac{h}{d}(1-e^{-\delta dt}). \tag{6}$$

By means of the Laplace transform, the solution of problem (5)-(6) can be expressed through an integral equation involving the fundamental solution $H(x,t)$. Indeed, in [14] it has been proved that the solution assumes the following form:

$$u(x,t) = \int_\Re H(x-\xi,t)\, u_0(\xi)\, d\xi + \int_0^t d\tau \int_\Re H(x-\xi,t-\tau)\, F[\xi,\tau,u(\xi,\tau)]\, d\xi. \tag{7}$$

Denoting by $J_1(z)$ the Bessel function of first kind and order 1, and considering the following functions:

$$H_1(x,t) = \frac{e^{-\frac{x^2}{4Dt}}}{2\sqrt{\pi D t}}\, e^{-at} + -\frac{1}{2} \int_0^t \frac{e^{-\frac{x^2}{4Dy}-ay}}{\sqrt{t-y}} \frac{\sqrt{\varepsilon}\, e^{-\beta\varepsilon(t-y)}}{\sqrt{\pi D}} J_1(2\sqrt{\varepsilon y(t-y)}\,)\,\}dy, \tag{8}$$

$$H_2 = \int_0^t H_1(x,y)\, e^{-\delta d(t-y)} \sqrt{\frac{\delta y}{t-y}} J_1(2\sqrt{\delta y(t-y)}\; dy, \tag{9}$$

one gets

$$H = H_1 - H_2. \tag{10}$$

3 Explicit Solutions

Several methods have been developed to find exact solutions of the partial differential equations [17–22].

Here, in order to find explicit solutions in the form of travelling solutions, from system (2) the following equation is deduced:

$$u_{tt} = Du_{xxt} - au_t + 2uu_t(a+1) - 3u^2u_t + \varepsilon\beta w - \varepsilon c - \varepsilon u - \delta u + \delta h - \delta dy. \tag{11}$$

Moreover, letting

$$\beta\varepsilon = \delta d,$$

one obtains

$$
\begin{aligned}
u_{tt} = Du_{xxt} - au_t + 2uu_t(a+1) - 3u^2u_t - \varepsilon c - \varepsilon u - \delta u + \delta h + \\
\varepsilon\beta(-u_t + Du_{xx} - au + u^2(a+1) - u^3 + I).
\end{aligned}
\tag{12}
$$

Now, if one introduces the variable wave

$$z = x - C\,t,$$

from (12) one gets

$$
\begin{aligned}
D\,C\,u_{zzz} + (C^2 - \varepsilon\beta\,D)u_{zz} - 3Cu^2u_z + 2C(a+1)u\,u_z + \varepsilon\beta u^3 + \\
-\,C\,(a\, + \varepsilon\beta)\,u_z - \,\varepsilon\beta(a+1)u^2 + \,\varepsilon\beta\,a\,u + (\varepsilon + \delta)\,u - K = 0,
\end{aligned}
\tag{13}
$$

where

$$K = (\delta h - \varepsilon\,c) + \varepsilon\beta\,I.$$

The solutions to be determined are of the type

$$u(z) = A\ f(z) +\ b\,, \tag{14}$$

where one assumes

$$f(z) = \frac{1}{1 + e^{(z - z_0)}}. \tag{15}$$

Since

$$f_z - f^2 + f = 0,$$

it results in

$$u_z =\ A\,f^2(z) - Af,$$

$$u_{zz} = 2\,Af^3 - 3\,A\,f^2 + Af,$$

$$u_{zzz} = 6Af^4(z) - 12\,A\,f^3(z) + 7Af^2 - Af,$$

$$u\,u_z = A^2\,f^3 + (-A^2 + Ab)f^2 - Abf,$$

$$u^2\,u_z = A^3f^4 + (2A^2\,b\, - A^3)f^3 + (Ab^2 - 2A^2b)f^2 - Ab^2f.$$

In order to satisfy equation (13), one has to assume

$$A^2 = 2D \tag{16}$$

and

$$\delta = -\varepsilon. \tag{17}$$

Moreover, under the assumption that

$$C > \sqrt{3}/4 \ \wedge\ D > 0$$

or

$$C < -\sqrt{3}/4 \wedge D > 0$$

or

$$0 < D < \frac{1}{12}\,(3 - \sqrt{3}\,\sqrt{3 - 16\,C^2} \ \wedge\ -\sqrt{3}/4 \ < \ C \ < \sqrt{3}/4$$

or

$$D > \frac{1}{12}\,(3 + \sqrt{3}\,\sqrt{3 - 16\,C^2} \ \wedge -\sqrt{3}/4 \ < \ C \ < \sqrt{3}/4\,,$$

constants a, b, and K must satisfy the following relationships:

$$b = \frac{1}{6\sqrt{2D}}\,\big(\sqrt{2}\ \sqrt{2C^2 + 6D^2 - 3D}\ + 2C - 6D + 3\sqrt{2D}\big),$$

$$a = 3\,b - \frac{C}{A} + \frac{3A}{2} - 1,$$

$$K = \varepsilon\beta b^3 - \varepsilon\beta(a+1)b^2 + [(\varepsilon + \delta) + \varepsilon\beta a]b.$$

4 Application

The previous analysis allows us to make some applications. Indeed, in order to point out the contribution of diffusion effects due to the second order term with the coefficient D, let us assume, for instance, the following values:

$$C = 1;\ \ z_0 = 0;\ \ \varepsilon\,\beta = 0.1\,.$$

In this way, this results in

$$b = \frac{\sqrt{6D^2 - 3D + 2}}{6\sqrt{D}} - \frac{\sqrt{D}}{\sqrt{2}} + \frac{1}{3\sqrt{2D}} + \frac{1}{2} \tag{18}$$

and consequently, one has

$$u(z) = \frac{\sqrt{2D}}{1 + e^z}\ +\ \frac{\sqrt{6D^2 - 3D + 2}}{6\sqrt{D}} - \frac{\sqrt{D}}{\sqrt{2}} + \frac{1}{3\sqrt{2D}} + \frac{1}{2}. \tag{19}$$

Plotting the graph of function (19), it is possible to note how the diffusion term influences the damping of the solution both when the coefficient D is equal to or less than 1 and when D is greater than 1.

5 Remarks

• The paper is concerned with the ternary autonomous dynamical system of FitzHugh-Rinzel (FHR) which, in biophysics, seems to be appropriate to describe some phenomena such as bursting oscillations. In this note, the FHR system under consideration includes a diffusion term, represented by a second order term, that derives directly from

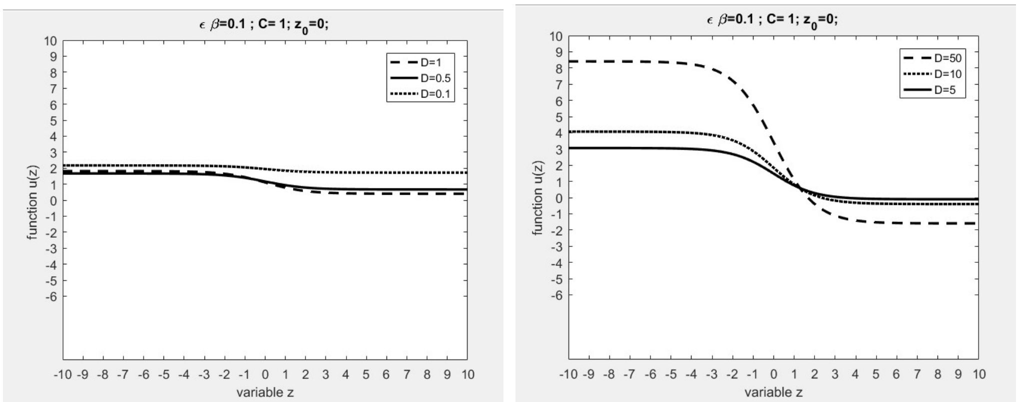

Figure 1: Solution u(z) when $\varepsilon\beta = 0.1$, $z_0 = 0$, $C = 1$. On the left: the values for the parameter D are such that $0 < D \leq 1$, while in the right-hand graph: we have considered $D > 1$.

the Hodgkin-Huxley theory for nerve membranes and that is frequently inserted in the FitzHugh-Nagumo model, too.

• Solutions can be expressed by means of an integral equation involving the fundamental solution. However, to give direct feedbacks related to the contribution due to the diffusion term D, by means of the method of travelling wave, explicit solutions have been determined.

• Once arbitrary parameters have been set, the trajectories of solutions are shown, whether the parameter D is less than 1 or D is greater than 1.

• Of course, as the chosen constants change, the behaviour of the various solutions can be pointed out.

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Existence of Solutions for the Debye-Hückel System with Low Regularity Initial Data in Critical Fourier-Besov-Morrey Spaces

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Abstract: This paper is devoted to studying the existence of solutions for the Cauchy problem of the Debye-Hückel system with low regularity initial data in critical Fourier-Besov-Morrey spaces. We show that there exists a unique local solution if the initial data belong to the Fourier-Morrey-Besov space $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$, and furthermore, if the initial data are sufficiently small, then the solution is global.

Keywords: *Debye-Hückel system; local existence; global existence; Littlewood-Paley theory; Fourier-Morrey-Besov spaces.*

Mathematics Subject Classification (2010): 35K45, 35Q99, 70k99, 93-00.

1 Introduction

In this paper, we consider the following Cauchy problem for the Debye-Hückel system in $\mathbb{R}^n \times \mathbb{R}^+$:

$$\begin{cases} \partial_t v = \Delta v - \nabla \cdot (v \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w = \Delta w + \nabla \cdot (w \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \Delta \phi = v - w & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1}$$

where the unknown functions $v = v(x,t)$ and $w = w(x,t)$ denote densities of the electron and the hole in electrolytes, respectively, $\phi = \phi(x,t)$ denotes the electric potential, $v_0(x)$ and $w_0(x)$ are the initial data. Throughout this paper, we assume that $n \geq 2$.

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Notice that the function ϕ is determined by the Poison equation in the third equation of (1), and it is given by $\phi(x,t) = (-\Delta)^{-1}(w-v)(x,t)$. So, the system (1) can be rewritten as the following system:

$$\begin{cases} \partial_t v - \Delta v = -\nabla \cdot \big(v\nabla(-\Delta)^{-1}(w-v)\big) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w - \Delta w = \nabla \cdot \big(w\nabla(-\Delta)^{-1}(w-v)\big) & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{2}$$

W. Nernst and M. Planck introduced the Debye-Hückel system at the end of the nineteenth century as a fundamental model for the ion diffusion in an electrolyte [9]. It can also be derived from the mathematical modeling of semiconductors [19], plasma physics [12], and chemotaxis [8]. Thus, (1) has been studied by many researchers. Ogawa, Takayoshi, and S. Shimizu in [18] established the local well-posedness for large initial data and the global well-posedness for small initial data in the critical Hardy space $\mathcal{H}^1\left(\mathbb{R}^2\right)$. In 2008, Kurokiba and Ogawa in [16] obtained similar results for the initial data in subcritical and critical Lebesgue and Sobolev spaces.

In the context of Besov spaces, Karch in [15] proved the existence of global solution of the system (1) with small initial data in the critical Besov space $\dot{\mathcal{B}}_{p,\infty}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ with $\frac{n}{2} \leq p < n$. Later, Deng and Li [10] showed that the system (1) is well-posed in $\dot{\mathcal{B}}_{4,2}^{-\frac{3}{2}}\left(\mathbb{R}^2\right)$, and ill-posed in $\dot{\mathcal{B}}_{4,r}^{-\frac{3}{2}}\left(\mathbb{R}^2\right)$ for $2 < r \leq \infty$. Zhao, Liu, and Cui [20] established the existence of global and local solution of the system (1) in the critical Besov space $\dot{\mathcal{B}}_{p,r}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ with $1 < p < 2n$ and $1 \leq r \leq \infty$ (see also [1–3]).

Inspired by the work [20], the purpose of this paper is to establish the existence of local solution to (1) for large initial data and global solution for small initial data in the critical Fourier-Besov-Morrey space $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$.

Let us firstly recall the scaling property of the systems:
if (v,w) solves (1) with the initial data (v_0,w_0) (ϕ can be determined by (v,w)), then (v_γ,w_γ) with $(v_\gamma(x,t),w_\gamma(x,t)) := \left(\gamma^2 v\left(\gamma x,\gamma^2 t\right), \gamma^2 w\left(\gamma x,\gamma^2 t\right)\right)$ is also a solution to (1) with the initial data

$$(v_{0,\gamma}(x), w_{0,\gamma}(x)) := \left(\gamma^2 v_0(\gamma x), \gamma^2 w_0(\gamma x)\right) \tag{3}$$

(ϕ_γ can be determined by (v_γ,w_γ)).

Definition 1.1 A critical space for the initial data of the system (1) is any Banach space $E \subset \mathcal{S}'(\mathbb{R}^n)$ whose norm is invariant under the scaling (3) for all $\gamma > 0$, i.e.,

$$\|(v_{0,\gamma}(x), w_{0,\gamma}(x))\|_E \approx \|(v_0(x), w_0(x))\|_E .$$

In accordance with these scales, we can show that the space pairs $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$ are critical for (1).

Throughout the paper, we use $\mathcal{FN}_{p,\lambda,q}^{s}$ to denote the homogenous Fourier-Besov-Morrey spaces, $(v,w) \in X$ to denote $(v,w) \in X \times X$ for a Banach space X $\Big($the product $X \times X$ will be endowed with the usual norm $\|(v,w)\|_{X\times X} := \|v\|_X + \|w\|_X$ $\Big)$, $\|(v,w)\|_X$ to denote $\|(v,w)\|_{X\times X}$, $V \lesssim W$ means that there exists a constant $C > 0$ such that $V \leq CW$, and p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq \infty$.

Now, our main results are stated below.

Theorem 1.1 *Let* $n \geq 2, \rho_0 > 2,\ \max\{n-(n-1)p, 0\} \leq \lambda < n,\ 1 \leq p < \infty,$ $q \in [1, \infty],\ (v_0, w_0) \in \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$ *and* $\frac{1}{\rho_0} + \frac{1}{\rho_0'} = 1.$
Then there exists $T \geq 0$ *such that the system* (1) *has a unique local solution* $(v, w) \in X_T$*, where*

$$X_T = \mathfrak{L}^{\rho_0}\left(0, T; \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}\right) \cap \mathfrak{L}^{\rho_0'}\left(0, T; \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}\right),$$

and

$$(v, w) \in \mathcal{C}\left(0, T; \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}\right).$$

Besides, there exixts $K \geq 0$ *such that if* (v_0, w_0) *satisfies* $\|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \leq K$*, then the above assertion holds for* $T = \infty$*; i.e., the solution* (v, w) *is global.*

2 Preliminaries

In this section, we give some notations and recall basic properties of Fourier-Besov-Morrey spaces, which will be used throughout the paper. The Fourier-Besov-Morrey spaces, presented in [11], are constructed by using a type of localization on Morrey spaces. The function spaces M_p^λ are defined as follows.

Definition 2.1 [11] Let $1 \leq p \leq \infty$ and $0 \leq \lambda < n$. The homogeneous Morrey space M_p^λ is the set of all functions $f \in L^p(B(x_0, r))$ such that

$$\|f\|_{M_p^\lambda} = \sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty, \tag{4}$$

where $B(x_0, r)$ is the open ball in $\mathbb{R}^n$ centered at x_0 and with radius $r > 0$. When $p = 1$, the L^1 -norm in (4) is understood as the total variation of the measure f on $B(x_0, r)$ and M_p^λ as a subspace of Radon measures. When $\lambda = 0$, we have $M_p^0 = L^p$.

The proofs of the results discussed in this work are based on a dyadic partition of unity in the Fourier variables, known as the homogeneous Littlewood-Paley decomposition. We present briefly this construction below. For more detail, we refer the reader to [4].
Let $f \in S'(\mathbb{R}^n)$. Define the Fourier transform as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

and its inverse Fourier transform as

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

Let $\varphi \in S(\mathbb{R}^d)$ be such that $0 \leq \varphi \leq 1$ and $\text{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and

$$\sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0.$$

We denote

$$\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right), \quad \psi_j(\xi) = \sum_{k\leq j-1} \varphi_k(\xi)$$

and

$$h(x)=\mathcal{F}^{-1}\varphi(x), \quad g(x)=\mathcal{F}^{-1}\psi(x).$$

We now present some frequency localization operators

$$\dot{\Delta}_j f=\varphi_j(D)f=2^{dj}\int_{\mathbb{R}^d} h\left(2^j y\right) f(x-y)dy$$

and

$$\dot{S}_j f=\sum_{k\leq j-1}\dot{\Delta}_k f=\psi_j(D)f=2^{dj}\int_{\mathbb{R}^d} g\left(2^j y\right) f(x-y)dy.$$

From the definition, one easily derives that

$$\begin{aligned} \dot{\Delta}_j\dot{\Delta}_k f&=0, \quad \text{if } |j-k|\geq 2,\\ \dot{\Delta}_j\left(\dot{S}_{k-1}f\dot{\Delta}_k f\right)&=0, \quad \text{if } |j-k|\geq 5. \end{aligned}$$

The following Bony paraproduct decomposition will be applied throughout the paper:

$$uv=\dot{T}_u v+\dot{T}_v u+R(u,v),$$

where

$$\dot{T}_u v=\sum_{j\in\mathbb{Z}}\dot{S}_{j-1}u\dot{\Delta}_j v, \quad \dot{R}(u,v)=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j u\tilde{\Delta}_j v, \quad \tilde{\Delta}_j v=\sum_{|j'-j|\leq 1}\dot{\Delta}_{j'}v.$$

Lemma 2.1 *[11] Let $1\leq p_1,p_2,p_3<\infty$ and $0\leq\lambda_1,\ \lambda_2,\ \lambda_3<n$.*
(i) (Hölder's inequality) Let $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{\lambda_3}{p_3}=\frac{\lambda_1}{p_1}+\frac{\lambda_2}{p_2}$, then we have

$$\|fg\|_{\mathrm{M}_{p_3}^{\lambda_3}}\leq\|f\|_{\mathrm{M}_{p_1}^{\lambda_1}}\|g\|_{\mathrm{M}_{p_2}^{\lambda_2}}. \tag{5}$$

(ii) (Young's inequality) If $\varphi\in L^1$ and $g\in\mathrm{M}_{p_1}^{\lambda_1}$, then

$$\|\varphi*g\|_{\mathrm{M}_{p_1}^{\lambda_1}}\leq\|\varphi\|_{L^1}\|g\|_{\mathrm{M}_{p_1}^{\lambda_1}}, \tag{6}$$

where $*$ denotes the standard convolution operator.

Now, we recall the Bernstein type lemma in Fourier variables in Morrey spaces.

Lemma 2.2 *[11] Let $1\leq q\leq p<\infty$, $0\leq\lambda_1,\lambda_2<n$, $\frac{n-\lambda_1}{p}\leq\frac{n-\lambda_2}{q}$ and let γ be a multi-index. If supp$(\widehat{h})\subset\{|\xi|\leq A2^j\}$, then there is a constant $C>0$ independent of h and j such that*

$$\|\widehat{D_\xi^\gamma h}\|_{\mathrm{M}_q^{\lambda_2}}\leq C2^{j|\gamma|+j(\frac{n-\lambda_2}{q}-\frac{n-\lambda_1}{p})}\|\widehat{h}\|_{\mathrm{M}_p^{\lambda_1}}. \tag{7}$$

We have now prepared all of the ingredients required to define the function spaces $\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^n)$, see [11].

Definition 2.2 (Homogeneous Fourier-Besov-Morrey spaces)
Let $1 \le p, q \le \infty$, $0 \le \lambda < n$ and $s \in \mathbb{R}$. The homogeneous Fourier-Besov-Morrey space $\mathcal{FN}^s_{p,\lambda,q}$ is defined as the set of all distributions $f \in \mathcal{S}'\backslash\mathcal{P}$, $\mathcal{P}$ is the set of all polynomials such that the norm $\|f\|_{\mathcal{FN}^s_{p,\lambda,q}}$ is finite, where

$$\|f\|_{\mathcal{FN}^s_{p,\lambda,q}} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \left\|\varphi_j \hat{f}\right\|^q_{\mathrm{M}^\lambda_p}\right)^{\frac{1}{q}} & for \quad q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \left\|\varphi_j \hat{f}\right\|_{\mathrm{M}^\lambda_p} & for \quad q = \infty. \end{cases} \tag{8}$$

Note that the space $\mathcal{FN}^s_{p,\lambda,q}(\mathbb{R}^n)$ equipped with the norm (8) is a Banach space. Since $\mathrm{M}^0_p = L^p$, we have $\mathcal{FN}^s_{p,0,q} = F\dot{B}^s_{p,q}$.

The definition of mixed space-time spaces is given below.

Definition 2.3 Let $s \in \mathbb{R}$, $1 \le p < \infty$, $1 \le q, \rho \le \infty$, $0 \le \lambda < n$, and $I = [0, T)$, $T \in (0, \infty]$. The space-time norm is defined on $u(t, x)$ by

$$\|u(t,x)\|_{\mathcal{L}^\rho(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} = \Big\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^q_{L^\rho(I,\mathrm{M}^\lambda_p)} \Big\}^{1/q},$$

and denote by $\mathcal{L}^\rho(I, \mathcal{FN}^s_{p,\lambda,q})$ the set of distributions in $S'(\mathbb{R} \times \mathbb{R}^n)/\mathcal{P}$ with finite $\|.\|_{\mathcal{L}^\rho(I,\mathcal{FN}^s_{p,\lambda,q})}$ norm.

According to the Minkowski inequality, it is easy to verify that

$$\begin{aligned} L^\rho\left(I; \mathcal{FN}^s_{p,\lambda,q}\right) &\hookrightarrow \mathcal{L}^\rho\left(I, \mathcal{FN}^s_{p,\lambda,q}\right) \quad \text{if } \rho \le q, \\ \mathcal{L}^\rho\left(I, \mathcal{FN}^s_{p,\lambda,q}\right) &\hookrightarrow L^\rho\left(I; \mathcal{FN}^s_{p,\lambda,q}\right) \quad \text{if } \rho \ge q, \end{aligned}$$

where $\|u(t,x)\|_{L^\rho(I;\mathcal{FN}^s_{p,\lambda,q})} := \left(\int_I \|u(\tau,\cdot)\|^\rho_{\mathcal{FN}^s_{p,\lambda,q}} d\tau\right)^{1/\rho}$.

At the end of this section, we will recall an existence and uniqueness result for an abstract operator equation in a Banach space that will be used to show Theorem 1.1 in the sequel. For the proof, we refer the reader to [4].

Lemma 2.3 *Let X be a Banach space with norm $\|.\|_X$ and $B : X \times X \longmapsto X$ be a bounded bilinear operator satisfying*

$$\|B(u,v)\|_X \le \eta \|u\|_X \|v\|_X$$

for all $u, v \in X$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < \frac{1}{4\eta}$ and if $y \in X$ so that $\|y\|_X \le \varepsilon$, the equation $x := y + B(x, x)$ has a solution $\overline{x}$ in X such that $\|\overline{x}\|_X \le 2\varepsilon$. This solution is the only one in the ball $\overline{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X < \varepsilon$, $x' = y' + B(x', x')$, and $\|x'\|_X \le 2\varepsilon$, then

$$\|\overline{x} - x'\|_X \le \frac{1}{1 - 4\varepsilon\eta} \|y - y'\|_X .$$

3 Linear Estimates in Fourier-Besov-Morrey Spaces

In this part, we give some crucial estimates in the proof of our main results.

Lemma 3.1 *[7] Let I=(0, T), $s \in \mathbb{R}$, $p, q, \rho \in [1,\infty]$ and $0 \leq \lambda < n$. There exists a constant $C > 0$ such that*

$$\|e^{t\Delta(.)}u_0\|_{\mathcal{L}^{\rho}\left([0,T),\mathcal{FN}^{s+\frac{2}{\rho}}_{p,\lambda,q}\right)} \leq C\|u_0\|_{\mathcal{FN}^{s}_{p,\lambda,q}}, \tag{9}$$

where $u_0 \in \mathcal{FN}^{s}_{p,\lambda,q}$.

Lemma 3.2 *[7] Let I=(0, T), $s \in \mathbb{R}$, $p, q, \rho \in [1,\infty]$ $0 \leq \lambda < n$ and $1 \leq r \leq \rho$. There exists a constant $C > 0$ such that*

$$\left\|\int_0^t e^{(t-\tau)\Delta}f(\tau)d\tau\right\|_{\mathcal{L}^{\rho}\left(I;\mathcal{FN}^{s+\frac{2}{\rho}}_{p,\lambda,q}\right)} \leq C\|f\|_{\mathcal{L}^{r}\left(I;\mathcal{FN}^{s-2+\frac{2}{r}}_{p,\lambda,q}\right)} \tag{10}$$

for all $f \in \mathcal{L}^{r}\left(I;\mathcal{FN}^{s-2+\frac{2}{r}}_{p,\lambda,q}\right)$.

4 Bilinear Estimates in Fourier-Besov-Morrey Spaces

Lemma 4.1 *Let $I = (0,T)$, $p, q \in [1,\infty]$, $\max\{n-(n-1)p, 0\} < \lambda < n$, $\rho_0 > 2$ and $\frac{1}{\rho_0} + \frac{1}{\rho_0'} = 1$. There exists a constant $C > 0$ such that*

$$\begin{aligned}\|\nabla.(f\nabla g)\|_{\mathfrak{L}^{1}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \leq C\Bigg[&\|f\|_{\mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|g\|_{\mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\\ &+ \|g\|_{\mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|f\|_{\mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\Bigg]\end{aligned}$$

for all $f \in \mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)$
and $g \in \mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)$.

Proof. Applying Bony's paraproduct decomposition and a quasi-orthogonality property for the Littlewood-Paley decomposition, for a fixed j, we obtain

$$\begin{aligned}\dot{\Delta}_j(f\nabla g) &= \sum_{|k-j|\leq 4}\dot{\Delta}_j(\dot{S}_{k-1}f\dot{\Delta}_k\nabla g) + \sum_{|k-j|\leq 4}\dot{\Delta}_j(\dot{S}_{k-1}\nabla g\dot{\Delta}_k f)\\ &+ \sum_{k\geq j-3}\dot{\Delta}_j(\dot{\Delta}_k f\widetilde{\dot{\Delta}}_k\nabla g)\\ &= I_j^1 + I_j^2 + I_j^3\,.\end{aligned}$$

Then, by the triangle inequalities in M_p^λ and in $l^q(\mathbb{Z})$, we have

$$\begin{aligned}
\|\nabla.(f\nabla g)\|_{\mathfrak{L}^1\left(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}\right)} &\leq \|f\nabla g\|_{\mathfrak{L}^1\left(I;\mathcal{FN}_{p,\lambda,q}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}\right)} \\
&\leq \{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{\dot{\Delta}_j(f\nabla g)}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}} \\
&\leq \{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^1}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}} \\
&+ \{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^2}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}} \\
&+ \{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^3}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}} \\
&:= J_1 + J_2 + J_3.
\end{aligned}$$

By using the Young inequality in Morrey spaces and the Bernstein-type inequality with $|\gamma| = 0$, we have

$$\left\|\varphi_j \hat{f}\right\|_{L^1} \leq C2^{j(\frac{n}{p'}+\frac{\lambda}{p})}\left\|\varphi_j \hat{f}\right\|_{M_p^\lambda}.$$

Then

$$\begin{aligned}
\|\widehat{I_j^1}\|_{L^1(I,M_p^\lambda)} &\leq \sum_{|k-j|\leq 4} \|\widehat{(\dot{S}_{k-1}f\dot{\Delta}_k\nabla g)}\|_{L^1(I,M_p^\lambda)} \\
&\leq \sum_{|k-j|\leq 4} \|\varphi_k\widehat{\nabla g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{l\leq k-2}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,L^1)} \\
&\lesssim \sum_{|k-j|\leq 4} 2^k\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{l\leq k-2} 2^{(\frac{n}{p'}+\frac{\lambda}{p})l}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,M_p^\lambda)} \\
&\lesssim \sum_{|k-j|\leq 4} 2^k\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{l\leq k-2} 2^{(-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0})l}2^{(2-\frac{2}{\rho_0})l}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,M_p^\lambda)} \\
&\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \sum_{|k-j|\leq 4} 2^k\Big(\sum_{l\leq k-2} 2^{l(2-\frac{2}{\rho_0})q'}\Big)^{\frac{1}{q'}}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \\
&\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \sum_{|k-j|\leq 4} 2^{k(3-\frac{2}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)},
\end{aligned}$$

where we have used the fact that $\rho_0 > 2$ in the last inequality.

Thus, by using the Young inequality, we have

$$\begin{aligned}
J_1 &\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})} \left\{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\Big(\sum_{|k-j|\le 4} 2^{k(3-\frac{2}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big)^q\right\}^{\frac{1}{q}}\\
&\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}\\
&\qquad\times\left\{\sum_{j\in\mathbb{Z}}\Big(\sum_{|k-j|\le 4} 2^{(j-k)(-1+\frac{n}{p'}+\frac{\lambda}{p})q}2^{k(2+\frac{n}{p'}+\frac{\lambda}{p}-\frac{2}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big)^q\right\}^{\frac{1}{q}}\\
&\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}\|g\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q})},
\end{aligned}$$

where we have used $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$.

Similarly, we get

$$J_2 \lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}\|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q})}.$$

For J_3, first we use the Young inequality in Morrey spaces, the Bernstein inequality ($|\gamma|=0$) together with the Hölder inequality, to get

$$\begin{aligned}
\|\widehat{I_j^3}\|_{L^1(I,M_p^\lambda)} &\le \sum_{k\ge j-3}\|\widehat{(\dot{\Delta}_k f\tilde{\dot{\Delta}}_k\nabla g)}\|_{L^1(I,M_p^\lambda)}\\
&\le \sum_{k\ge j-3}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\sum_{|l-k|\le 1}\|\varphi_l\widehat{\nabla g}\|_{L^{\rho_0}(I,L^1)}\\
&\lesssim \sum_{k\ge j-3}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\sum_{|l-k|\le 1}2^l2^{l(\frac{n}{p'}+\frac{\lambda}{p})}\|\varphi_l\widehat{g}\|_{L^{\rho_0}(I,M_p^\lambda)}\\
&\lesssim \sum_{k\ge j-3}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big(\sum_{|l-k|\le 1}2^{l(1-\frac{2}{\rho_0})q'}\Big)^{\frac{1}{q'}}\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}\\
&\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}\sum_{k\ge j-3}2^{k(1-\frac{2}{\rho_0})}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}.
\end{aligned}$$

Then, applying the Hölder inequality for the series, we obtain

$$\begin{aligned}
J_3 &\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}\left\{\sum_{j\in\mathbb{Z}}2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\Big(\sum_{k\ge j-3}2^{k(1-\frac{2}{\rho_0})}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big)^q\right\}^{\frac{1}{q}}\\
&\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}\left\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k\ge j-3}2^{(j-k)(-1+\frac{n}{p'}+\frac{\lambda}{p})}2^{k(\frac{n}{p'}+\frac{\lambda}{p}-\frac{2}{\rho_0})}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big)^q\right\}^{\frac{1}{q}}\\
&\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}\|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q})}\sum_{i\le 3}2^{i(-1+\frac{n}{p'}+\frac{\lambda}{p})}\\
&\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}\|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q})},
\end{aligned}$$

where we have used the condition $\lambda > n-(n-1)p$ to ensure that the series $\sum_{i\le 3} 2^{i(-1+\frac{n}{p'}+\frac{\lambda}{p})}$ converges. Thus, we finished the proof of Lemma 4.1.

5 Proof of Theorem 1.1

To ensure the existence of the global and local solution of the system (1), we will use Lemma 2.3 with the linear and bilinear estimate that we have established in Sections 3 and 4. Let $\rho_0 > 2$ be any given real number and $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$. Note that the space X_T defined in Theorem 1.1 is a Banach space equipped with the norm

$$\|u\|_{X_T} = \|u\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} + \|u\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.$$

For $T>0$ to be determined later. Given $(v,w)\in X_T$, we define $\mathfrak{F}(v,w)=(\bar{v},\bar{w})$ to be the solution of the following initial value problem:

$$\begin{cases} \partial_t\bar{v}-\Delta\bar{v} = -\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right), & \bar{v}(x,0)=v_0(x), \\ \partial_t\bar{w}-\Delta\bar{w} = \nabla\cdot\left(w\nabla(-\Delta)^{-1}(w-v)\right), & \bar{w}(x,0)=w_0(x). \end{cases} \tag{11}$$

Obviously, (v,w) is a solution of (1) if and only if it is a fixed point of $\mathfrak{F}$.

Lemma 5.1 *Let $(v,w)\in X_T$. Then $(\bar{v},\bar{w})\in X_T$. Moreover, there exist two constants $C_0>0$ and $C_1>0$ such that*

$$\|(\bar{v},\bar{w})\|_{X_T} \le C_0\,\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + C_1\|(v,w)\|_{X_T}^2. \tag{12}$$

Proof. By Duhamel's principle, the system (2) is equivalent to the following integral system:

$$\begin{cases} \bar{v}(t) = e^{t\Delta}v_0 - \displaystyle\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau \\ \bar{w}(t) = e^{t\Delta}w_0 + \displaystyle\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(w\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau. \end{cases} \tag{13}$$

Set

$$B_1(v,w) := -\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau,$$

$$B_2(v,w) := \int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(w\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau,$$

then the equivalent integral system (13) can be rewritten as

$$(\bar{v}(t),\bar{w}(t)) = (e^{t\Delta}v_0, e^{t\Delta}w_0) + (B_1(v,w),B_2(v,w)). \tag{14}$$

According to Lemma 3.1 with $s=-2+\frac{n}{p'}+\frac{\lambda}{p}$, $I=[0,\infty)$ and $\rho=\rho_0$ (*or* ρ_0'), we obtain

$$\|e^{t\Delta}v_0\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}\right)} \lesssim \|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}$$

and

$$\|e^{t\Delta}v_0\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}\right)} \lesssim \|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}},$$

which implies

$$\|e^{t\Delta}v_0\|_{X_T} \lesssim \|v_0\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.$$

Similary,

$$\|e^{t\Delta}w_0\|_{X_T} \lesssim \|w_0\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.$$

Thus

$$\|(e^{t\Delta}v_0, e^{t\Delta}w_0)\|_{X_T} \leq C_0 \|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}. \tag{15}$$

Applying Lemma 3.2 with $s=-2+\frac{n}{p'}+\frac{\lambda}{p}$ and $r=1$, and Lemma 4.1, we obtain

$$\begin{aligned}
&\|B_1(v,w)\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \\
&= \|\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \\
&\lesssim \|\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)\|_{\mathfrak{L}^{1}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \\
&\lesssim \|v\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|(-\Delta)^{-1}(w-v)\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \\
&\qquad + \|(-\Delta)^{-1}(w-v)\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|v\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \\
&\lesssim \|v\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|w-v\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \\
&\qquad + \|w-v\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|v\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \\
&\lesssim \left(\|(v,w)\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|(v,w)\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\right) \\
&\lesssim \|(v,w)\|^2_{X_T}.
\end{aligned}$$

Analogously, we get

$$\|B_1(v,w)\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \lesssim \|(v,w)\|^2_{X_T}.$$

Thus, we obtain

$$\|B_1(v,w)\|_{X_T} \lesssim \|(v,w)\|^2_{X_T}.$$

Similary,

$$\|B_2(v,w)\|_{X_T} \lesssim \|(v,w)\|^2_{X_T}.$$

Finally,

$$\|(B_1(v,w),B_2(v,w))\|_{X_T} \leq C_1\|(v,w)\|^2_{X_T}. \tag{16}$$

Combining (15) and (16), we obtain

$$\|(\bar{v},\bar{w})\|_{X_T} \leq C_0 \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + C_1\|(v,w)\|_{X_T}^2.$$

And we have completed the proof of Lemma 5.1, as desired.

The last lemma ensures that $\mathfrak{F}$ is well-defined and maps X_T into itself.

We begin by showing the global existence for small initial data. For this purpose, we choose $T=\infty$. We have from Lemma 5.1

$$\begin{aligned}\|\mathfrak{F}(v,w)\|_{X_\infty} &\leq C_0 \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + C_1\|(v,w)\|_{X_\infty}^2 \\ &\leq C_0 \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + 4C_1\varepsilon^2.\end{aligned}$$

Choosing $\varepsilon < \frac{1}{8\max\{C_0,C_1\}}$ for any $(v_0,w_0)\in\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$ with $\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} < \frac{\varepsilon}{8\max\{C_0,C_1\}}$, we get

$$\|\mathfrak{F}(v,w)\|_{X_\infty} < \varepsilon.$$

Finally, by using Lemma 2.3 we can obtain a unique global solution for small initial data in the closed ball $\bar{B}(0,2\varepsilon)=\{x\in X_\infty \ : \ \ \|x\|_{X_\infty}\leq 2\varepsilon\}$.

For the local existence, we shall decompose the initial data v_0 into two terms

$$v_0 = \mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\hat{v}_0\right) + \mathcal{F}^{-1}\left(\chi_{B^C(0,\delta)}\hat{v}_0\right) := v_{0,1}+v_{0,2},$$

where $\delta=\delta(v_0)>0$ is a real number. Similary, we decompose w_0:

$$w_0 = \mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\hat{w}_0\right) + \mathcal{F}^{-1}\left(\chi_{B^C(0,\delta)}\hat{w}_0\right) := w_{0,1}+w_{0,2}.$$

Since

$$\begin{cases} v_{0,2}\longrightarrow 0 \ \text{ in } \ \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}} \ \text{ when } \quad \delta\to+\infty, \\ w_{0,2}\longrightarrow 0 \ \text{ in } \ \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}} \ \text{ when } \quad \delta\to+\infty, \end{cases}$$

there exists δ large enough such that

$$C_0\|(v_{0,2},w_{0,2})\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \leq \frac{\varepsilon}{2}.$$

We get

$$\left\|\left(e^{t\Delta}v_0,e^{t\Delta}w_0\right)\right\|_{X_T} \leq \left\|\left(e^{t\Delta}v_{0,1},e^{t\Delta}w_{0,1}\right)\right\|_{X_T}+\frac{\varepsilon}{2}. \tag{17}$$

We have

$$\begin{aligned}&\left\|\left(e^{t\Delta}v_{0,1},e^{t\Delta}w_{0,1}\right)\right\|_{X_T} \\ &= \left\|\left(e^{t\Delta}v_{0,1},e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} + \left\|\left(e^{t\Delta}v_{0,1},e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.\end{aligned}$$

Using the fact that $|\xi| \approx 2^j$ for all $j \in \mathbb{Z}$, we obtain

$$\begin{aligned}
&\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \\
&= \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0})q} \|\varphi_j \widehat{e^{t\Delta}v_{0,1}}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\
&\quad + \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0})q} \|\varphi_j \widehat{e^{t\Delta}w_{0,1}}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\
&= \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q} 2^{j(\frac{2}{\rho_0})q} \|\varphi_j |\xi|^2 \chi_{B(0,\delta)} \hat{v_0}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\
&\quad + \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q} 2^{j(\frac{2}{\rho_0})q} \|\varphi_j |\xi|^2 \chi_{B(0,\delta)} \hat{w_0}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\
&\lesssim \delta^{2+\frac{2}{\rho_0}} \Big(\Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q} \|\varphi_j \hat{v_0}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\
&\quad + \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q} \|\varphi_j \hat{w_0}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\Big) \\
&\lesssim \delta^{2+\frac{2}{\rho_0}} T^{\frac{1}{\rho_0}} \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.
\end{aligned}$$

Thus

$$\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \le C_2\delta^{2+\frac{2}{\rho_0}} T^{\frac{1}{\rho_0}} \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Similary,

$$\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})} \le C_2\delta^{2+\frac{2}{\rho_0'}} T^{\frac{1}{\rho_0'}} \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Hence,
$$\begin{aligned}
\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{X_T} &\le C_2\delta^{2+\frac{2}{\rho_0}} T^{\frac{1}{\rho_0}} \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \\
&\quad + C_2\delta^{2+\frac{2}{\rho_0'}} T^{\frac{1}{\rho_0'}} \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.
\end{aligned}$$

We choose T small enough so that

$$\begin{cases}
C_2\delta^{2+\frac{2}{\rho_0}} T^{\frac{1}{\rho_0}} \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \le \frac{\varepsilon}{4} \\
and \\
C_2\delta^{2+\frac{2}{\rho_0'}} T^{\frac{1}{\rho_0'}} \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \le \frac{\varepsilon}{4}.
\end{cases}$$

So, if

$$T \le \min\Big\{\Big(\frac{\varepsilon}{4C_2\delta^{2+\frac{2}{\rho_0}} \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}}\Big)^{\rho_0}, \Big(\frac{\varepsilon}{4C_2\delta^{2+\frac{2}{\rho_0'}} \|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}}\Big)^{\rho_0'}\Big\},$$

then $\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{X_T} \leq \frac{\varepsilon}{2}$. This result with (5.2) yields that $\left\|\left(e^{t\Delta}v_0, e^{t\Delta}w_0\right)\right\|_{X_T} \leq \varepsilon$. Therefore, applying Lemma 2.3 again, we get a fixed point of $\mathfrak{F}$ in the closed ball $\bar{B}(0,2\varepsilon) = \{x \in X_T \;:\; \|x\|_{X_T} \leq 2\varepsilon\}$. Thus, for any arbitrary $(v_0, w_0) \in \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$, (1) has a unique local solution in $\bar{B}(0,2\varepsilon)$.

Regularity: We khow if $(v,w) \in X_T \times X_T$ is a solution of (1), then we can show that

$$\nabla\cdot(v\nabla\phi)\,,\;\; \nabla\cdot(w\nabla\phi) \in \mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right).$$

By using the definition of the Fourier-Besov-Morrey spaces, we have

$$\begin{aligned}
&\|v(t_1) - v(t_2)\|^q_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&\leq \sum_{j\leq N}\left(2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})}\|\hat{v}_j(t_1) - \hat{v}_j(t_2)\|_{\mathrm{M}^\lambda_p}\right)^q + 2\sum_{j>N}\left(2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})}\|\hat{v}_j(t)\|_{L^\infty(I,\mathrm{M}^\lambda_p)}\right)^q,
\end{aligned}$$

where $\hat{v_j} = \varphi_j\hat{v}$. For any small constant $\varepsilon > 0$, let N be large enough so that

$$\sum_{j>N} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\|\hat{v}_j(t)\|^q_{L^\infty(I,\mathrm{M}^\lambda_p)} \leq \frac{\varepsilon}{4}.$$

According to Taylor's formula and using the same arguments as in [20], we get

$$\begin{aligned}
&\sum_{j\leq N}\left(2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})}\|\hat{v}_j(t_1) - \hat{v}_j(t_2)\|_{\mathrm{M}^\lambda_p}\right)^q \\
&\lesssim |t_1-t_2|^q \sum_{j\leq N} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\left\|(\widehat{\partial_t u})_j\right\|^q_{L^1(I,\mathrm{M}^\lambda_p)} \\
&\lesssim |t_1-t_2|^q \times \|\partial_t u\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \\
&\lesssim |t_1-t_2|^q \times \left(\|\Delta v\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} + \|\nabla\cdot(v\nabla\phi)\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)}\right) \\
&\lesssim |t_1-t_2|^q \times \left(\|v\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} + \|\nabla\cdot(v\nabla\phi)\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)}\right) \\
&\lesssim |t_1-t_2|^q \times \left(\|v_0\|^q_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + 2\|\nabla\cdot(v\nabla\phi)\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)}\right).
\end{aligned}$$

Thus, we obtain the continuity of v in time t.

Similarly, we use the same discussion to get the continuity of w in time t. Hence $(v,w) \in C\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)$, and we are done.

6 Conclusion

In this work, we considered the Debye-Hückel system. The homogeneous Littlewood-Paley decomposition and Bony's paraproduct decomposition are the important tools to obtain the (global and local) well-posedness result for such system. Our results extend and complement the previous ones of Zhao, Jihong, Liu, and Cui [20].

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A New Fractional-Order 3D Chaotic System Analysis and Synchronization

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Abstract: In this work, a fractional-order form of a novel 3D chaotic system is introduced. Firstly, this fractional system can display chaotic behavior for a given minimal commensurate order. Secondly, theoretical and numerical solution representation is given by exploiting the Adams–Bashforth–Moulton algorithm for the presented novel fractional-order system. Thirdly, we have studied full-state hybrid projective synchronization (FSHPS) type the novel 3D fractional-order system and the fractional-order hyper-chaotic Lorenz system based on the definition of this kind of synchronization and the Lyapunov theory of stability of linear fractional-order systems. Finally, numerical simulations are given to show the effectiveness of the proposed controller via the improved Adams–Bashforth–Moulton algorithm.

Keywords: *fractional-order system; chaotic system; FSHPS; Lyapunov theory; synchronization.*

Mathematics Subject Classification (2010): 37B55, 34C28, 34D08, 37B25, 37D45, 93C40, 93D05.

1 Introduction

During these years the synchronization of chaotic dynamical systems has generated a great interest among researchers in nonlinear sciences, in view of its practical applications, many types of synchronization have been reported in the literature; these include complete, generalized, anticipated, lag, measure, projective, phase, reduced order and adaptive synchronizations. These concepts of synchronization have led to the creation of many methods of controlling chaos and synchronization by many researchers, including

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the active control method [2], sliding mode control [3, 40], backstepping control [4], adaptive control [5, 39, 40], fuzzy control [6], passive control [7], projective synchronization method [8], function projective synchronization method [9], etc.

In recent years, fractional-order systems have become a hot research field addressed by many researchers such as [10-16, 25-33]. Study of chaos and synchronization in fractional-order dynamical systems has attracted considerable attention [17-19] due to its powerful potential applications in different fields such as secure communication, telecommunications, cryptography [20-21]. Diverse techniques of different types of synchronization have been proposed and developed for fractional-order systems. These include the sliding mode control [22-23], active control technique [24-25], function projective synchronization [26], and modified projective synchronization [27], hybrid projective synchronization [28], full state hybrid projective synchronization [29], inverse full state hybrid projective synchronization [35-36] and others, see, for example, [30-34, 40-41].

This paper is organized as follows. In Section 2, the Caputo fractional-order derivative definition is given with some notes. In Section3, we described the new form of the three-dimensional system in fractional order, Section 4 is devoted to the study of the FSHP synchronization between the novel 3D fractional-order system and the fractional-order hyper-chaotic Lorenz system based on the definition of this kind of synchronization and the Lyapunov theory of stability. In Section 5, we present the numerical results to verify the effectiveness of the method. Finally, the conclusion is mentioned in Section 6.

2 Preliminaries

In general, for fractional derivatives there are three well known definitions, that is, the Riemann-Liouville, Grünwald-Letnikov and Caputo definitions. The Caputo fractional derivative is much preferred since it is more popular in real application and the initial conditions for fractional-order differential equations with the Caputo derivative are in the same form as for integer-order differential equations.

The Caputo fractional derivative of $f(t)$ is given as

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(\tau - n)^{q-n+1}} d\tau \tag{1}$$

for $n-1 < q \leq n, n \in N, t > 0$. $\Gamma(\cdot)$ is the gamma function.

Some other important properties of the fractional derivatives and integrals can be found in several works ([10-15], etc). Geometric and physical interpretation of fractional integration and fractional differentiation were exactly described in [16].

2.1 Problem Formulation

We consider the drive system given by

$$D_t^{q_i} x_i(t) = f_i(X(t)), i = 1, .., n, \tag{2}$$

where $X(t) = (x_1, x_2, ..., x_n)^T$ is the state vector of the system (2), $f_i : R^n \longrightarrow R^n$, for $i = 1, .., n$, are nonlinear functions, and the response system is given by

$$D_t^{q_i} y_i(t) = \sum_{j=1}^{m} b_{ij} y_j(t) + g_i(Y(t)) + u_i, i = 1, .., m, \tag{3}$$

where $Y(t) = (y_1, y_2, ..., y_m)^T$ is the state vector of the system (3), $g_i : R^m \longrightarrow R^m$, for $i = 1, .., m$, are nonlinear functions, $0 < q_i < 1, D_t^{q_i}$ is the Caputo fractional derivative of order q_i. u_i, $i = 1, .., m$, are controllers to be designed so that the system (2) and the system (3) to be synchronized.

Lemma 2.1 *The trivial solution of the following fractional-order system* [35]*:*

$$D_t^q X(t) = F(X(t)), \tag{4}$$

where D_t^q is the Caputo fractional derivative of order q , $0 < q \leq 1$, $F : R^n \longrightarrow R^n$, is asymptotically stable if there exists a positive definite function $V(X(t))$ such that $D_t^q V(X(t)) < 0$ for all $t > 0$.

Lemma 2.2 $\forall X(t) \in R^n, \forall q \in [0,1]$, *and* $\forall\ t > 0$

$$\frac{1}{2} D_t^q \left(X^T(t) X(t) \right) \leq X^T(t) D_t^q (X(t)). \tag{5}$$

Now, we introduce the definition of FSHPS between the master and slave systems.

Definition 2.1 FSHPS occurs between the master and slave systems (2) and (3) when there exist controllers u_i, $i = 1, 2, ..., n$, and given real numbers $(\alpha_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ such that the synchronization errors

$$e_i(t) = y_i(t) - \sum_{j=1}^{n} \alpha_{ij} x_j(t), i = 1, .., m \tag{6}$$

satisfy $\lim_{t\rightarrow+\infty} e_i(t) = 0$.

3 Description of the Novel Chaotic System

In this research work, we consider the fractional-order form of the integer-order 3D chaotic system introduced in [38] and given by

$$\begin{cases} D_t^{q_1} x = a(y - x) + byz, \\ D_t^{q_2} y = (c - a)x + cy - xz, \\ D_t^{q_3} z = xy - z, \end{cases} \tag{7}$$

where a, b, c are positive real parameters with $b \neq 1$. In the first part of this paper, we shall show that the system (7) is chaotic when the parameters a, b and c take the values

$$a = 15, b = 8/3, c = 10. \tag{8}$$

3.1 Dynamical behavior

For the values of parameters (8), the system (7) has five equilibrium points

$$\begin{cases} E_0 = (0, 0, 0), \\ E_{1,3} = (\pm 4.049, \mp 3.1659, -12.819), \\ E_{2,4} = (\pm 1.7464, \pm 1.2563, 2.194). \end{cases} \tag{9}$$

For E_0, we obtain the eigenvalues

$$\lambda_1 = -1, \lambda_2 = -11.514, \lambda_3 = 6.5139. \tag{10}$$

This implies that E_0 is an unstable saddle point.

With the same method, the eigenvalues of the Jacobian at E_1 and E_3 are

$$\lambda_1 = 3.3706 - 8.3184i, \lambda_2 = 3.3706 + 8.3184i, \lambda_3 = -12.741 \tag{11}$$

and the eigenvalues of the Jacobian at E_2 and E_4 are

$$\lambda_1 = 1.0824 + 4.5105i, \lambda_2 = 1.0824 - 4.5105i, \lambda_3 = -8.1648, \tag{12}$$

then E_1 and E_3 are two unstable saddle-focus points and E_2 and E_4 are two unstable saddle-focus points because none of the eigenvalues have real part zero and λ_1, λ_2 are complex.

In the case of the comensurate-order system, where $q_1 = q_2 = q_3$, a necessary condition for the fractional-order nonlinear system (7) to be chaotic is $q > \frac{2}{\pi} \arctan\left(\frac{|Im(\lambda)|}{Re(\lambda)}\right)$, where λ are the eigenvalues of the saddle equilibrium point of index two in system (7). From the above eigenvalues we can determine a minimal commensurate order to keep the system (7) chaotic and it is $q > 0.75491$ for E_1 and E_3 and $q > 0.85006$ for E_2 and E_4. Thus, the necessary condition of existence of chaos in fractional-order system (7) is $q > 0.85006$.

The necessary condition for the system (7) to exhibit chaotic oscillations in the incommensurate case is $\frac{\pi}{2M} - min_i(|arg(\lambda_i\,(J_E))|) > 0$, where $\lambda_i\,(J_E)$, $i = 1, 2, 3$, are the eigenvalues of the Jacobian matrix J_E of the system (7) at the equilibrium E, M is the LCM of the fractional orders. For example, if $q_1 = 0.9, q_2 = 0.9, q_3 = 0.8$, then $M = 10$. The characteristic equation of the system evaluated at the equilibrium E_i is $det(diag[\lambda^{Mq_1}, \lambda^{Mq_2}, \lambda^{Mq_3}] - J_{E_i}) = 0$, i.e., $det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{E_i}) = 0$, $i = 1, 2, 3, 4$.

We get $det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{E_0}) = 0, det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{E_{1,3}}) = 0, det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{E_{2,4}}) = 0$. This yields: $\lambda^{26} + \lambda^{18} + 5\lambda^{17} + 5\lambda^9 - 75\lambda^8 - 75 = 0, \lambda^{26} + \lambda^{18} + 5\lambda^{17} - 5.3334\lambda^9 - 3.04 \times 10^{-4}\lambda^8 + 1026.4 = 0$, $\lambda^{26} + \lambda^{18} + 5\lambda^{17} + 3.8412\lambda^9 + 2.094 \times 10^{-3}\lambda^8 + 175.67 = 0$.

From the roots of the above equations, we find $\lambda = 1.2315$ whose argument is zero, which is the minimum argument, and hence the necesary stability condition holds because $\frac{\pi}{2M} - 0 > 0$.

3.2 Application of Adams–Bashforth–Moulton algorithm to the system (7)

By exploiting the Adams–Bashforth–Moulton algorithm [37], the novel fractional-order chaotic system (7) can be written as

$$\begin{cases} x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)} \begin{pmatrix} a\left(y^p_{n+1} - x^p_{n+1}\right) + by^p_{n+1}z^p_{n+1} \\ +\sum_{j=1}^{n} a_{1,j,n+1}\left(a\left(y_j - x_j\right) + by_jz_j\right) \end{pmatrix}, \\ y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)} \begin{pmatrix} (c-a)\,x^p_{n+1} + cy^p_{n+1} - x^p_{n+1}z^p_{n+1} \\ +\sum_{j=1}^{n} a_{2,j,n+1}\left((c-a)\,x_j + cy_j - x_jz_j\right) \end{pmatrix}, \\ z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)} \left(x^p_{n+1}y^p_{n+1} - z^p_{n+1} + \sum_{j=1}^{n} a_{3,j,n+1}\left(x_jy_j - z_j\right)\right) \end{cases} \tag{13}$$

in which

$$\left\{\begin{array}{c} x^p_{n+1} = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=1}^{n} b_{1,j,n+1} \left(a\left(y_j - x_j\right) + b y_j z_j\right), \\ y^p_{n+1} = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=1}^{n} b_{2,j,n+1} \left(\left(c-a\right) x_j + c y_j - x_j z_j\right), \\ z^p_{n+1} = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=1}^{n} b_{3,j,n+1} \left(x_j y_j - z_j\right) \end{array}\right. \tag{14}$$

with

$$\left\{\begin{array}{c} b_{1,j,n+1} = \frac{h^{q_1}}{q_1} \left((n-j+1)^{q_1} - (n-j)^{q_1}\right), \\ b_{2,j,n+1} = \frac{h^{q_2}}{q_2} \left((n-j+1)^{q_2} - (n-j)^{q_2}\right), \\ b_{3,j,n+1} = \frac{h^{q_3}}{q_3} \left((n-j+1)^{q_3} - (n-j)^{q_3}\right) \end{array}\right. \tag{15}$$

and

$$a_{i,j,n+1} = \left\{\begin{array}{c} (n)^{q_i+1} - (n-q_i)(n+1)^{q_i}, j = 0, \\ (n-j+2)^{q_i+1} - (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, 1 \le j \le n, \quad i = 1,2,3, \\ 1, j = n+1. \end{array}\right. \tag{16}$$

Applying the above algorithm, numerical solution of a fractional-order system can be computed.

Fig.1 depicts the simulation result (double scroll-attractor) for the fractional-order system (7) projected onto the $x - z$ plane, computed for the simulation time $Tsim = 100s$ and $q_1 = q_2 = q_3 = 0.88$ and time step $h = 0.005$, $a = 15$, $b = 8/3$, $c = 10$, $(x(0), y(0), z(0)) = (1, -1, 2)$.

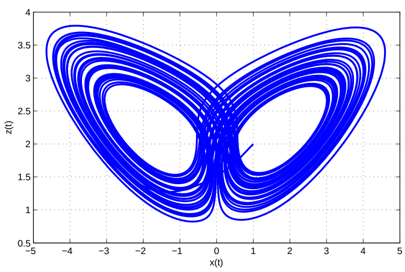

Figure 1: Simulation result for the fractional-order system (7) projected onto the $x - z$ plane.

Also, the simulation result for the fractional-order system (7) projected onto the $z - y$ plane for $a = 15, b = 8/3, c = 10$, $q_1 = q_2 = q_3 = 0.95, (x(0), y(0), z(0)) = (-2, 5, -10)$ is shown in Fig.2.

4 FSHP Synchronization of Fractional-Fractional Order Systems

In this section, the new fractional-order chaotic system (7) and fractional-order hyperchaotic Lorenz system are used to achieve FSHPS between these systems. Thus, as the driving system, we consider the novel fractional-order chaotic system given by

$$\left\{\begin{array}{l} D_t^{q_1} x_1 = a(x_2 - x_1) + b x_2 x_3, \\ D_t^{q_2} x_2 = (c-a) x_1 + c x_2 - x_1 x_3, \\ D_t^{q_3} x_3 = x_1 x_2 - x_3, \end{array}\right. \tag{17}$$

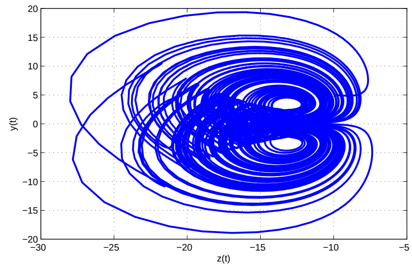

Figure 2: Simulation result for the fractional-order system (7) projected onto the $z-y$ plane.

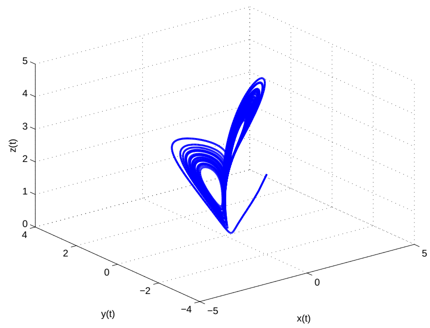

Figure 3: Simulation result for the fractional-order system (7) projected onto the $x-y-z$ plane for $q_1 = q_2 = 0.9, q_3 = 0.8, (x(0), y(0), z(0)) = (1, -1, 2)$.

where $a = 15, b = 8/3, c = 10$, and as the response system, we consider the controlled fractional-order hyper-chaotic Lorenz system given by

$$\begin{cases} D_t^{q_1} y_1 = a\left(y_2 - y_1\right) + y_4 + u_1, \\ D_t^{q_2} y_2 = cy_1 - y_2 - y_1 y_3 + u_2, \\ D_t^{q_3} y_3 = -by_3 + y_1 y_2 + u_3, \\ D_t^{q_4} y_4 = -y_2 y_3 + dy_4 + u_4, \end{cases} \tag{18}$$

where $a = 10, b = 28, c = 8/3, d = -1$ and with the fractional-orders of the system $(q_1, q_2, q_3, q_4) = (0.98, 0.98, 0.98, 0.98)$.

In view of Definition 2.1, the state errors for (17) and (18) are

$$e_i = y_i - \sum_{j=1}^{3} \alpha_{ij} x_j, i = 1, 2, 3, 4. \tag{19}$$

This gives

$$D_t^{q_i} e_i = D_t^{q_i} y_i - D_t^{q_i} \left(\sum_{j=1}^{3} \alpha_{ij} x_j \right), i = 1, 2, 3, 4. \tag{20}$$

Consequently, the error dynamic system is given by

$$\begin{cases} D_t^{q_1} e_1 = D_t^{q_1} y_1 - D_t^{q_1} \left(\sum_{j=1}^{3} \alpha_{1j} x_j \right), \\ D_t^{q_2} e_2 = D_t^{q_2} y_2 - D_t^{q_2} \left(\sum_{j=1}^{3} \alpha_{2j} x_j \right), \\ D_t^{q_3} e_3 = D_t^{q_3} y_3 - D_t^{q_3} \left(\sum_{j=1}^{3} \alpha_{3j} x_j \right), \\ D_t^{q_4} e_4 = D_t^{q_4} y_4 - D_t^{q_4} \left(\sum_{j=1}^{3} \alpha_{4j} x_j \right), \end{cases} \tag{21}$$

i.e.,

$$\begin{cases} D_t^{q_1} e_1 = a (y_2 - y_1) + y_4 + u_1 - D_t^{q_1} (\alpha_{11} x_1 + \alpha_{12} x_2 + \alpha_{13} x_3), \\ D_t^{q_2} e_2 = c y_1 - y_2 - y_1 y_3 + u_2 - D_t^{q_2} (\alpha_{21} x_1 + \alpha_{22} x_2 + \alpha_{23} x_3), \\ D_t^{q_3} e_3 = -b y_3 + y_1 y_2 + u_3 - D_t^{q_3} (\alpha_{31} x_1 + \alpha_{32} x_2 + \alpha_{33} x_3), \\ D_t^{q_4} e_4 = -y_2 y_3 + d y_4 + u_4 - D_t^{q_4} (\alpha_{41} x_1 + \alpha_{42} x_2 + \alpha_{43} x_3). \end{cases} \tag{22}$$

The system (22) can be described as

$$\begin{cases} D_t^{q_i} e_i = \sum_{j=1}^{4} b_{ij} e_j (t) + R_i + u_i, i = 1, 2, 3, 4, \end{cases} \tag{23}$$

where

$$\begin{cases} R_1 = -D_t^{q_1} (\alpha_{11} x_1 + \alpha_{12} x_2 + \alpha_{13} x_3) + \sum_{j=1}^{4} b_{1j} (y_j (t) - e_j (t)), \\ R_2 = -y_1 y_3 - D_t^{q_2} (\alpha_{21} x_1 + \alpha_{22} x_2 + \alpha_{23} x_3) + \sum_{j=1}^{4} b_{2j} (y_j (t) - e_j (t)), \\ R_3 = y_1 y_2 - D_t^{q_3} (\alpha_{31} x_1 + \alpha_{32} x_2 + \alpha_{33} x_3) + \sum_{j=1}^{4} b_{3j} (y_j (t) - e_j (t)), \\ R_4 = -y_2 y_3 - D_t^{q_4} (\alpha_{41} x_1 + \alpha_{42} x_2 + \alpha_{43} x_3) + \sum_{j=1}^{4} b_{4j} (y_j (t) - e_j (t)). \end{cases} \tag{24}$$

Rewrite error system (23) in the compact form

$$D_t^q e = Be + R + U, \tag{25}$$

where $B = (b_{ij})_{4\times4}$ and $e = (e_1, e_2, e_3, e_4)^T, R = (R_i)_{1\leq i\leq 4}, U = (u_i)_{1\leq i\leq 4}$.

Theorem 4.1 *FSHPS between the master system* (17) *and the slave system* (18) *occurs under the following control law:*

$$U = -(R + De), \tag{26}$$

where D is a 4×4 feedback gain matrix selected so that $B - D$ is a negative definite matrix.

Proof. By inserting (26) into (25), we get

$$D_t^q e = (B - D) e, \tag{27}$$

where $B = (b_{ij}), D = (d_{ij})$ are two 4×4 matrices and $e = (e_1, e_2, e_3, e_4)^T$ is the error vector of the system. If we choose matrix D such that $B - D$ is negative, then all the eigenvalues $\lambda_i, i = 1, 2, 3, 4$, of $B - D$ stay in the left-half plane, i.e., $Re(\lambda_i) < 0$, and if a candidate Lyapunov function is chosen as

$$V = \sum_{i=1}^{4} \frac{1}{2} e_i^2, \tag{28}$$

then the time Caputo fractional derivative of order 0.98 of V along the trajectory of system (27) is as follows:

$$D_t^{0.98}V = \sum_{j=1}^{4} D_t^{0.98}\left(\frac{1}{2}e_i^2\right). \tag{29}$$

Using Lemma 2.2, we get

$$\begin{aligned} D_t^{0.98}V &\leq \sum_{j=1}^{4} e_i D_t^{0.98} e_i \qquad (30)\\ &= \lambda_1 e_1^2 + \lambda_2 e_2^2 + \lambda_3 e_3^2 + \lambda_4 e_4^2 < 0, \qquad (31)\end{aligned}$$

which ensures, according to Lemma 2.1, that the trivial solution of the fractional-order system (27) is asymptotically stable. Hence the FSHP synchronization between the system (17) and the system (18) is achieved. This completes the proof.

5 Numerical Simulation

According to the above method, for FSHPS we have

$$B = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 8/3 & -1 & 0 & 0 \\ 0 & 0 & -28 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{32}$$

and

$$\begin{cases} R_1 = 10e_1 - 10e_2 - e_4 + 25x_1 - 35x_2 + 5x_3 - 10y_1 + 10y_2 + y_4 - 5x_1x_2 + 2x_1x_3 - \frac{8}{3}x_2x_3, \\ R_2 = e_2 - \frac{8}{3}e_1 + 35x_1 - 40x_2 - 3x_3 + \frac{8}{3}y_1 - y_2 + 3x_1x_2 + x_1x_3 - \frac{16}{3}x_2x_3 - y_1y_3, \\ R_3 = 28e_3 + 75x_1 - 90x_2 + 2x_3 - 28y_3 - 2x_1x_2 + 3x_1x_3 - \frac{32}{3}x_2x_3 + y_1y_2, \\ R_4 = e_4 + 100x_1 - 110x_2 + 7x_3 - y_4 - 7x_1x_2 + 2x_1x_3 - 16x_2x_3 - y_2y_3, \end{cases} \tag{33}$$

and

$$(u_1, u_2, u_3, u_4)^T = -\left(R + D\,(e_1, e_2, e_3, e_4)^T\right), \tag{34}$$

i.e.,

$$\begin{cases} u_1 = 35x_2 - 25x_1 - 15e_1 - 5x_3 + 10y_1 - 10y_2 - y_4 + 5x_1x_2 - 2x_1x_3 + \frac{8}{3}x_2x_3, \\ u_2 = 40x_2 - 35x_1 - 5e_2 + 3x_3 - \frac{8}{3}y_1 + y_2 - 3x_1x_2 - x_1x_3 + \frac{16}{3}x_2x_3 + y_1y_3, \\ u_3 = 90x_2 - 75x_1 - 30e_3 - 2x_3 + 28y_3 + 2x_1x_2 - 3x_1x_3 + \frac{32}{3}x_2x_3 - y_1y_2, \\ u_4 = 110x_2 - 100x_1 - 10e_4 - 7x_3 + y_4 + 7x_1x_2 - 2x_1x_3 + 16x_2x_3 + y_2y_3 \end{cases} \tag{35}$$

for the chosen $(\alpha_{ij})_{4\times 3} = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & -3 \\ 4 & 3 & 2 \\ 6 & 2 & 7 \end{pmatrix}, D = \begin{pmatrix} 5 & 10 & 0 & 1 \\ 8/3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}.$

Then the error system is given by

$$\begin{pmatrix} D_t^{q_1} e_1 \\ D_t^{q_2} e_2 \\ D_t^{q_3} e_3 \\ D_t^{q_4} e_4 \end{pmatrix} = \begin{pmatrix} -15 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & -10 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \tag{36}$$

and then the eigenvalues of the matrix $(B - D)$ are given by $\lambda_1 = -15$, $\lambda_2 = -5$, $\lambda_3 = -30$, $\lambda_4 = -10$, which all are negative. Hence the error system is asymptotically stable and the synchronization between the two systems (17) and (18) is achieved.

We used the improved classical Adams–Bashforth–Moulton method to show the effectiveness of the proposed controller by solving the system (36) for $(q_1, q_2, q_3, q_4) = (0.98, 0.98, 0.98, 0.98)$ and with the initial conditions chosen as $(e_1(0), e_2(0), e_3(0), e_4(0)) = (-2, 1, -5, 1)$. In Figs.4-7, the time-history of the synchronization errors $e_1(t)$; $e_2(t)$; $e_3(t)$; $e_4(t)$ is depicted.

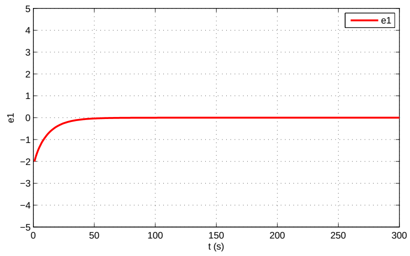

Figure 4: Time-response of the error $e_1(t)$ under the controller (35) $for FSHP$.

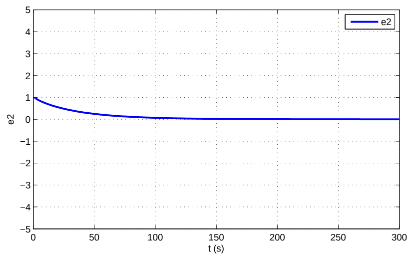

Figure 5: Time-response of the error $e_2(t)$ under the controller (35) $for FSHP$.

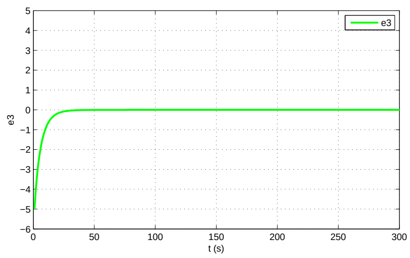

Figure 6: Time-response of the error $e_3(t)$ under the controller (35) $for FSHP$.

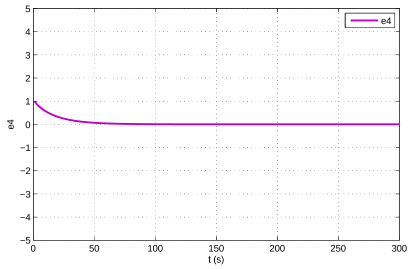

Figure 7: Time-response of the error $e_4(t)$ under the controller (35) $forFSHP$.

6 Conclusion

In this work, we have presented a fractional-order form of a new 3D chaotic system based on the Caputo derivative definition, where we have proven that this system has chaotic behavior starting with a specific value of minimal commensurate order. A theoretical and numerical solution representation was given using the Adams–Bashforth–Moulton algorithm for this system. Also, full-state hybrid projective synchronization (FSHPS) between the novel 3D fractional-order system and the fractional-order hyper-chaotic Lorenz system has been studied based on the definition of this kind of synchronization and the Lyapunov theory of stability for linear fractional-order systems. Numerical simulations are given to validate the effectiveness of the proposed controller via the improved Adams–Bashforth–Moulton algorithm in Matlab. Further studies regarding practical applications of this fractional system will be carried out in our next works.

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Study of a Non-Isothermal Hooke Operator in Thin Domain with Friction on the Bottom Surface

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Abstract: This work is focused on the study of the asymptotic behavior of a coupled problem that consists of an elastic body and the change of heat. The friction exerted on the body is nonlinear of Coulomb type in a thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$. As a first step, we give the variational formulation of the problem and the establishment of the existence and uniqueness results for the weak solution. We proceed to the asymptotic analysis. To do this, we use the scale change following the third component and new unknowns to conduct the study on a domain Ω independent of ε. Then we prove some estimates for the displacement and the temperature. Finally, these estimates allow us to have the limit problem and prove the uniqueness of the solution.

Keywords: *a priori inequalities; boundary conditions; Coulomb law; coupled problem; elastic body; Fourier law.*

Mathematics Subject Classification (2010): 35R35, 76F10, 78M35, 70K45, 70K20, 93-00, 70K20.

1 Introduction

In solid mechanics, thin structures are widely used in several fields of industry, for example, in underwater industry, aerospace, civil engineering and in common constructions, in the field of energy, industrial design, and even in the living world. We also find the use of thin structures in the metallurgical industry, in particular in the rolling process of thin sheets etc. More details can be seen in [1]. In mathematical literature, the problems in thin areas and especially in the elasticity of thin films, plates and shells have already been studied for more than a century. For example, Ciarlet in [10] and Destuynder in [12] have studied the equilibrium states of a thin plate $\Omega \times (-\varepsilon, +\varepsilon)$ under external forces, where Ω is a smooth domain in $\mathbb{R}^2$ and ε is a small parameter, to justify the

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two-dimensional model of the plates. In recent years, many authors have applied asymptotic methods in three-dimensional or two-dimensional elasticity and viscosity problems to derive new two-dimensional or one-dimensional reduced models. The importance of asymptotic methods is that they can be used in place of full three-dimensional models when the thickness is small enough. In addition, two-dimensional models are simpler than their three-dimensional counterpart, which facilitates their study. They also allow less costly digital simulations than the three-dimensional ones.

Our goal in this paper is to give the asymptotic behavior of a non-isothermal Hooke operator in a thin domain with Coulomb friction on the bottom surface. One of the objectives of this study is to obtain a two-dimensional equation that allows a reasonable description of the phenomenon occurring in a three-dimensional domain by passing the limit to 0 on the small thickness of the domain (3D). The scientific research in mechanics are articulated around two main components: one devoted to the laws of behavior and the other one to the boundary conditions imposed on the body. Here we describe real phenomena which transform into mathematical problems with boundary conditions and with certain types of friction, the type of problem that is presented here is very common in application. The physical domains are defined, where the height is much smaller then the length, as the problem of elasticity and viscoelasticity of a tire. We consider a non isothermal elastic body with Coulomb free boundary friction conditions in the stationary regime occupying a bounded, homogeneous domain $\Omega^\varepsilon \subset \mathbb{R}^3$, where $(0 < \varepsilon < 1)$ is a small parameter that will tend to zero. The boundary of Ω^ε will be noted $\Gamma^\varepsilon = \overline{\Gamma_1^\varepsilon} \cup \overline{\Gamma_L^\varepsilon} \cup \overline{w}$ and assumed to be Lipschitz, such that Γ_1^ε is the upper surface of the equation $x_3 = \varepsilon h(\acute{x})$, Γ_L^ε is the lateral surface and w is a fixed bounded domain of $\mathbb{R}^3$ with $x_3 = 0$, which is a bottom of the domain Ω^ε. Several works have been done on the mechanical contact with the various laws of behavior and various boundary conditions of friction close to our problem, yet these items were based only on the existence and uniqueness of the weak solution. Let us mention, for example, in [2], Bayada et al. are engaged in the asymptotic and numerical analysis for the unilateral contact problem with Coulomb friction between two general elastic bodies and a thin elastic soft layer. Paumier in [21] performs an asymptotic modeling of a unilateral problem of a thin plate. He demonstrates that this three-dimensional problem with friction tends towards another two-dimensional one without friction. The justification by the asymptotic analysis for the elastic plates is given by Gilbert in [16] and for the shells is given by Chacha in [9].

More recently, some research papers have been written dealing with the asymptotic analysis of a boundary value problem governed by the elasticity system. For example, the asymptotic behavior of the dynamical problem of isothermal and non-isothermal elasticity with non linear friction of Tresca type was studied in [5, 24]. Also, the authors in [4] carried out the asymptotic analysis of a frictionless contact between two elastic bodies in a stationary regime in a three-dimensional thin domain with friction. The reader can also consult certain works concerning partial differential equations posed in different thin domains, see, for example, [15, 17, 18, 22, 23].

This paper is organized as follows. As a first step, we give the variational formulation of the problem and demonstrate the results of existence and uniqueness for the weak solution, then we move on to the asymptotic analysis. For this, using the change of scale according to the third component we conduct the study on a domain Ω which does not depend on ε. Then, by the use of different inequalities, we prove some estimates for the displacement and the temperature, which allow us to go to the limit when ε tends towards zero in the variational formulation. Finally, our main result is the proof of the existence and uniqueness of the limit of a weak solution to the problem described in the abstract.

2 Statement of the Problem

In this section, we first define the thin domain and some sets necessary to study the asymptotic behavior of the solutions. Next, we introduce the problem considered in the thin domain. We finish this section giving the weak variational formulations of our problem.

2.1 The domain

We consider a mathematical problem governed to the stationary equation for an elasticity system in three dimensional bounded domain $\Omega^\varepsilon \subset \mathbb{R}^3$ with boundary $\Gamma^\varepsilon = \overline{\Gamma}_L^\varepsilon \cup \overline{\Gamma}_1^\varepsilon \cup \overline{w}$. We denote by $|.|$ the Euclidean norm on $\mathbb{R}^2$. Let w be a fixed, bounded domain of $\mathbb{R}^3$ with $x_3 = 0$. We suppose that w has a Lipschitz continuous boundary and is the bottom of the domain. The upper surface Γ_1^ε is defined by $x_3 = \varepsilon h(\acute{x}) = \varepsilon h(x_1, x_2)$. We introduce a small parameter ε, that will tend to zero, where h is a function of class C^1 defined on w such that $0 < h_* < h(\acute{x}) < h^*$, for all $(\acute{x}, 0) \in \omega$, with

$$\Omega^\varepsilon = \left\{(\acute{x}, z) \in \mathbb{R}^3 \colon \ (\acute{x}, 0) \in \omega, 0 < x_3 < \varepsilon h(\acute{x})\right\}.$$

We introduce the following functional framework:

$$H^1(\Omega^\varepsilon)^3 = \left\{ v \in (L^2(\Omega^\varepsilon))^3; \qquad \frac{\partial v_i}{\partial x_j} \in L^2(\Omega^\varepsilon); \quad \forall i, j = 1, 2, 3 \right\}.$$

We define the closed non-empty convex of $H^1(\Omega^\varepsilon)^3$:

$$V^\varepsilon = \left\{ \varrho \in \left(H^1(\Omega^\varepsilon)\right)^3; \ \ \varrho = G^\varepsilon \text{ on } \Gamma_L^\varepsilon \ , \ \varrho = 0 \text{ on } \Gamma_1^\varepsilon \text{ and } \varrho.n = 0 \text{ on } w \right\},$$

where G^ε is defined below. We note by $H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}$ the vector sub-space of $H^1(\Omega^\varepsilon)$:

$$H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega^\varepsilon) = \left\{ \varrho \in H^1(\Omega^\varepsilon) : \varrho = 0 \text{ on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon \ \right\}.$$

The spaces Ω^ε, $H^1(\Omega^\varepsilon)^3$, $V^{\ \varepsilon}$ and $H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega^\varepsilon)$ are the domain in which we study the asymptotic behavior of elasticity, the Sobolev space, the closed convex, and the vectorial sub-space of $H^1(\Omega^\varepsilon)$ are endowed with their natural norms and scalar product.

2.2 The problem

We assume that the deformations of an elastic body are governed by the following equations. The law of conservation of momentum is $div(\sigma^\varepsilon) + f^\varepsilon = 0$, we designate by $\sigma^\varepsilon = \left(\sigma_{i,j}^\varepsilon\right)_{1 \le i,j \le 3}$ the stress tensor and by $D = (d_{i,j})_{1 \le i,j \le 3}$ the tensor of deformation: $d_{i,j}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$; $1 \le i, j \le 3$.

It is supposed that the law of behavior follows the law of Hooke

$$\sigma_{i,j}^\varepsilon(u) = 2\mu(T^\varepsilon) d_{i,j}(u^\varepsilon) + \lambda(T^\varepsilon) d_{kk}(u^\varepsilon)\delta_{ij}.$$

$u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ is the displacement of the elastic body, λ and μ are the coefficients of Lamé with $\lambda + \mu \geq 0$, T^ε is the temperature, $u_\tau^\varepsilon, u_n^\varepsilon$ are the tangential and normal components of u^ε on the boundary w given by $u_n^\varepsilon = u^\varepsilon.n$, $u_{\tau_i}^\varepsilon = u_i^\varepsilon - u_n^\varepsilon.n_i$ and $\sigma_\tau^\varepsilon, \sigma_n^\varepsilon$ are the tangential and normal components of σ^ε given by

$$\sigma_n^\varepsilon = (\sigma^\varepsilon.n_i)\,.n_j, \sigma_{\tau_i}^\varepsilon = \sigma_{ij}^\varepsilon.n_j - \sigma_n^\varepsilon.n_i.$$

The law of conservation of energy is given by

$$\begin{cases} -\nabla(K^\varepsilon \nabla T^\varepsilon) = \sigma^\varepsilon : D(u^\varepsilon) + r^\varepsilon(T^\varepsilon), \\ \sigma^\varepsilon : D(u^\varepsilon) = \sum\limits_{i,j=1}^{3} \sigma_{i,j}^\varepsilon d_{i,j}(u^\varepsilon), \end{cases}$$

where K^ε is the thermal conductivity and $r^\varepsilon(T^\varepsilon)$ is the heat source.

To describe the boundary conditions, let us introduce first a vector function $g = (g_1, g_2, g_3)$ in $H^{1/2}(\Gamma^\varepsilon)$ such that $\int_{\Gamma^\varepsilon} g.nds = 0$, then according to ([6]) there exists a function G^ε:

$$G^\varepsilon \in (H^1(\Omega^\varepsilon))^3 \quad \text{with} \quad G^\varepsilon = g \text{ on } \Gamma^\varepsilon.$$

Also, we suppose that

$$g_3 = u_3 = 0 \text{ and } s = g \text{ on } \omega.$$

- On Γ_1^ε, no slip condition is given. The upper surface is assumed to be fixed so that $u^\varepsilon = 0$.
- On Γ_L^ε, the displacement is unknown and parallel to the $w-$plane: $u^\varepsilon = g$ with $g_3 = 0$.
- On w, there is no flux condition across w so that $u^\varepsilon.n = 0$.
- The tangential velocity on w is unknown and satisfies the Coulomb friction law:

$$\begin{cases} |\sigma_\tau^\varepsilon| < F^\varepsilon\,|\sigma_n^\varepsilon| \Rightarrow u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = F^\varepsilon\,|\sigma_n^\varepsilon| \Rightarrow \exists\beta \geq 0 \text{ such that } u_\tau^\varepsilon = s - \beta\sigma_\tau^\varepsilon, \end{cases}$$

where $F^\varepsilon \geq 0$ is the coefficient of friction.

For the temperature, we assume that

$$\begin{cases} T^\varepsilon = 0 \quad \text{on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \\ \frac{\partial T^\varepsilon}{\partial n} = 0 \text{ on } w. \end{cases}$$

The complete problem consists of finding the displacement field u^ε and the temperature T^ε which satisfy the following equations and boundary conditions:

$$div(\sigma^\varepsilon) + f^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \tag{2.1}$$

$$\sigma_{i,j}^\varepsilon(u^\varepsilon) = 2\mu^\varepsilon(T^\varepsilon)d_{i,j}(u^\varepsilon) + \lambda^\varepsilon(T^\varepsilon)d_{kk}(u^\varepsilon)\delta_{ij}, \text{ in } \Omega^\varepsilon, \tag{2.2}$$

$$-\nabla(K^\varepsilon \nabla T^\varepsilon) = \sigma^\varepsilon : D(u^\varepsilon) + r^\varepsilon(T^\varepsilon) \quad \text{in } \Omega^\varepsilon, \tag{2.3}$$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_1^\varepsilon, \tag{2.4}$$

$$u^\varepsilon = g \quad \text{with } g_3 = 0 \quad \text{on } \Gamma_L^\varepsilon, \tag{2.5}$$

$$\begin{cases} |\sigma_\tau^\varepsilon| < F^\varepsilon\,|\sigma_n^\varepsilon| \Rightarrow u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = F^\varepsilon\,|\sigma_n^\varepsilon| \Rightarrow \exists\beta \geq 0 \text{ such that } u_\tau^\varepsilon = s - \beta\sigma_\tau^\varepsilon \end{cases} \text{ on } w. \tag{2.6}$$

$$T^\varepsilon = 0 \quad \text{on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \tag{2.7}$$

$$\frac{\partial T^\varepsilon}{\partial n} = 0 \quad \text{on } w. \tag{2.8}$$

2.3 Weak variational formulations

We finish this section by giving the equivalent weak variational formulation of problem (2.1) – (2.8) which will be useful in the next sections. By standard calculations, the variational formulation of the problem (2.1) – (2.8) is given as follows.

Problem $(\mathbf{P}_v)$ Find a displacement field $u^\varepsilon \in V^\varepsilon(\Omega^\varepsilon)$ and a temperature $T^\varepsilon \in H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega^\varepsilon)$ such that

$$a(T^\varepsilon, u^{\ \varepsilon}, \varrho - u^\varepsilon) + j^\varepsilon(\varrho) - j^\varepsilon(u^\varepsilon) \geq (f^{\ \varepsilon}, \varrho - u^{\ \varepsilon}), \quad \forall \varrho \in V^{\ \varepsilon}. \tag{2.9}$$

$$b(T^\varepsilon, \psi) = c(u^\varepsilon, T^\varepsilon, \psi), \quad \forall \psi \in H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}, \tag{2.10}$$

where

$$a(T^\varepsilon, u^\varepsilon, v) = \sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon) d_{ij}(u^\varepsilon) d_{ij}(v) d\acute{x} dx_3 + \int_{\Omega^\varepsilon} \lambda^\varepsilon(T^\varepsilon)\ div(u^\varepsilon)\ div(v) d\acute{x} dx_3, \tag{2.11}$$

$$(f^\varepsilon, v) = \int_{\Omega^\varepsilon} f^{\ \varepsilon} v d\acute{x} dx_3 = \sum_{i=1}^{3} \int_{\Omega^\varepsilon} f_i^{\ \varepsilon} v_i d\acute{x} dx_3, \tag{2.12}$$

$$j^\varepsilon(v) = \int_w F^\varepsilon \left|\sigma_n^\varepsilon\right| \left|v_T - s\right| d\acute{x} \text{ with } \ S(\sigma_n^\varepsilon) = \left|\sigma_n^\varepsilon\right|, (S \text{ is given below}) \tag{2.13}$$

$$b(T^\varepsilon, \psi) = \int_{\Omega^\varepsilon} K^\varepsilon \frac{\partial T^\varepsilon}{\partial x_i} \frac{\partial \psi}{\partial x_i} d\acute{x} dx_3, \tag{2.14}$$

$$\begin{gathered} c(u^\varepsilon, T^\varepsilon, \psi) = \textstyle\sum_{i=1}^{3} \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon) d_{ij}^2(u^\varepsilon) \psi d\acute{x} dx_3 + \\ \textstyle\int_{\Omega^\varepsilon} \lambda^\varepsilon(T^\varepsilon)\ div(u^\varepsilon)\ div(u^\varepsilon) \psi d\acute{x} dx_3 + \int_{\Omega^\varepsilon} r^\varepsilon(T^\varepsilon) \psi d\acute{x} dx_3. \end{gathered} \tag{2.15}$$

Remark 2.1 ([13, 14]) If we have only $u^\varepsilon \in V^\varepsilon$ and σ_n^ε is defined by duality as an element of $H^{-\frac{1}{2}}(w)$ has no sense, then the integral $j^\varepsilon(v)$ has no meaning. Then from the mathematical point of view it is necessary that $S(\sigma_n^\varepsilon) = |\sigma_n^\varepsilon|$ with S being a regularization operator from $H^{-\frac{1}{2}}(w)$ into $L^2_+(w)$ defined by

$$S(\tau)(x) = \left| \langle \tau, \varphi(x-\tau) \rangle_{H^{-\frac{1}{2}}(w), H^{\frac{1}{2}}_{00}(w)} \right|, \text{ for all} \in H^{-\frac{1}{2}}(w) \text{ and } S(\tau) \in L^2_+(w),$$

where φ is a given positive function of class C^∞ with support in w, and $H^{-\frac{1}{2}}(w)$ is the dual space to

$$H^{\frac{1}{2}}_{00}(w) = \left\{ \varphi_{|w} : \varphi \in H^1(\Omega^\varepsilon);\ \varphi = 0 \quad \text{on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon \right\}.$$

$L^2_+(w)$ is the subspace of $L^2(w)$ of non-negative functions.

Lemma 2.1 *If u^ε and T^ε are solutions of the problem* (2.1) – (2.8), *then they satisfy the variational problem* $(\mathbf{P}_v)$.

Proof. Multiply the equation (2.1) by $(\varrho - u^\varepsilon)$, where $\varrho \in V^\varepsilon$. By performing an integration by parts on Ω^ε, using the Green formula and (2.4)-(2.8), we obtain the variational problem (2.9). For the proof of (2.10), multiplying the equation (2.3) by ψ, where $\psi \in H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega^\varepsilon)$ and using the Green formula, we find

$$\sum_{i=1}^{3} \int_{\Omega^\varepsilon} K^\varepsilon \frac{\partial T^\varepsilon}{\partial x_i} \frac{\partial \psi}{\partial x_i} d\acute{x} dx_3 = \sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} 2\mu^\varepsilon (T^\varepsilon) d^2_{ij}(u^\varepsilon) \psi d\acute{x} dx_3 +$$

$$\int_{\Omega^\varepsilon} \lambda^\varepsilon (T^\varepsilon) \; div(u^\varepsilon) \; div(u^\varepsilon) \psi d\acute{x} dx_3 + \int_{\Gamma^\varepsilon} K^\varepsilon \frac{\partial T^\varepsilon}{\partial n_i} \psi ds + \int_{\Omega^\varepsilon} r^\varepsilon (T^\varepsilon) \psi d\acute{x} dx_3.$$

Now, for the boundary condition (2.8), we get $b(T^\varepsilon, \psi) = c(u^\varepsilon, T^\varepsilon, \psi), \quad \forall \psi \in H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}$.

Theorem 2.1 *If $f^\varepsilon \in (L^2(\Omega^\varepsilon))^3$ and the friction coefficient F^ε is a non-negative function in $L^\infty(w)$, then there exists $u^\varepsilon \in V^\varepsilon(\Omega^\varepsilon)$ which is a solution to the problem* (2.9)-(2.10). *Moreover, for small F^ε, the solution is unique.*

Proof. The proof is similar to that in [2], and we shall not reproduce it in full giving only a sketch here. Firstly, for the existence of solution u^ε we apply Tichonov's fixed point theorem (the proof can be found in [11]), then to prove the uniqueness of u^ε we use the same procedure as in [2, 20].

3 Problem in Transpose Form and Variational Problem

We shall now focus our attention on the asymptotic analysis of problem (2.1) – (2.8). For this analysis, we use the change of variable $z = \dfrac{x_3}{\varepsilon}$ to transform the initial problem in Ω^ε into a new problem posed in the fixed domain Ω which does not depend on ε:

$$\Omega = \left\{ (\acute{x}, z) \in \mathbb{R}^3 : (\acute{x}, 0) \in \omega, 0 < z < h(\acute{x}) \right\}$$

and we denote by $\Gamma = \overline{\Gamma}_L \cup \overline{\Gamma}_1 \cup \overline{w}$ its boundary. In addition, we define the following functions on Ω:

$$\begin{cases} u_i^\varepsilon(\acute{x}, x_3) = \hat{u}_i^\varepsilon(\acute{x}, z), \quad i = 1, 2, \\ \varepsilon^{-1} u_3^\varepsilon(\acute{x}, x_3) = \hat{u}_3^\varepsilon(\acute{x}, z), \quad T^{\,\varepsilon}(\acute{x}, x_3) = \hat{T}^{\,\varepsilon}(\acute{x}, z), \end{cases} \tag{3.1}$$

$$\hat{f}^{\,\varepsilon}(\acute{x}, z) = \varepsilon^2 f^{\,\varepsilon}(\acute{x}, x_3), \quad \hat{g}\,(\acute{x}, z) = g\,(\acute{x}, x_3), \tag{3.2}$$

$$\hat{K} = K^{\,\varepsilon}, \quad \hat{r} = \varepsilon\, r^\varepsilon, \quad \hat{\lambda} = \lambda^\varepsilon, \quad \hat{\mu} = \mu^\varepsilon, \; \hat{F} = \varepsilon^{-1} F^\varepsilon, \tag{3.3}$$

with $\hat{\mu}, \hat{\lambda}, \hat{f}, \hat{K}, \hat{F}$ and $\hat{g}$ independent of ε. So, the revaluation G^ε of g is defined by

$$\begin{cases} \varepsilon \hat{G}_3(\acute{x}, z) = G_3^\varepsilon(\acute{x}, x_3), \\ \hat{G}_i(\acute{x}, z) = G_i^\varepsilon(\acute{x}, x_3), \quad i = 1, 2. \end{cases} \tag{3.4}$$

We introduce the functional framework in Ω:

$$\begin{aligned} V &= \left\{ \varrho \in \left(H^1(\Omega)\right)^3 ; \quad \varrho = \hat{G} \text{ on } \Gamma_L \,,\; \varrho = 0 \text{ on } \Gamma_1^\varepsilon \text{ and } \varrho.n = 0 \text{ on } w \right\}, \\ \Pi(V) &= \left\{ \varrho \in H^1(\Omega)^2 : \varrho = (\varrho_1, \varrho_2)\,, \;\; \varrho_i = g \text{ on } \Gamma_L \text{ and } \varrho_i = 0 \text{ on } \Gamma_1, i = 1, 2 \right\}, \\ V_z &= \left\{ v = (v_1, v_2) \in L^2(\Omega)^2 ; \;\; \frac{\partial v_i}{\partial z} \in L^2(\Omega)\,, i = 1, 2; v = 0 \text{ on } \Gamma_1 \right\}, \\ H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega) &= \left\{ \varrho \in H^1(\Omega) : \varrho = 0 \text{ on } \Gamma_L \cup \Gamma_1 \right\}. \end{aligned}$$

It is clear that V_z is a Banach space with the norm

$$\|v\|_{V_z} = \left(\sum_{i=1}^{2} \|v\|^2_{L^2(\Omega)} + \left\| \frac{\partial v_i}{\partial z} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

With the change of scale defined in (3.1) – (3.3), the problem ($\mathbf{P}_v$) becomes as follows.

Find the displacement $\hat{u}^\varepsilon \in V$ and the temperature $\hat{T}^\varepsilon \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega)$ such that

$$a(\hat{T}^\varepsilon, \hat{u}^{\,\varepsilon}, \hat{\varrho} - \hat{u}^\varepsilon) + j(\hat{\varrho}) - j(\hat{u}^\varepsilon) \geq (\hat{f}^{\,\varepsilon}, \hat{\varrho} - \hat{u}^{\,\varepsilon}), \quad \forall \hat{\varrho} \in V, \tag{3.5}$$

$$b(\hat{T}^\varepsilon, \hat{\psi}) = c(\hat{u}^\varepsilon, \hat{T}^\varepsilon, \hat{\psi}), \quad \forall \hat{\psi} \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega), \tag{3.6}$$

where

$$\begin{aligned} a(\hat{T}^\varepsilon, \hat{u}^\varepsilon, \hat{\varrho} - \hat{u}^\varepsilon) &= \varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial x_j} + \frac{\partial \hat{u}^\varepsilon_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} (\hat{\varrho}_i - \hat{u}^\varepsilon_i) d\acute{x} dz \\ &+ \sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}^\varepsilon_3}{\partial x_i} \right) \frac{\partial}{\partial z} (\hat{\varrho}_i - \hat{u}^\varepsilon_i) \\ &+ \sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}^\varepsilon_3}{\partial x_i} \right) \varepsilon^2 \frac{\partial}{\partial x_i} (\hat{\varrho}_i - \hat{u}^\varepsilon_3) d\acute{x} dz \\ &+ \varepsilon^2 \int_\Omega 2\hat{\mu}(\hat{T}^\varepsilon) \frac{\partial \hat{u}^\varepsilon_3}{\partial z} \frac{\partial}{\partial z} (\hat{\varrho}_3 - \hat{u}^\varepsilon_3) d\acute{x} dz \\ &+ \varepsilon^2 \int_\Omega \hat{\lambda}(\hat{T}^{\,\varepsilon}) \; div(\hat{u}^\varepsilon) \; div(\hat{\varrho} - \hat{u}^\varepsilon) d\acute{x} dz, \end{aligned}$$

$$\begin{aligned} (\hat{f}^{\,\varepsilon}, \hat{\varrho} - \hat{u}^\varepsilon) &= \sum_{i=1}^{2} \int_\Omega \hat{f}_i^{\,\varepsilon} (\hat{\varrho}_i - \hat{u}^\varepsilon_i) \, d\acute{x} dz + \varepsilon \int_\Omega \hat{f}_3^{\,\varepsilon} (\hat{\varrho}_3 - \hat{u}^\varepsilon_3) \, d\acute{x} dz \\ b(\hat{T}^\varepsilon, \hat{\psi}) &= \sum_{i=1}^{2} \int_\Omega \hat{K} \varepsilon^2 \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \frac{\partial \hat{\psi}}{\partial x_i} d\acute{x} dz + \int_\Omega \hat{K} \frac{\partial \hat{T}^\varepsilon}{\partial z} \frac{\partial \hat{\psi}}{\partial z} d\acute{x} dz \\ j(\hat{\varrho}) &= \int_w \hat{F} S(\sigma^\varepsilon_n) \, |\hat{\varrho}_T - s| \, d\acute{x}, \end{aligned}$$

$$\begin{aligned} c(\hat{u}^\varepsilon, \hat{T}^\varepsilon, \hat{\psi}) &= \sum_{i,j=1}^{2} \frac{1}{2} \int_\Omega \varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial x_j} + \frac{\partial \hat{u}^\varepsilon_j}{\partial x_i} \right)^2 \hat{\psi} d\acute{x} dz \\ &+ \sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}^\varepsilon_3}{\partial x_i} \right)^2 \hat{\psi} d\acute{x} dz + \int_\Omega 2\varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_3}{\partial z} \right)^2 \hat{\psi} d\acute{x} dz \\ &+ \int_\Omega \varepsilon^2 \hat{\lambda}(\hat{T}^\varepsilon) \; div(\hat{u}^\varepsilon) \; div(\hat{u}^\varepsilon) \hat{\psi} d\acute{x} dz + \int_\Omega \hat{r}(\hat{T}^\varepsilon) \hat{\psi} d\acute{x} dz. \end{aligned}$$

In the next subsection, we will do the estimates of $(u^\varepsilon, T^\varepsilon)$ solution of our variational problem in fixed domain.

3.1 A priori estimates of the displacement

It is enough to prove the following essential result.

Lemma 3.1 *Assume that* $f \in \left(L^2(\Omega)\right)^3$, *the coefficient of friction* $F^{\varepsilon} > 0$ *in* $L^{\infty}(w)$ *and there are strictly positive constants* $\mu_*, \mu^*, \lambda_*, \lambda^*$ *such that*

$$0 < \mu_* \leq \mu(a) \leq \mu^* \text{ and } 0 < \lambda_* \leq \lambda(b) \leq \lambda^*, \quad \forall a, b \in \mathbb{R}. \tag{3.7}$$

Then there is a strictly positive constant C *independent of* ε *such that*

$$\sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon \frac{\partial \hat{u}_3^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \left(\left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon^2 \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right) \leq C. \tag{3.8}$$

Proof. Let u^{ε} be the solution of the problem $(\mathbf{P}_v)$ so that

$$a(T^{\varepsilon}, u^{\varepsilon}, u^{\varepsilon}) \leq a(T^{\varepsilon}, u^{\varepsilon}, \varrho) + (f^{\,\varepsilon}, u^{\varepsilon}) + j^{\varepsilon}(\varrho) - (f^{\,\varepsilon}, \varrho). \tag{3.9}$$

Because $j^{\varepsilon}(u^{\varepsilon})$ is positive and as $\sum_{i,j=1}^{2} |d_{ij}(u^{\varepsilon})|^2 \leq |\nabla u^{\varepsilon}|^2$, $|\, div(u^{\varepsilon})|^2 \leq |\nabla u^{\varepsilon}|^2$, so, according to the inequality of Korn (from [19]), there exists C_K independent of ε such that

$$a(T^{\varepsilon}, u^{\varepsilon}, u^{\varepsilon}) \geq 2\mu_* C_K \left\| \nabla u^{\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2. \tag{3.10}$$

Applying the Hölder and Young inequalities, we find the following:

$$a(T^{\varepsilon}, u^{\varepsilon}, \varrho) \leq \frac{3\,\mu_* C_K}{8} \left\| \nabla u^{\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2 + \left(\frac{4\,(\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K}\right) \left\| \nabla \varrho \right\|_{L^2(\Omega^{\varepsilon})}^2. \tag{3.11}$$

Then

$$(f^{\,\varepsilon}, u^{\varepsilon}) \leq \frac{(\varepsilon h^*)^2}{2\,\mu_* C_K} \left\| \nabla f^{\,\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2 + \frac{\mu_* C_K}{2} \left\| \nabla u^{\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2. \tag{3.12}$$

$$(f^{\,\varepsilon}, \varrho) \leq \frac{(\varepsilon h^*)^2}{2\,\mu_* C_K} \left\| \nabla f^{\,\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2 + \frac{\mu_* C_K}{2} \left\| \nabla \varrho \right\|_{L^2(\Omega^{\varepsilon})}^2. \tag{3.13}$$

By $(3.10) - (3.13)$ and choosing $\varrho = G^{\varepsilon}$, we get the variational equation

$$\begin{aligned} \frac{9}{8}\mu_* C_K \left\| \nabla u^{\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2 \quad \leq \quad & \frac{(\varepsilon h^*)^2}{\mu_* C_K} \left\| \nabla f^{\,\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2 + \\ & \left(\frac{4\,(\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K} + \frac{\mu_* C_K}{2} \right) \left\| \nabla G^{\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2. \end{aligned}$$

As $\varepsilon^2 \left\| \nabla f^{\,\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2 = \varepsilon^{-1} \left\| \nabla \hat{f}^{\,\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2$ and $\varepsilon \left\| \nabla G^{\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2 = \left\| \nabla \hat{G} \right\|_{L^2(\Omega^{\varepsilon})}^2$, then

$$\begin{aligned} \varepsilon \left\| \nabla u^{\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2 = \varepsilon^2 \textstyle\sum_{i,j=1}^{2} & \left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_J} \right\|_{L^2(\Omega)}^2 + \varepsilon^2 \left\| \frac{\partial \hat{u}_3^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)}^2 \\ + \textstyle\sum_{i=1}^{2} & \left(\left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)}^2 + \varepsilon^4 \left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right) \leq C \end{aligned}$$

with

$$C = \frac{8}{9\mu_* C_K} c_0 \text{ and } c_0 = \frac{(h^*)^2}{\mu_* C_K} \left\| \nabla \hat{f}^{\,\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})}^2 + \left(\frac{4\,(\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K} + \frac{\mu_* C_K}{2} \right) \left\| \nabla \hat{G} \right\|_{L^2(\Omega^{\varepsilon})}^2.$$

3.2 A priori estimates of the temperature

In this subsection, we look for an a priori estimate of the temperature $\hat{T}^{\varepsilon}$, for this we need to establish the following lemma which is a direct consequence of the Poincaré inequality.

Lemma 3.2 *The temperature* $\hat{T}^{\varepsilon}$ *is increased by*

$$\left\|\hat{T}^{\varepsilon}\right\|_{L^2(\Omega)} \leq h^* \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}. \tag{3.14}$$

Lemma 3.3 *Suppose that the hypotheses of Lemma 3.1 are verified. Furthermore, suppose there are*

two strictly positive constants K_* *and* K^* *such that*

$$0 \leq K_* \leq K\left(\acute{x}, z\right) \leq K^*, \forall \left(\acute{x}, z\right) \in \Omega, \tag{3.15}$$

a positive constant $\hat{r}^*$ *such that*

$$\hat{r}(a) \leq \hat{r}^*, \tag{3.16}$$

then there exists a positive constant C *independent of* ε *such that*

$$\varepsilon^2 \sum_{i=1}^{2} \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial x_i}\right\|_{L^2(\Omega)} + \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)} \leq C. \tag{3.17}$$

Proof. In the variational equation (2.17), we choose $\psi = \hat{T}^{\varepsilon}$, we get

$$\sum_{i=1}^{3} I_i = \sum_{i=1}^{2} \int_{\Omega} \varepsilon^2 \hat{K} \frac{\partial \hat{T}^{\varepsilon}}{\partial x_i} \frac{\partial \hat{T}^{\varepsilon}}{\partial x_i} d\acute{x}dz + \int_{\Omega} \frac{\partial \hat{T}^{\varepsilon}}{\partial z} \frac{\partial \hat{T}^{\varepsilon}}{\partial z} d\acute{x}dz$$

with

$$\begin{aligned} I_1 \;\; = \;\; & \sum_{i,j=1}^{2} \frac{1}{2} \int_{\Omega} \varepsilon^2 \hat{\mu}(\hat{T}^{\varepsilon}) \left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j} + \frac{\partial \hat{u}_j^{\varepsilon}}{\partial x_i}\right)^2 \hat{T}^{\varepsilon} d\acute{x}dz + \sum_{i=1}^{2} \int_{\Omega} \left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i}\right)^2 \hat{T}^{\varepsilon} d\acute{x}dz \\ & + \int_{\Omega} 2\varepsilon^2 \hat{\mu}(\hat{T}^{\varepsilon}) \left(\frac{\partial \hat{u}_3^{\varepsilon}}{\partial z}\right)^2 \hat{T}^{\varepsilon} d\acute{x}dz, \end{aligned}$$

$$I_2 = \int_{\Omega} \hat{r}(\hat{T}^{\varepsilon}) \hat{T}^{\varepsilon} d\acute{x}dz, \quad I_3 = \int_{\Omega} \varepsilon^2 \hat{\lambda}(\hat{T}^{\varepsilon})\; div(\hat{u}^{\varepsilon})\; div(\hat{u}^{\varepsilon}) \hat{T}^{\varepsilon} d\acute{x}dz.$$

By the Cauchy-Schwartz and the Young inequalities and Lemma 3.2, we find

$$|I_1| \leq 2\hat{\mu}^* C \left\|\hat{T}^{\varepsilon}\right\|^2_{L^2(\Omega)} \leq 2\hat{\mu}^* C h^* \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}. \tag{3.18}$$

The analogue of I_1 gives

$$|I_2| \leq \hat{r}^* \left\|\hat{T}^{\varepsilon}\right\|^2_{L^2(\Omega)} \leq \hat{r}^* h^* \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}, \tag{3.19}$$

$$|I_3| \leq \hat{\lambda}^* C \left\| \hat{T}^\varepsilon \right\|^2_{L^2(\Omega)}. \tag{3.20}$$

On the other hand, by the use of (3.14)-(3.15), we find

$$b(\hat{T}^\varepsilon, \hat{T}^\varepsilon) = \sum_{i=1}^{2} \int_\Omega \varepsilon^2 \hat{K} \left| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right|^2 d\acute{x}dz + \int_\Omega \hat{K} \left| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right|^2 d\acute{x}dz.$$

This implies

$$\hat{K}_* \varepsilon^2 \left\| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|^2_{L^2(\Omega)} + \hat{K}_* \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} \leq b(\hat{T}^\varepsilon, \hat{T}^\varepsilon) \leq C_1 \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)}, \tag{3.21}$$

where C_1 is a constant independent of ε given by $C_1 = 2\hat{\mu}^* h^* C + \hat{r}^* h^* + \hat{\lambda}^* C h^*$, thus

$$\left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} \leq \hat{K}_*^{-1} C_1. \tag{3.22}$$

By injecting this last estimate in (3.21), we deduce (3.17).

3.3 Convergence results

In this part, we will establish the following theorem.

Theorem 3.1 *Under the same assumptions as in Lemmas* 3.1 *and* 3.3 *there exist* $u^* = (u_1^*, u_2^*)$ *in* V_z *and* T^* *in* V_z *such that for sub suites of* $\hat{u}^\varepsilon$*(resp* $\hat{T}^\varepsilon$*) noted again* $\hat{u}^\varepsilon$*(resp* $\hat{T}^\varepsilon$*), we have the following convergence results*:

$$\hat{u}_i^\varepsilon \rightharpoonup u_i^* \quad \text{weakly in} \quad V_z(\Omega), 1 \leq i \leq 2, \tag{3.23}$$

$$\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega), 1 \leq i, j \leq 2, \tag{3.24}$$

$$\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega), \tag{3.25}$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \ \text{weakly in} \quad L^2(\Omega), 1 \leq i \leq 2, \tag{3.26}$$

$$\varepsilon \hat{u}_3^\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega), \tag{3.27}$$

$$\hat{T}^\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad V_z(\Omega), \tag{3.28}$$

$$\varepsilon \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \rightharpoonup 0 \ \text{weakly in} \quad L^2(\Omega), 1 \leq i \leq 2. \tag{3.29}$$

Proof. The convergences of (3.23) to (3.27) are a direct result from the inequality (3.8). By using the estimate (3.17), we deduce that $\left\| \hat{T}^{\ \varepsilon} \right\| \leq h^* \left\| \frac{\partial \hat{T}^{\ \varepsilon}}{\partial z} \right\| \leq h^* C_2$. So, $\hat{T}^{\ \varepsilon}$ is bounded in $V_z(\Omega)$, which shows the existence of $T^{\ *}$ in $V_z(\Omega)$. In addition, $\varepsilon \left\| \frac{\partial \hat{T}^{\ \varepsilon}}{\partial x_i} \right\| \leq C_2$, thus $\left(\varepsilon \ \frac{\partial \hat{T}^{\ \varepsilon}}{\partial x_i} \right)$ converges to $\frac{\partial T^{\ *}}{\partial x_i}$ and $\hat{T}^\varepsilon$ converges to T^* in $V_z(\Omega)$, then $\varepsilon \ \frac{\partial \hat{T}^{\ \varepsilon}}{\partial x_i}$ weakly converges to 0 in $V_z(\Omega)$.

4 Study of the Limit Problem

To reach the desired goal, we need in the rest of this paragraph the results of previous convergences.

Lemma 4.1 *There exists a subsequence of $S(\sigma_n^\varepsilon(u^\varepsilon))$ converging strongly towards $S(\sigma_n^*(u^*))$ in $L^2(w)$*

Proof. To prove this lemma, we use the same technique as in [2] (Lemma 5.1) and in [6] (Lemma 5.2).

Theorem 4.1 *$u_i^\varepsilon \to u_i^*$ strongly in $V_z(\Omega)$, $i = 1, 2$, and with the same assumptions as in Theorem 3.1, the solution (u^*, T^*) satisfies*

$$\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^*)\frac{\partial u_i^*}{\partial z}\frac{\partial}{\partial z}(\hat{\varrho}_i - u_i^*)d\acute{x}dz + j(\hat{\varrho}) - j(u^*) \geq \sum_{i=1}^{2}\left(\hat{f}_i{}^\varepsilon, \hat{\varrho}_i - u_i^*\right), \ \ \forall \hat{\varrho} \in \Pi(V), \quad (4.1)$$

$$-\frac{\partial}{\partial z}\left(\hat{\mu}(T^*)\frac{\partial u_i^*}{\partial z}\right) = \hat{f}_i{}^\varepsilon, \ \ \textit{for } i = 1, 2 \text{ in } L^2(\Omega), \quad (4.2)$$

$$-\frac{\partial}{\partial z}\left(\hat{K}\frac{\partial T^*}{\partial z}\right) = \sum_{i=1}^{2}\hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2 + \hat{r}(T^*) \ \textit{in } L^2(\Omega). \quad (4.3)$$

Proof. For $u_i^\varepsilon \to u_i^*$ strongly in V_z, we use the same methods as in [6] (proof of Theorem 4.2). By applying the convergence results of Theorem 3.1 to the variational equality (3.5) and using the fact that j is convex and lower semi-continuous, we obtain

$$\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^*)\frac{\partial u_i^*}{\partial z}\frac{\partial}{\partial z}(\hat{\varrho}_i - u_i^*)d\acute{x}dz + j(\hat{\varrho}) - j(u^*) \geq \sum_{i=1}^{2}\left(\hat{f}_i{}^\varepsilon, \hat{\varrho}_i - u_i^*\right). \quad (4.4)$$

From [7] (Lemma 5.3), we can choose in (4.4)

$$\hat{\varrho}_i = u_i^* \pm \psi_i, \psi_i \in H_0^1(\Omega) \text{ for } i = 1, 2 \text{ and } \hat{\varrho}_3 = u_3^*,$$

then we get

$$\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^*)\frac{\partial u_i^*}{\partial z}\frac{\partial \psi_i}{\partial z}d\acute{x}dz = \sum_{i=1}^{2}\left(\hat{f}_i{}^\varepsilon, \psi_i\right).$$

Using Green's formula and choosing $\psi_1 = 0$ and $\psi_2 \in H_0^1(\Omega)$, then $\psi_2 = 0$ and $\psi_1 \in H_0^1(\Omega)$, we obtain

$$-\int_\Omega \hat{\mu}(T^*)\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial z}\right)d\acute{x}dz = \int_\Omega \hat{f}_i{}^\varepsilon \psi_i d\acute{x}dz,$$

thus

$$-\hat{\mu}(T^*)\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial z}\right) = \hat{f}_i{}^\varepsilon, \text{for } i = 1, 2 \text{ in } H^{-1}(\Omega), \quad (4.5)$$

and as $\hat{f}_i{}^\varepsilon \in L^2(\Omega)$, then (4.5) is true in $L^2(\Omega)$.

On the other hand, going to the limit in (3.6) and using (3.28)-(3.29), we find

$$\int_\Omega \hat{K}\frac{\partial T^*}{\partial z}\frac{\partial \psi}{\partial z}d\acute{x}dz = \sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2 \psi d\acute{x}dz + \int_\Omega \hat{r}(T^*)\psi d\acute{x}dz, \forall \psi \in H^1_{\Gamma_L \cup \Gamma_1}(\Omega).$$

Now, by the formula of Green, we get

$$\int_\Omega \frac{\partial}{\partial z}\left(\hat{K}\frac{\partial T^*}{\partial z}\right)\psi d\acute{x}dz = \sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2\psi d\acute{x}dz + \int_\Omega \hat{r}(T^*)\psi d\acute{x}dz, \forall\psi \in H^1_{\Gamma_L\cup\Gamma_1}(\Omega).$$

Consequently,

$$\frac{\partial}{\partial z}\left(\hat{K}\frac{\partial T^*}{\partial z}\right)\psi d\acute{x}dz = \sum_{i=1}^{2}\hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2\psi d\acute{x}dz + \hat{r}(T^*)\psi d\acute{x}dz, \text{ in } H^1_{\Gamma_L\cup\Gamma_1}(\Omega). \quad (4.6)$$

The formula (4.6) is valid in $L^2(\Omega)$ since $\hat{\mu}$ and $\hat{r}$ are two bounded functions in $\mathbb{R}$ and $\left(\frac{\partial u_i^*}{\partial z}\right)^2$ is an element of $L^2(\Omega)$.

Theorem 4.2 *Under the same assumptions as in the previous theorem, we have*

$$\int_w \hat{F}\left|S(\sigma_n^*(s^*))\right|(|\psi + s^* - s| - |s^* - s|)d\acute{x} - \int_w \hat{\mu}(\varsigma^*)\hat{\xi}.\psi d\acute{x} \geq 0, \forall\psi \in L^2(w) \quad (4.7)$$

$$\left\{\begin{array}{c} \hat{\mu}(\varsigma^*)\hat{\xi}^* < \hat{F}(S(\sigma_n^*(s^*)) \Rightarrow s^* = s, \\ \hat{\mu}(\varsigma^*)\hat{\xi}^* = \hat{F}(S(\sigma_n^*(s^*) \Rightarrow \exists\beta \geq 0 \text{ such that } s^* = s - \beta\hat{\xi}^* \end{array}\right. \text{ on } w \quad (4.8)$$

with

$$s^*(\acute{x}) = u^*(\acute{x},0), \varsigma^* = T^*(\acute{x},0) \text{ and } \hat{\xi}^* = \frac{\partial u^*}{\partial z}(\acute{x},0).$$

Another u^* and T^* satisfy the following weak form:

$$\begin{array}{c} \int_w \left(\int_0^h F(\acute{x},z)dz - hs^*(\acute{x},0) - \hat{\mu}(\varsigma^*)\hat{\xi}^*\int_0^h A(\acute{x},z)dz\right)\nabla\psi d\acute{x} \\ + \int_w\int_0^h u^*(\acute{x},z)\nabla\psi dz d\acute{x} = 0, \forall\psi \in H^1(w), \end{array} \quad (4.9)$$

where

$$A(\acute{x},z) = \int_0^z \frac{d\gamma}{\hat{\mu}\left(T^*(\acute{x},\gamma)\right)}, \quad F(\acute{x},z) = \int_0^z\int_0^\gamma \frac{\hat{f_i}^\varepsilon(\acute{x},\eta)}{\hat{\mu}\left(T^*(\acute{x},\gamma)\right)}d\eta d\gamma.$$

Proof. The proofs of (4.7)-(4.8) are similar to those given in the case of the problem fluids, see [3]. To demonstrate (4.9) by integrating twice the equation (4.2) from 0 to z, we find

$$u^*(\acute{x},z) = s^*(\acute{x},0) + \hat{\mu}(\varsigma^*)\hat{\xi}^*A(\acute{x},z) - F(\acute{x},z), \quad (4.10)$$

and as $u^*(\acute{x},h(x)) = 0$, we have

$$s^*(\acute{x},0) + \hat{\mu}(\varsigma^*)\hat{\xi}^*A(\acute{x},z) = F(\acute{x},z). \quad (4.11)$$

By integrating (4.10) from 0 to h, we obtain

$$\int_0^h u^*(\acute{x},z)dz = s^*(\acute{x},0)h + \hat{\mu}(\varsigma^*)\hat{\xi}^*\int_0^h A(\acute{x},z)dz - \int_0^h F(\acute{x},z)dz. \quad (4.12)$$

So,

$$\int_0^h u^*(\acute{x},z)dz - s^*(\acute{x},0)h - \hat{\mu}(\varsigma^*)\hat{\xi}^*\int_0^h A(\acute{x},z)dz + \int_0^h F(\acute{x},z)dz = 0.$$

From (4.11) and (4.12), we deduce (4.9). This ends the proof requested.
Before studying the existence and uniqueness of the solution, we need the following sets:

$$W_z = \left\{ v \in V_z; \frac{\partial^2 v}{\partial z^2} \in L^2(\Omega) \right\} \text{ and } B_c = \left\{ v \in W_z \times W_z; \left\| \frac{\partial v}{\partial z} \right\|_{V_z} \leq c \right\}.$$

Theorem 4.3 *Under the assumptions of Theorem 3.1 and if there exists a positive sufficiently small constant F^* such that $\|\hat{F}\|_{L^\infty(\omega)} \leq F^*$, then the solution (u^*, T^*) of the limit problem (4.1)-(4.3) is unique in $B_c \times W_z$.*

Proof. For the uniqueness of solution, we follow the same steps and results as in [2,8]. Suppose there are solutions (u^1, T^1) and (u^2, T^2) to the problem limit (4.1) and (4.3) for every $\psi \in H^1_{\Gamma_L \cup \Gamma_l}(\Omega)$, we have

$$\int_\Omega -\hat{K} \frac{\partial T^1}{\partial z} \frac{\partial \psi}{\partial z} d\acute{x} dz = \sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^1) \left(\frac{\partial u_i^1}{\partial z} \right)^2 \psi d\acute{x} dz + \int_\Omega \hat{r}(T^1) \psi d\acute{x} dz, \tag{4.12}$$

$$\int_\Omega -\hat{K} \frac{\partial T^2}{\partial z} \frac{\partial \psi}{\partial z} d\acute{x} dz = \sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^2) \left(\frac{\partial u_i^2}{\partial z} \right)^2 \psi d\acute{x} dz + \int_\Omega \hat{r}(T^2) \psi d\acute{x} dz. \tag{4.13}$$

By subtracting (4.12) and (4.13), we get

$$\begin{aligned} \textstyle\int_\Omega -\hat{K} \frac{\partial}{\partial z} \left(T^1 - T^2 \right) \frac{\partial \psi}{\partial z} d\acute{x} dz = \textstyle\sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) \left(\frac{\partial u_i^1}{\partial z} \right)^2 - \hat{\mu}(T^2) \left(\frac{\partial u_i^2}{\partial z} \right)^2 \right] \psi d\acute{x} dz \\ + \textstyle\int_\Omega \left[\hat{r}(T^1) - \hat{r}(T^2) \right] \psi d\acute{x} dz. \end{aligned} \tag{4.14}$$

In (4.14), we add and subtract the term $\hat{\mu}(T^1) \left(\frac{\partial u_i^2}{\partial z} \right)^2$, we find

$$\begin{aligned} \int_\Omega \hat{K} \frac{\partial}{\partial z} \left(T^1 - T^2 \right) \frac{\partial \psi}{\partial z} d\acute{x} dz \quad = \quad & \sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) \frac{\partial}{\partial z} \left(u_i^1 + u_i^2 \right) \frac{\partial}{\partial z} \left(u_i^1 - u_i^2 \right) \right] \psi d\acute{x} dz + \\ & \sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) - \hat{\mu}(T^2) \right] \left(\frac{\partial u_i^2}{\partial z} \right)^2 \psi d\acute{x} dz + \\ & \int_\Omega \left[\hat{r}(T^1) - \hat{r}(T^2) \right] \psi d\acute{x} dz. \end{aligned}$$

By choosing $\psi = \left(T^1 - T^2 \right) \in H^1_{\Gamma_L \cup \Gamma_l}(\Omega)$, we get

$$\int_\Omega \hat{K} \frac{\partial}{\partial z} \left| T^1 - T^2 \right|^2 d\acute{x} dz = \sum_{i=1}^{3} R_k \tag{4.15}$$

with

$$\begin{aligned} R_1 \quad &= \quad \sum_{i=1}^{2} R_1^i = \sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) \frac{\partial}{\partial z} \left(u_i^1 + u_i^2 \right) \frac{\partial}{\partial z} \left(u_i^1 - u_i^2 \right) \right] \left(T^1 - T^2 \right) d\acute{x} dz, \\ R_2 \quad &= \quad \sum_{i=1}^{2} R_2^i = \sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) - \hat{\mu}(T^2) \right] \left(\frac{\partial u_i^2}{\partial z} \right)^2 \left(T^1 - T^2 \right) d\acute{x} dz, \\ R_3 \quad &= \quad \int_\Omega \left[\hat{r}(T^1) - \hat{r}(T^2) \right] \left(T^1 - T^2 \right) d\acute{x} dz, \end{aligned}$$

and similarly,

$$\int_{\Omega} \hat{K} \frac{\partial}{\partial z} \left|T^1 - T^2\right|^2 d\acute{x}dz \geq K_* \left[1 + (h^*)^2\right]^{-1} \left\|T^1 - T^2\right\|_{V_z}. \tag{4.16}$$

Now, by the Hölder inequality we get

$$\left|R_1^i\right| \leq \mu^* \left\|\frac{\partial}{\partial z}\left(u_i^1 + u_i^2\right)\right\|_{L^4(\Omega)} \left\|\frac{\partial}{\partial z}\left(u_i^1 - u_i^2\right)\right\|_{L^2(\Omega)} \left\|T^1 - T^2\right\|_{L^4(\Omega)},$$

as the compact injection of $V_z(\Omega)$ in $L^4(\Omega)$ is continuous, then there is a constant $\alpha > 0$ such that

$$\left|R_1^i\right| \leq \mu^* \alpha^2 \left\|\frac{\partial}{\partial z}\left(u_i^1 + u_i^2\right)\right\|_{V_z} \left\|\left(u_i^1 - u_i^2\right)\right\|_{V_z} \left\|T^1 - T^2\right\|_{V_z}.$$

And since u_i^1 and u_i^2 are two elements of B_c, we get

$$\left|R_1^i\right| \leq 2\mu^* \alpha^2 c \left\|\left(u_i^1 - u_i^2\right)\right\|_{V_z} \left\|T^1 - T^2\right\|_{V_z},$$

using the Young inequality ($\alpha_1 + \alpha_2 \leq \sqrt{2}\,(\alpha_1 + \alpha_2)^{\frac{1}{2}}$ for $\alpha_1, \alpha_2 \geq 0$), we have

$$\left|R_1^i\right| \leq 2\mu^* \alpha^2 c \left\|T^1 - T^2\right\|_{V_z} \sum_{i=1}^{2} \left\|\left(u_i^1 - u_i^2\right)\right\|_{V_z} \leq 2\sqrt{2}\mu^* \alpha^2 c \left\|T^1 - T^2\right\|_{V_z} \left\|\left(u_i^1 - u_i^2\right)\right\|_{V_z \times V_z},$$

then

$$|R_1| \leq 2\sqrt{2}\mu^* \alpha^2 c \left\|T^1 - T^2\right\|_{V_z} \left\|\left(u^1 - u^2\right)\right\|_{V_z \times V_z}. \tag{4.17}$$

And

$$\left|R_2^i\right| \leq C_\mu \int_{\Omega} \left|T^1 - T^2\right|^2 \left|\frac{\partial u_i^2}{\partial z}\right| d\acute{x}dz \leq C_\mu \alpha^4 c^2 \left\|T^1 - T^2\right\|_{V_z}^2,$$

thus

$$|R_2| \leq 2C_\mu \alpha^4 c^2 \left\|T^1 - T^2\right\|_{V_z}^2, \tag{4.18}$$

as the function $\hat{r}$ is Lipschitz on $\mathbb{R}$, there exists a constant $C_{\hat{r}}$ such that

$$|R_3| \leq C_{\hat{r}} \left\|T^1 - T^2\right\|_{V_z}^2. \tag{4.19}$$

By injecting (4.14) – (4.19) in (4.13) we have

$$K_* \left[1 + (h^*)^2\right]^{-1} \left\|T^1 - T^2\right\|_{V_z}^2 \leq \left(2C_{\hat{\mu}} \alpha^4 c^2 + C_{\hat{r}}\right) \left\|T^1 - T^2\right\|_{V_z}^2 + 2\sqrt{2}\mu^* \alpha^2 c \left\|T^1 - T^2\right\|_{V_z} \left\|\left(u^1 - u^2\right)\right\|_{V_z \times V_z},$$

we suppose that $c < c_0 = \left[2C_{\hat{\mu}} \alpha^4\right]^{\frac{-1}{2}} \left(K_* \left[1 + (h^*)^2\right]^{-1} - C_{\hat{r}}\right)^{\frac{1}{2}}$, provided that $K_* > \left[1 + (h^*)^2\right] C_{\hat{r}}$, then

$$\left\|T^1 - T^2\right\|_{V_z}^2 \leq 2\sqrt{2}\mu^* \alpha^{-2} C_{\hat{\mu}}^{-1} c (c_0^2 - c^2)^{-1} \left\|\left(u^1 - u^2\right)\right\|_{V_z \times V_z}. \tag{4.20}$$

We also have the following two inequalities:

$$\begin{aligned}&\textstyle\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varrho}_i^1-u_i^1\right)d\acute{x}dz+\int_w \hat{F}S(\sigma_n^*(u_i^1))\left|\hat{\varrho}_i^1-s\right|d\acute{x}\\ &\quad-\textstyle\int_w \hat{F}S(\sigma_n^*(u_i^1))\left|u_i^1-s\right|d\acute{x}\geq\sum_{i=1}^{2}\left(\hat{f}_i{}^{\varepsilon},\hat{\varrho}_i^1-u_i^1\right),\end{aligned}\tag{4.21}$$

$$\begin{aligned}&\textstyle\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varrho}_i^2-u_i^2\right)d\acute{x}dz+\int_w \hat{F}S(\sigma_n^*(u_i^2))\left|\hat{\varrho}_i^2-s\right|d\acute{x}\\ &\quad-\textstyle\int_w \hat{F}S(\sigma_n^*(u_i^2))\left|u_i^2-s\right|d\acute{x}\geq\sum_{i=1}^{2}\left(\hat{f}_i{}^{\varepsilon},\hat{\varrho}_i^2-u_i^2\right).\end{aligned}\tag{4.22}$$

We choose $\hat{\varrho}_i^1 = u_i^2$ in (4.21) and $\hat{\varrho}_i^2 = u_i^1$ in (4.22), and after summing up the two inequalities, it comes to $W = u_i^2 - u_i^1$

$$\sum_{i=1}^{2}\int_{\Omega}\left[\hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial W}{\partial z}-\hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}\right]d\acute{x}dz+\int_w \hat{F}S(\sigma_n^*(u_i^1)\left(\left|u_i^2-s\right|-\left|u_i^1-s\right|\right)d\acute{x}$$
$$-\int_w \hat{F}S(\sigma_n^*(u_i^2)\left(\left|u_i^2-s\right|-\left|u_i^1-s\right|\right)d\acute{x}\geq 0,$$

so, for the next term, we have

$$\begin{aligned}&\int_w \hat{F}S(\sigma_n^*(u_i^1)\left(\left|u_i^2-s\right|-\left|u_i^1-s\right|\right)d\acute{x}-\int_w \hat{F}S(\sigma_n^*(u_i^2)\left(\left|u_i^2-s\right|-\left|u_i^1-s\right|\right)d\acute{x}\\ \leq\;&\int_w \left|\hat{F}\left(S(\sigma_n^*(u_i^1)-S(\sigma_n^*(u_i^2))\right|\left|u_i^2-u_i^1\right|d\acute{x}.\end{aligned}$$

According to the inequality of Cauchy-Schwartz we obtain

$$\int_w \left|\hat{F}\left(S(\sigma_n^*(u_i^1)-S(\sigma_n^*(u_i^2))\right|\left|u_i^2-u_i^1\right|d\acute{x}\leq\left\|\hat{F}\right\|_{L^\infty(w)}C\left\|u_i^2-u_i^1\right\|_{V_z}^2\leq F^*C\left\|u_i^2-u_i^1\right\|_{V_z}^2.$$

By the previous theorem the term $F^*C\left\|u_i^2-u_i^1\right\|_{V_z}^2$ tends to 0, then we have

$$\sum_{i=1}^{2}\int_{\Omega}\left[\hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial W}{\partial z}-\hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}\right]d\acute{x}dz\geq 0,\tag{4.23}$$

we add and subtract the term $\hat{\mu}(T^1)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}$ from the equation (4.23), we get

$$\sum_{i=1}^{2}\int_{\Omega}\left[\hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial W}{\partial z}-\hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}\right]d\acute{x}dz+\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}(T^1)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz$$
$$-\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}(T^1)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz\geq 0.$$

So,

$$\sum_{i=1}^{2}\int_{\Omega}-\hat{\mu}(T^1)\frac{\partial W}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz+\sum_{i=1}^{2}\int_{\Omega}(\hat{\mu}(T^1)-\hat{\mu}(T^2))\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz\geq 0,\tag{4.24}$$

and

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}(T^1)\frac{\partial W}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz\geq\frac{\mu_*}{2}\left\|W\right\|_{V_z}^2.\tag{4.25}$$

Using the Hölder inequality, and the results of [8], we find

$$\left| \sum_{i=1}^{2} \int_{\Omega} (\hat{\mu}(T^1) - \hat{\mu}(T^2)) \frac{\partial u_i^2}{\partial z} \frac{\partial W}{\partial z} d\acute{x} dz \right| \leq \sqrt{2} c \alpha^2 C_{\hat{\mu}} \left\| T^1 - T^2 \right\|_{V_z} \|W\|_{V_z}. \tag{4.26}$$

By injecting (4.25) into (4.24), we get

$$\frac{\mu_*}{2} \|W\|_{V_z} \leq \sqrt{2} c \alpha^2 C_{\hat{\mu}} \left\| T^1 - T^2 \right\|_{V_z}. \tag{4.27}$$

Returning to (4.20), we obtain

$$\begin{aligned} \left\| T^1 - T^2 \right\|_{V_z} &\leq 2\sqrt{2} \mu^* c \alpha^{-2} (c_0^2 - c^2)^{-1} \left\| u^2 - u^1 \right\|_{V_z \times V_z} \\ &\leq 8 \mu_*^{-1} \mu^* C_{\hat{\mu}}^{-1} c^2 (c_0^2 - c^2)^{-1} \left\| T^1 - T^2 \right\|_{V_z} \leq 0, \end{aligned}$$

provided that $0 < c < c_1 = (1 + 8\mu_*^{-1}\mu^*)^{-\frac{1}{2}} c_0$. Therefore, $\left\| T^1 - T^2 \right\|_{V_z} = 0$.

So, there exists $T^1 = T^2$ almost everywhere in V_z. According to (4.27), we deduce that $u^1 = u^2$ almost everywhere in V_z.

5 Conclusion

The purpose of this paper is to study the asymptotic behavior of a non-isothermal elasticity system in a thin domain with Coulomb friction on the bottom surface. One of the objectives of this study is to obtain a two-dimensional equation that allows a reasonable description of the phenomenon occurring in the three-dimensional domain by passing the limit to 0 on the small thickness of the domain (3D). As a first step, we gave the variational formulation of the problem and demonstrate the results of existence and uniqueness of the weak solution, then we moved on to the asymptotic analysis. For this, by using the change of scale according to the third component we conduct the study on a domain Ω which does not depend on ε. Then, by different inequalities, we proved some estimates for the displacement and the temperature, which allow us to go to the limit when ε tends towards zero in the variational formulation. Finally, we have reached our main result concerning the proof of the convergence results and uniqueness of the limit problem.

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Fractional Discrete Neural Networks with Different Dimensions: Coexistence of Complete Synchronization, Antiphase Synchronization and Full State Hybrid Projective Synchronization

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Abstract: This paper aims to present the coexistence of complete synchronization, antiphase synchronization and full state hybrid projective synchronization in two fractional discrete neural networks with different dimensions. A new theorem is proved, which assures the coexistence of these synchronization types in different dimensional fractional discrete neural networks. Finally, simulation results are reported to confirm the effectiveness of the synchronization approach illustrated herein.

Keywords: *neural networks; synchronization; discrete-fractional calculus.*

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1 Introduction

Chaos synchronization is a key issue in the study of nonlinear dynamical systems described by differential equations (continuous-time systems) or difference equations (discrete-time systems). Starting from that concept, in the last three decades, different types of chaos synchronization have been conceived for both integer-order systems and fractional-order systems described by non-integer order derivative [1–6, 8, 9]. For example, when each response system variable synchronizes with a linear combination of drive system variables, full state hybrid projective synchronization is achieved [2]. On the other hand, when each response system variable synchronizes with the opposite of the drive system variable, antiphase synchronization is obtained [4]. These synchronization types have been applied to both integer-order systems [1–4] and fractional systems described by non-integer order derivative [5–7]. In particular, a recent paper has analyzed the coexistence of some synchronization types in two chaotic systems described by fractional derivatives [7].

It should be noted that, differently from fractional systems described by non-integer order derivative, few synchronization types have been introduced for fractional systems described by non-integer order difference operators [10–14]. For example, in [10], the complete synchronization of two chaotic fractional Grass-Miller maps is proved. In [11], full state hybrid projective synchronization is achieved between two fractional discrete systems of different dimensions. In [12], the complete synchronization of two fractional forms of the discrete double scroll is illustrated. In [13], the full state hybrid projective synchronization between an integer-order discrete system and a fractional-order discrete system is achieved. In [14], the complete synchronization between two fractional forms of a novel generalized Hénon map is reported. When considering fractional discrete-time neural networks, which represent a class of discrete systems described by non-integer order difference operators [15–19], it should be noted that very few examples of synchronization have been published to date. For example, in [20], the complete synchronization of fractional discrete neural networks with time delays is discussed. In [21], the complete synchronization of discrete-time fractional-order complex-valued neural networks has been illustrated. To the best of the authors' knowledge, no synchronization method for fractional discrete neural networks, different from the complete synchronization, has been reported to date. Additionally, the topic related to the coexistence of different synchronization types in fractional discrete neural networks is unexplored.

The paper is organized as follows. In Section 2, some basic notions regarding discrete fractional calculus are given. In Section 3, the two-dimensional fractional discrete neural network, considered as a master system, and the three-dimensional fractional discrete neural network, considered as a slave system, are illustrated. In Section 4, a new theorem is proved, which assures the coexistence of complete synchronization, antiphase synchronization and full state hybrid projective synchronization in the two neural networks with different dimensions considered in the previous section. Finally, simulation results are reported to confirm the effectiveness of the synchronization approach illustrated herein.

2 Discrete Fractional Operators

In the following we recall some definitions and preliminaries of the discrete fractional calculus. The notation $\mathbb{N}_a$ denotes the isolated time scale and $\mathbb{N}_a = \{a, a+1, a+2, ...\}$, $(a \in \mathbb{R} \text{ fixed})$.

Definition 2.1 [22] The ν–th fractional sum is defined by

$$\Delta_a^{-\nu} X(t) = \frac{1}{\Gamma(\upsilon)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} X(s), \quad t \in \mathbb{N}_{a+\nu},\ \nu > 0, \tag{1}$$

where $X(t) : \mathbb{N}_a \to \mathbb{R}$, and the term $t^{(\upsilon)}$ denotes the falling function defined in terms of the Gamma function Γ as

$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}. \tag{2}$$

Definition 2.2 [23] For $0 < \nu < 1$, and $X(t)$ defined on $\mathbb{N}_a$, the Caputo-like delta difference is defined by

$${}^C\Delta_a^{\nu} X(t) = \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t-s-1)^{(-\nu)} \Delta X(s), \quad t \in \mathbb{N}_{a+1-\nu}. \tag{3}$$

To deal with fractional-order systems in discrete time and to employ our numerical tools, we recall the following result.

Theorem 2.1 *[24] For the delta fractional difference equation*

$$\begin{cases} {}^C\Delta_a^{\nu} u(t) = f(t+\upsilon-1, u(t+\nu-1)), \\ \Delta^k = u_k,\ n = [\nu]+1,\ k = 0, 1, ..., n-1, \end{cases} \tag{4}$$

the equivalent discrete integral equation can be obtained as

$$u(t) = u_0(t) + \frac{1}{\Gamma(\nu)} \sum_{s=a+n-\nu}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s+\nu-1, u(s+\nu-1)), t \in \mathbb{N}_{\alpha+n}, \tag{5}$$

where

$$u_0(t) = \sum_{k=0}^{m-1} \frac{(t-a)^k}{k} \Delta^k u(a). \tag{6}$$

3 Master and Slave Fractional Discrete Neural Networks

In this section, we will endeavour to show that the two fractional maps, even though they have different dimensions, can still be synchronized within time in the light of an appropriate synchronization scheme.

3.1 The master system

One may consider the two-dimensional fractional discrete neural network introduced in [19] as a master system and distinguish its states by typing the subscript m for each of them. That is,

$$\begin{cases} {}^C\Delta_a^{\nu} x_{1m}(t) = -0.7x_{1m}(t+\nu-1) + a_{11}\tanh(x_{1m}(t+\nu-1)) + a_{12}\tanh(x_{2m}(t+\nu-1)), \\ {}^C\Delta_a^{\nu} x_{2m}(t) = -0.85x_{2m}(t+\nu-1) + a_{21}\tanh(x_{1m}(t+\nu-1)) + a_{22}\tanh(x_{2m}(t+\nu-1)), \end{cases} \tag{7}$$

where ${}^{C}\Delta_a^{\nu}$ denotes the Caputo difference operator, $0 \le \nu \le 1$, $t \in \mathbb{N}_{a+(1-\nu)}$, a is the starting point and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.1 & 0.11 \\ 0.12 & 0.13 \end{pmatrix}.$$

According to Theorem 2.1, the equivalent implicit discrete formula can be written in the form

$$\begin{cases} x_{1m}(n) = x_{1m}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \left[-0.7x_{1m}(j) + a_{11}\tanh(x_{1m}(j)) + a_{12}\tanh(x_{2m}(j)\right], \\ x_{2m}(n) = x_{2m}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \left[-0.85x_{2m}(j) + a_{21}\tanh(x_{1m}(j)) + a_{22}\tanh(x_{2m}(j))\right]. \end{cases} \tag{8}$$

3.2 The slave system

On the other hand, the three-dimensional fractional discrete neural network, proposed in [16], is considered as slave and all its states are distinguished by typing another subscript, say s, for each of them, i.e.,

$$\begin{cases} {}^{C}\Delta_a^{\nu} y_{1s}(t) = -y_{1s}(t+\nu-1) + b_{11}\tanh(y_{1s}(t+\nu-1)) + b_{12}\tanh(y_{2s}(t+\nu-1)) \\ \qquad\qquad + b_{13}\tanh(y_{3s}(t+\nu-1)) + \mathbf{C}_1, \\ {}^{C}\Delta_a^{\nu} y_{2s}(t) = -y_{2s}(t+\nu-1) + b_{21}\tanh(y_{1s}(t+\nu-1)) + b_{22}\tanh(y_{2s}(t+\nu-1)) \\ \qquad\qquad + b_{23}\tanh(y_{3s}(t+\nu-1)) + \mathbf{C}_2, \\ {}^{C}\Delta_a^{\nu} y_{3s}(t) = -y_{3s}(t+\nu-1) + b_{33}\tanh(y_{1s}(t+\nu-1)) + b_{32}\tanh(y_{2s}(t+\nu-1)) \\ \qquad\qquad + b_{33}\tanh(y_{3s}(t+\nu-1)) + \mathbf{C}_3, \end{cases} \tag{9}$$

where the system parameters are given as

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{31} & b_{33} \end{pmatrix} = \begin{pmatrix} 2 & -1.2 & 0 \\ 2 & 1.71 & 1.15 \\ -4.75 & 0 & 1.1 \end{pmatrix},$$

and $\mathbf{C}_1, \mathbf{C}_2$, $\mathbf{C}_3$ are synchronization controllers that have to be designed. On the other hand, the numerical formulae can be given accordingly

$$\begin{cases} y_{1s}(n) = y_{1s}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} - y_{1s}(j) + 2\tanh(y_{1s}(j)) + 1.2\tanh(y_{2s}(j)), \\ y_{2s}(n) = y_{2s}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} - y_{2s}(j) + 2\tanh(y_{1s}(j)) + 1.71\tanh(y_{2s}(j)) \\ \qquad\qquad +1.15\tanh(y_{3s}(j)), \\ y_{3s}(n) = y_{3s}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} - y_{3s}(j) - 4.75\tanh(y_{1s}(j)) + 1.1\tanh(y_{3s}(j)). \end{cases} \tag{10}$$

4 Synchronization

In this section, we will endeavor to show that the two fractional discrete neural networks, even though they have different dimensions, can still be synchronized within time in the light of an appropriate synchronization scheme. Actually, the process of picking up an adaptive controll law $(\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3)^T$ aims to compel the following synchronization errors:

$$\begin{cases} e_1 = y_{1s} - x_{1m}, \\ e_2 = y_{2s} + x_{2m}, \\ e_3 = y_{3s} - (x_{1m} + x_{2m}) \end{cases} \tag{11}$$

to be asymptotically tended to the origin, i.e.,

$$\lim_{t \to +\infty} |e_i(t)| = 0, \qquad \text{for } i = 1, 2, 3. \tag{12}$$

Remark 4.1 From the error system (11), it is obvious that states y_{1s} and x_{1m} are complete synchronized, y_{2s} is anti-synchronized with x_{2m}, and y_{3s} is full state synchronized with x_{1m} and x_{2m}. So, complete synchronization, anti-synchronization and full state hybrid projective synchronization co-exist between the fractional discrete neural networks (7) and the slave system (9).

Before stating the proposed control law and establishing its stability, it is important to state the following theorem, which is essential for our proof. Interested readers are referred to [25] for the proof of this result.

Theorem 4.1 *The zero equilibrium of the linear fractional-order discrete–time system*

$$^C\Delta_a^\nu X(t) = \mathbf{M} X(t + \nu - 1), \tag{13}$$

where $X(t) = (x_1(t), ..., x_n(t))^T$, $0 < \nu \leq 1$, $\mathbf{M} \in \mathbb{R}^{n \times n}$ *and* $\forall t \in \mathbb{N}_{a+1-\nu}$, *is asymptotically stable if*

$$\lambda \in \left\{ z \in \mathbb{C} : \ |z| < \left(2 \cos \frac{|\arg z| - \pi}{2 - \nu} \right)^\nu \ \textit{and} \ |\arg z| > \frac{\nu \pi}{2} \right\} \tag{14}$$

for all the eigenvalues λ *of* $\mathbf{M}$.

With this stability result, we are now ready to state the following theorem which is considered the main result of this work.

Theorem 4.2 *Complete synchronization, antiphase synchronization and full state hybrid projective synchronization coexist in the synchronization of the master system (7) and the slave system (9) if the control law is selected as follows:*

$$\begin{cases} \mathbf{C}_1(t) = 0.3x_{1m}(t) - b_{11} tanh(y_1(t)) - b_{12} tanh(y_2(t)) \\ \qquad\qquad + a_{11} tanh(x_{1m}(t)) + a_{12} tanh(x_{2m}(t)), \\ \mathbf{C}_2(t) = -b_{21} tanh(y_{1s}(t)) - b_{22} tanh(y_{2s}(t)) \\ \qquad\qquad - b_{23} tanh(y_{3s}(t)) - 0.15x_{2m}(t) \\ \qquad\qquad - a_{21} tanh(x_{1m}(t)) - a_{22} tanh(x_{2m}(t)), \\ \mathbf{C}_3(t) = 0.3x_{1m}(t) + 0.15x_{2m}(t) \\ \qquad\qquad - b_{31} tanh(y_{1s}(t)) - b_{33} tanh(y_{3s}(t)) \\ \qquad\qquad + a_{11} tanh(x_{1m}(t)) + a_{12} tanh(x_{2m}(t)) \\ \qquad\qquad + a_{21} tanh(x_{1m}(t)) + a_{22} tanh(x_{2m}(t)), \end{cases} \tag{15}$$

where $t \in \mathbb{N}_{a+1-\nu}$.

Proof. For the purpose of establishing an asymptotic convergence of the synchronization errors, given in (11), to zero, we start applying the Caputo-type fractional-order differences to (11), which yields

$$\begin{cases} {}_{h}^{C}\Delta_{a}^{\nu}e_1(t) = -y_{1s}\left(t+\nu-1\right) + b_{11}\tanh(y_{1s}\left(t+\nu-1\right)) \\ \qquad + b_{12}\tanh(y_{2s}\left(t+\nu-1\right)) \\ \qquad + 0.7x_{1m}\left(t+\nu-1\right) - a_{11}\tanh(x_{1m}\left(t+\nu-1\right)) \\ \qquad - a_{12}\tanh(x_{2m}\left(t+\nu-1\right)) + \mathbf{C}_1, \\ {}_{h}^{C}\Delta_{a}^{\nu}e_2(t) = -y_{2s}\left(t+\nu-1\right) + b_{21}\tanh(y_{1s}\left(t+\nu-1\right)) \\ \qquad + b_{22}\tanh(y_{2s}\left(t+\nu-1\right)) + b_{23}\tanh(y_{3s}\left(t+\nu-1\right)) \\ \qquad - 0.85x_{2m}\left(t+\nu-1\right) + a_{21}\tanh(x_{1m}\left(t+\nu-1\right)) \\ \qquad + a_{22}\tanh(x_{2m}\left(t+\nu-1\right)) + \mathbf{C}_2, \\ {}_{h}^{C}\Delta_{a}^{\nu}e_3(t) = -y_{3s}\left(t+\nu-1\right) + b_{31}\tanh(y_{1s}\left(t+\nu-1\right)) \\ \qquad + b_{33}\tanh(y_{3s}\left(t+\nu-1\right)) + 0.7x_{1m}\left(t+\nu-1\right) \\ \qquad - a_{11}\tanh(x_{1m}\left(t+\nu-1\right)) - a_{12}\tanh(x_{2m}\left(t+\nu-1\right)) \\ \qquad + 0.85x_{2m}\left(t+\nu-1\right) - a_{21}\tanh(x_{1m}\left(t+\nu-1\right)) \\ \qquad - a_{22}\tanh(x_{2m}\left(t+\nu-1\right)) + \mathbf{C}_3. \end{cases} \tag{16}$$

Substituting the proposed control law given in (15) into (16) leads to the following new discrete system:

$$\begin{cases} {}^{C}\Delta_{a}^{\nu}e_1\left(t\right) = -e_1\left(t+\nu-1\right), \\ {}^{C}\Delta_{a}^{\nu}e_2\left(t\right) = -e_2\left(t+\nu-1\right), \\ {}^{C}\Delta_{a}^{\nu}e_3\left(t\right) = -e_3\left(t+\nu-1\right). \end{cases} \tag{17}$$

It can be described more compactly as

$${}^{C}\Delta_{a}^{\nu}\left(e_1, e_2, e_3\right)\left(t\right)^{T} = \mathbf{M}\times\left(e_1, e_2, e_3\right)\left(t+\nu-1\right)^{T}, \tag{18}$$

where

$$\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{19}$$

We aim to show that the zero solution of (18) is globally asymptotically stable, which guarantees that all states converge towards zero at infinite time. In order to do so, we make use of the stability theory described in Theorem 4.2. Simply, we can show that the eigenvalues of the matrix $\mathbf{M}$ are: $\lambda_1 = \lambda_2 = \lambda_3 = -1$. It is easy to see that all the eigenvalues of the matrix $\mathbf{M}$ satisfy $|\arg\lambda_i| = \pi > \frac{\nu\pi}{2}$ and $|\lambda_i| < \left(2\cos\frac{|\arg\lambda_i|-\pi}{2-\nu}\right)^{\nu}$ for $i = 1, 2, 3$. According to Theorem 4.2, the zero solution of (18) is globally asymptotically stable. Hence, the master system (7) and the slave system (9) has been synchronized by means of control laws (15).

Here, some numerical experiments are considered to verify the effectiveness of the proposed approach. The initial values of the master system (7) are set as $x_{1m}(0) = 0.2$, $x_{2m}(0) = 0.3$, and the initial values of the slave system are considered as $y_{1s} = 0.02, y_{2s} = -0.5$ and $y_{3s} = 0.8$. Figure 1 illustrates the hybrid synchronization error system (14) with $\mu = 0.05$. It is very clear that the error states e_1, e_2 and e_3 can converge to zero when the controller functions C_1, C_2 and C_3 are added to the slave systems, which implies that the synchronization is realized. Furthermore, we plot the evolution of states of the master and slave systems for the fractional-order value $\mu = 0.05$. Figure 2 illustrates the results. Clearly, the state variables of the master and slave systems are synchronized completely. Thus, the numerical results show very well that the two fractional maps achieve hybrid synchronization.

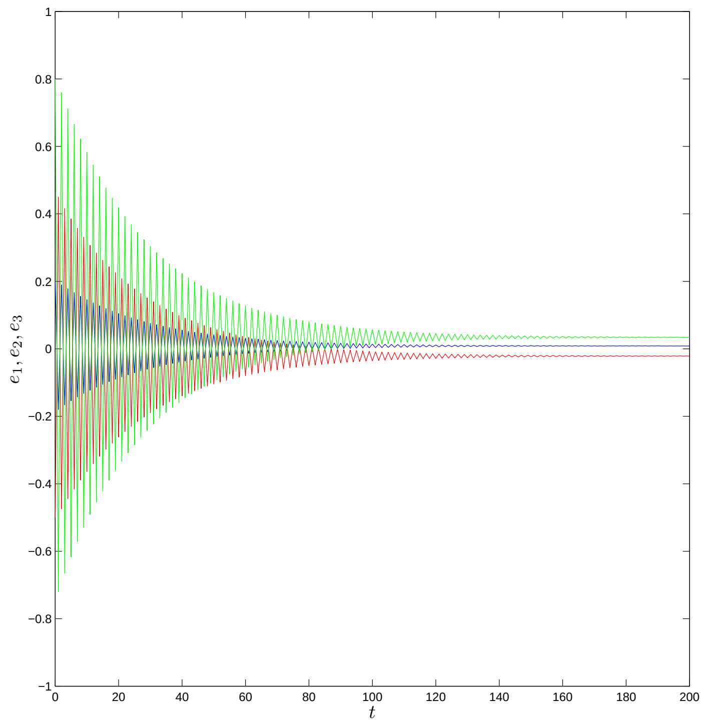

Figure 1: Hybrid synchronization error between the master system (7) and the slave system (9) with $\nu = 0.05$.

5 Conclusion

This paper has shown the coexistence of some synchronization types in two fractional discrete neural networks with different dimensions. At first, a two-dimensional fractional discrete neural network (considered as a master system) and a three-dimensional fractional discrete neural network (considered as a slave system) have been introduced. Then a new theorem has been proved, which assures the coexistence of complete synchronization, antiphase synchronization and full state hybrid projective synchronization in different dimensional fractional discrete neural networks. Finally, simulation results have been obtained to highlight the effectiveness of the conceived approach.

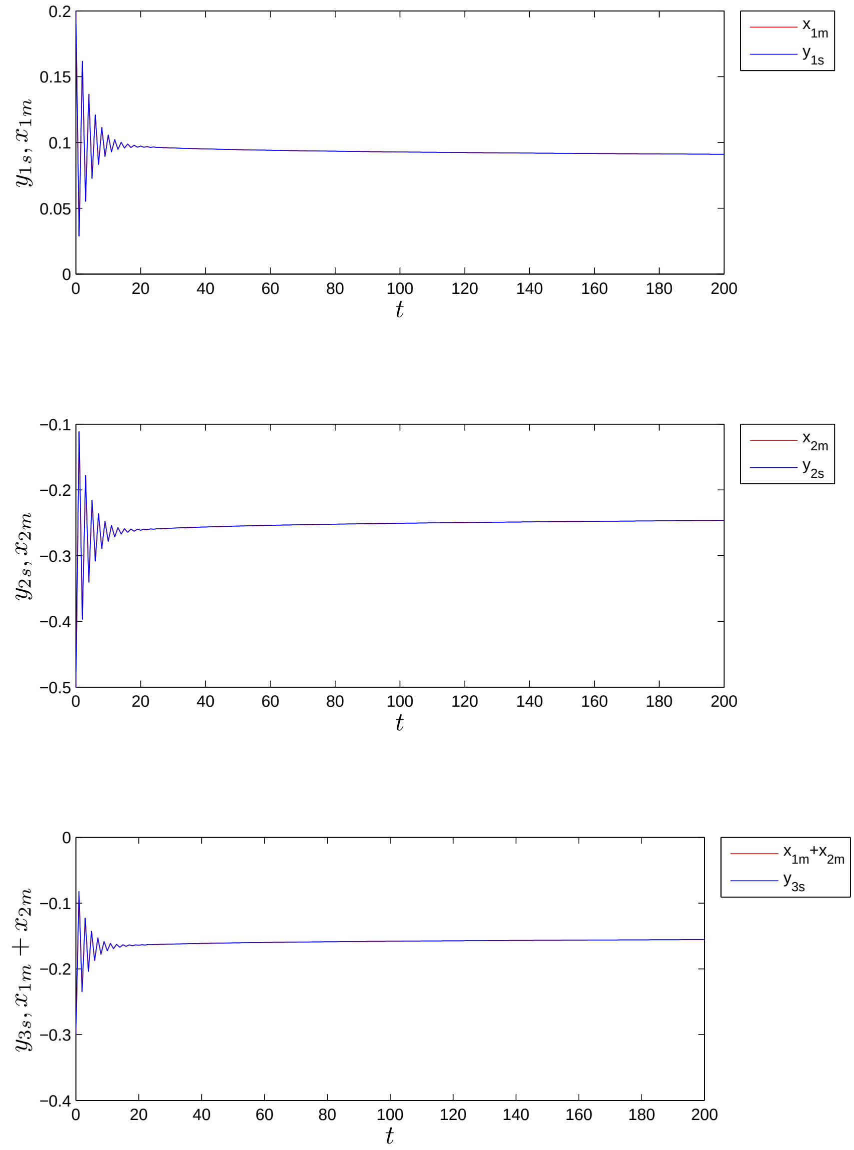

Figure 2: The time history of the master system (7) and the slave system (9) with $\nu = 0.05$.

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Master-Slave Synchronization of a Planar 2-DOF Model of Robotic Leg

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Abstract: In this paper, synchronization in the master-slave coupling scheme of two mechanical lower limbs, or legs, is numerically presented. In particular, we use the sliding mode control approach for trajectory tracking of the master's end-effector and, in addition, the slave's end-effector synchronizes with the master's end-effector. The synchronization studies reported in the literature have two main interesting results: phase synchronization and anti-phase synchronization. It is our perception that these two synchronization types appear in human movements such as jumping, sitting or standing (as phase movements), and walking, running or swimming (as anti-phase movements). This work pretends to replicate some of these movements in a prosthetic leg, where the prosthetic leg is the slave and the natural leg is the master. The contribution of this work is the use of the master-slave synchronization scheme, in conjunction with the sliding mode control method, conceived in the particular problem of people with an amputated leg. Simulation studies performed on two mechanical dynamical models of 2-DOF are presented to demonstrate the viability and performance of the proposed master-slave synchronization scheme.

Keywords: *synchronization; nonlinear control; position control; robotics; sliding mode control.*

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1 Introduction

Many years and efforts have been required to develop humanoids. From the robot *Knight* to the robot *Advanced Step in Innovative Mobility (ASIMO)* or the robot *ATLAS* [1–3]. The *Knight* robot was developed by the Italian genius Leonardo Da Vinci in 1495, the ASIMO robot by Honda in 1986, and the ATLAS robot by Boston Dynamics in 2013. Both the ASIMO robot and the ATLAS robot involve, in addition to the mechanical engineering of the *Knight* robot, other disciplines such as electrical engineering, computation engineering, and control theory, among others. Despite the locomotive level reached by ASIMO and/or ATLAS, they still lack the grace, speed, and performance of the human being. As the locomotion of these humanoids resembles that of humans, we can use them as assistant robots [4] or assistants [5], for example, people with a leg amputation regain the ability to walk, swim, or jump [6–10]. On the other hand, the synchronization theory is of great interest in the scientific community due to the physical phenomena it can explain. For example, in nature, we can see the synchronization of the flash of fireflies, electronically reproduced in [11]. In communication systems, along with the theory of chaos, it is possible to encrypt any kind of information [12] that can be implemented with micro-controllers as in [13]. Synchronization patterns for inter-human coordination have been found in humanoid research [14]. For example, for sitting the limbs are in phase synchronized [15] and for walking they are in anti-phase synchronized [16]. Rodriguez-Angeles and Nijmeijer in [17] mention that, "The problem of robot synchronization can be seen as tracking paths between systems with an additional challenge that is not considered in tracking controllers of the trajectory". In their publication, they synchronize two manipulator robots by using the state estimation for feedback, E. Cicek in [18] has used an adaptive controller, and Bondhus et. al. in [19] have used a PID controller. The main differences between this work and [17–19] are: 1) the control method used, 2) the synchronization time, and 3) the conception of the work in the particular problem in conjunction with the approach and potential application in the rehabilitation of people with an amputated leg. The contribution of this work is the synchronization of trajectories between two similar 2-degree of freedom (2-DOF) manipulator robot systems, under the master-slave coupling scheme, making use of the method of sliding mode control (SMC), conceived for the potential application for people with an amputated leg. In this case, the human's limb (*thigh* and *shank*) is the master system and a 2-serial *link* mechanical system is the slave system.

In a potential experimental evaluation, the master lower extremity would send its position to the slave lower extremity (using sensors), which must follow the movements of the master system until the synchronization is achieved, either in phase or in anti-phase. For simulation reasons, the master system is also modeled as a 2-DOF serial link mechanical system. This work is organized as follows. In Section 2, the non-linear systems synchronization is explained. In Section 3, we describe the control scheme for tracking the paths of the master system and how the slave system couples with the master one to achieve synchronization. In Section 4, we present the dynamic model of a system with 2-DOF, which is used to model the thigh and shank. In Section 5, the control by the SMC method is presented, applied to both the master and the slave systems. Also, the stability proof based on Lyapunov's theory is presented. In Section 6, the results obtained are shown in the simulation for a circular path, where the synchronization of the master-slave scheme is confirmed. Finally, in Section 7 some conclusions are reported.

2 Master-Slave Synchronization

Huygen's observations regarding the synchronization of two weakly coupled mechanical parameters reveal five types of synchronization [20]: 1) full or identical synchronization, 2) generalized synchronization, 3) phase synchronization, 4) anticipated or delay synchronization, and 5) envelope amplitude synchronization. The present work focuses on the phase synchronization for two mechanical systems with 2-DOF.

For two nonlinear dynamic systems synchronization, suppose a system described by

$$\dot{\boldsymbol{q}}_m = \boldsymbol{f}(\boldsymbol{q}_m), \tag{1}$$

where $\boldsymbol{f}$ is the non-linear vector at least twice differentiable and with a smooth curve. And $\boldsymbol{q}_m \in R^n$ is the *master system* state vector. The second, *slave system*, is defined by

$$\dot{\boldsymbol{q}}_s = \boldsymbol{h}(\boldsymbol{q}_s, \boldsymbol{u}), \tag{2}$$

where $\boldsymbol{u} \in R^n$ is the input signal to the system, $\boldsymbol{h}$ is the non-linear vector that, like $\boldsymbol{f}$, is at least twice differentiable. Let the synchronization error be given by

$$\boldsymbol{e}_s = \boldsymbol{q}_m - \boldsymbol{q}_s, \tag{3}$$

then the control objective is to design a signal $\boldsymbol{u} \in R^n$ in such a way that

$$\lim_{t\to\infty} \|\boldsymbol{e}_s(t)\| = 0, \tag{4}$$

where $\|\cdot\|$ is the Euclidean norm. This means that, when $t \to \infty$, the systems (1) and (2) are synchronized.

3 Synchronization Strategy

Two stages are needed to achieve synchronization for two 2-DOF mechanical systems under the master-slave scheme. In the first stage, the master system follows the position of the desired path, and in the second stage, the slave system follows the position of the master system, Figure 1 shows these two stages. The first stage is represented by white blocks, while the gray blocks represent the second stage.

In the first stage, the value of $\boldsymbol{\tau}_m$ is increased or decreased until $\boldsymbol{q}_d - \boldsymbol{q}_m = \boldsymbol{e}_m \approx 0$, which means that the desired trajectory is reached. In the second stage, the slave system takes $\boldsymbol{q}_m$ as the desired position and compares it with $\boldsymbol{q}_s$, when $\boldsymbol{q}_m - \boldsymbol{q}_s = \boldsymbol{e}_s \approx 0$, both systems are in phase synchronized. The master system follows the path indicated by the vector $\boldsymbol{x}_d = [x_d \quad y_d]^T$ with the Cartesian (x, y) values. The inverse *kinematics* convert them to angular values that are required by the master's system joints.

3.1 Direct and inverse kinematic

With the kinematics analysis, it is possible to calculate the end-effector's position, speed, and acceleration without considering the forces and torques causing the movement. The geometric relationship between the system link's joints and the reference frame is established. If the length and angles of the links are known, it is possible to compute the end-effector position in the Cartesian plane through the *direct kinematics.* On the other hand, if the lengths of the links and end-effector position in the Cartesian plane are

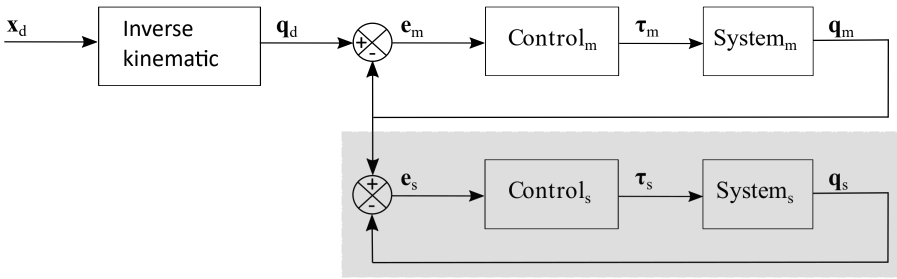


Figure 1: Master-slave synchronization scheme.

known, it is possible to compute the value of the link's angles through the *inverse kinematics*. For a two-link manipulator system in the Cartesian plane, as shown in Figure 2, the *direct kinematic* equations are:

$$x_m = l_{m1}\cos(q_{m1}) + l_{m2}\cos(q_{m1} + q_{m2}), \tag{5}$$

$$y_m = l_{m1}\sin(q_{m1}) + l_{m2}\sin(q_{m1} + q_{m2}), \tag{6}$$

where the master system link length variables are l_{m1} and l_{m2}, while the joint's angular variables are q_{m1} and q_{m2}.

On the other hand, the *inverse kinematics* equations to compute the joint's angular values are defined as follows:

$$q_{m2} = \tan^{-1}\left(\frac{\pm\sqrt{1-D^2}}{D}\right), \tag{7}$$

$$q_{m1} = \tan^{-1}(\frac{y_m}{x_m}) - \tan^{-1}\left(\frac{l_{m2}\sin(q_{m2})}{l_{m1} + l_{m2}\cos(q_{m2})}\right), \tag{8}$$

where

$$D = \frac{x_m^2 + y_m^2 - l_{m1}^2 - l_{m1}^2}{2l_{m1}l_{m2}}.$$

In this way it is possible to express the desired position $\boldsymbol{x}_d$ using either (5)-(6) or (7)-(8).

4 Dynamical Model of the 2-DOF Mechanism

The mechanism being used consists of two links connected in series with a revolute-type joint, therefore, it is a two-degree-of-freedom (2-DOF) mechanism. The synchronization strategy, described in Section 3, was simulated with an actuated 2-DOF mechanism that can perform any smooth trajectories in the Cartesian plane (x, y). One 2-DOF mechanism for the master and another 2-DOF one for the slave. The dynamic model which represents N mechanical systems with 2-DOF is given by

$$H_i(\boldsymbol{q}_i)\ddot{\boldsymbol{q}}_i + C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\dot{\boldsymbol{q}}_i + \boldsymbol{g}_i(\boldsymbol{q}_i) = \boldsymbol{\tau}_i, \quad i = m, s\,, \tag{9}$$

where the sub-index m is for the master system and the sub-index s is for the slave system, therefore, $\boldsymbol{q}_i = [q_{i1}\ q_{i2}]^T$, $\boldsymbol{\tau}_i = [\tau_{i1}\ \tau_{i2}]^T$ and

$$H_i = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, C_i = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \boldsymbol{g}_i = \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix}, \tag{10}$$

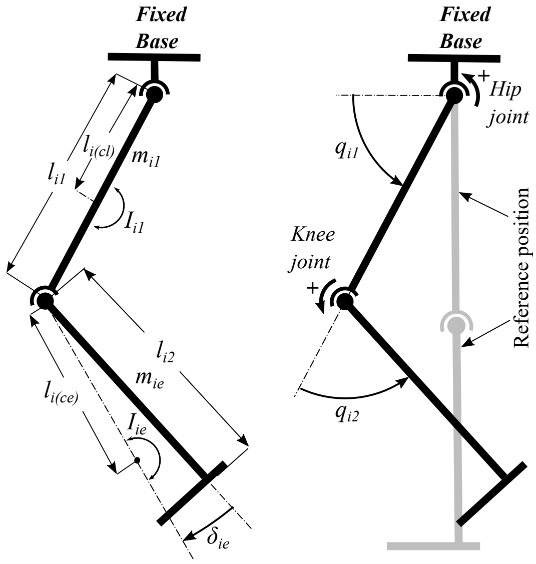


Figure 2: Master's mechanical system: thigh ($\boldsymbol{l_{i1}}$) and shank ($\boldsymbol{l_{i2}}$). The sub-index $\boldsymbol{i}$ is for either the master or the slave system.

where H_i is the inertia matrix, C_i is the Coriolis matrix, and $\boldsymbol{g}_i$ is the gravitational vector. Both τ_{i1} and τ_{i2} are the torques and moments of the joint's manipulator robot, i.e., the thigh and shank. Lastly, q_{i1} and q_{i2} are the angular positions of the thigh and shank as shown in Figure 2.

The elements of the matrices indicated in (9) are

$$
\begin{aligned}
&H_{11} = \alpha_i + 2\varepsilon_i \cos(q_{i2}) + 2\eta_i \sin(q_{i2}),\\
&H_{12} = \beta_i + \varepsilon_i \cos(q_{i2}) + \eta_i \sin(q_{i2}),\\
&H_{21} = \beta_i + \varepsilon_i \cos(q_{i2}) + \eta_i \sin(q_{i2})\\
&H_{22} = \beta_i,\\
&C_{11} = -2\varepsilon_i \sin(q_{i2} + 2\eta_i \cos(q_{i2}))\dot{q_{i2}},\\
&C_{12} = -\varepsilon_i \sin(q_{i2} + \eta_i \cos(q_{i2}))\dot{q_{i2}},\\
&C_{21} = \varepsilon_i \sin(q_{i2} + \eta_i \cos(q_{i2}))\dot{q_{i1}},\\
&C_{22} = 0,\\
&g_{11} = \varepsilon_i \rho_{i2} \cos(q_{i1} + q_{i2}) + \eta_i \rho_{i2} \sin(q_{i1} + q_{i2}) + (\alpha_i - \beta_i + \rho_{i1})\rho_{i2} \cos(q_{i1}),\\
&g_{12} = \varepsilon_i \rho_{i2} \cos(q_{i1} + q_{i2}) + \eta_i \rho_{i2} \sin(q_{i1} + q_{i2}).
\end{aligned}
$$

The variables α_i, β_i, ε_i, and η_i are related to the physical link's parameters [21], and are defined as follows:

$$\begin{aligned}
\alpha_i &= I_{i1} + m_{i1} l_{i(cl)}^2 + I_{ie} + m_{ie} l_{i(ce)}^2 + m_{ie} l_{i1}^2, \\
\beta_i &= I_{ie} + m_{ie} l_{i(ce)}^2, \\
\varepsilon_i &= m_{ie} l_{i1} l_{i(ce)} \cos(\delta_{ie}), \\
\eta_i &= m_{ie} l_{i1} l_{i(ce)} \sin(\delta_{ie}), \\
\rho_{i1} &= l_{i1} l_{i(cl)} - I_{i1} - m_{i1} l_{i1}^2, \\
\rho_{i2} &= g/l_{i1},
\end{aligned}$$

where

$$\begin{aligned}
m_{i1} &- \textit{thigh mass}, \\
l_{i1} &- \textit{thigh length}, \\
l_{i2} &- \textit{shank length}, \\
l_{i(cl)} &- \textit{thigh center mass point}, \\
I_{i1} &- \textit{thigh inertia}, \\
m_{ie} &- \textit{shank mass}, \\
l_{ie} &- \textit{shank length}, \\
l_{i(ce)} &- \textit{shank center mass point}, \\
I_{ie} &- \textit{shank inertia}, \\
\delta_{ie} &- \textit{angle between the shank and shank center mass point}.
\end{aligned}$$

It is important to say that the shank center mass point changes with a passive foot attached.

5 Controller Design

The control objective for many robot systems is to reach either a position, speed, or acceleration, and keep it within a specified range. A manipulator robot described in (9) is highly nonlinear, susceptible to the external disturbance or changes, and it is time variable. That is, the model's parameter values can vary for different positions, altitudes, loads, and model uncertainties [22–24]. The classical control theory for systems with these characteristics has low performance, and the asymptotic stability on follow-up tasks is not guaranteed. These issues are avoided with robust control approaches [25]. The sliding mode control (SMC) is a robust control that ensures the control objective even with the system's uncertainties [26]. Another characteristic of this control approach is that it can reduce the representation of a non-linear system by one order, which makes it easier to control compared to the classic control like PID control, see [27]. Some improvements of this control approach, mainly in mechanical systems, are presented in [28], which we use in this work.

5.1 Sliding mode control based on input-output stability

For a system like the one shown in (9), and assuming that α_i, β_i, ε_i, and η_i are known values, the master's and slave's position errors for the desired position path $\boldsymbol{q}_d$ are

$$\boldsymbol{e}_m = \boldsymbol{q}_d - \boldsymbol{q}_m, \quad \boldsymbol{e}_s = \boldsymbol{q}_m - \boldsymbol{q}_s. \tag{11}$$

Define

$$\dot{\boldsymbol{q}}_{mr} = \dot{\boldsymbol{q}}_d + \Lambda(\boldsymbol{q}_d - \boldsymbol{q}_m), \;\; \dot{\boldsymbol{q}}_{sr} = \dot{\boldsymbol{q}}_m + \Lambda(\boldsymbol{q}_m - \boldsymbol{q}_s), \tag{12}$$

where Λ is a positive diagonal matrix.

Since the dynamics of the robot is linear with respect to its parameters [28], we have

$$H_i(\boldsymbol{q}_i)\ddot{\boldsymbol{q}}_{ir} + C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\dot{\boldsymbol{q}}_{ir} + \boldsymbol{g}_i(\boldsymbol{q}_i) = Y_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i, \dot{\boldsymbol{q}}_{ir}, \ddot{\boldsymbol{q}}_{ir})\boldsymbol{p}_i, \tag{13}$$

where

$$\boldsymbol{p}_i = [\alpha_i \quad \beta_i \quad \varepsilon_i \quad \eta_i]^T, \tag{14}$$

$$Y_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i, \dot{\boldsymbol{q}}_{ir}, \ddot{\boldsymbol{q}}_{ir}) = \begin{bmatrix} Y_{i1} & Y_{i2} & Y_{i3} & Y_{i4} \\ Y_{i5} & Y_{i6} & Y_{i7} & Y_{i8} \end{bmatrix} \tag{15}$$

with

$$\begin{aligned}
Y_{i1} &= \ddot{q}_{ir1} + \rho_{i2}\cos(q_{i1}),\\
Y_{i2} &= \ddot{q}_{ir2} - \rho_{i2}\cos(q_{i1}),\\
Y_{i3} &= 2\cos(q_{i2})\ddot{q}_{ir1} + \cos(q_{i2})\ddot{q}_{ir2} - 2\sin(q_{i2})\dot{q}_{i2}\dot{q}_{ir1} - \sin(q_{i2})\dot{q}_{i2}\dot{q}_{ir2} + \rho_{i2}\cos(q_{i1} + q_{i2}),\\
Y_{i4} &= 2\sin(q_{i2})\ddot{q}_{ir1} + \sin(q_{i2})\ddot{q}_{ir2} + 2\cos(q_{i2})\dot{q}_{i2}\dot{q}_{ir1} + \cos(q_{i2})\dot{q}_{i2}\dot{q}_{ir2} + \rho_{i2}\sin(q_{i1} + q_{i2}),\\
Y_{i5} &= 0,\\
Y_{i6} &= \ddot{q}_{ir1} + \ddot{q}_{ir2},\\
Y_{i7} &= \cos(q_{i2})\ddot{q}_{ir1} + \sin(q_{i2})\dot{q}_{i1}\ddot{q}_{ir1} + \rho_{i2}\cos(q_{i1} + q_{i2}),\\
Y_{i8} &= \sin(q_{i2})\ddot{q}_{ir1} - \cos(q_{i2})\dot{q}_{i1}\dot{q}_{ir1} + \rho_{i2}\sin(q_{i1} + q_{i2}).
\end{aligned}$$

The matrix $Y_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i, \dot{\boldsymbol{q}}_{ir}, \ddot{\boldsymbol{q}}_{ir})$ is known as the *dynamic regressor matrix.*

Using (11) and (12), the *sliding* variable for the master and slave ($i = m, s$) is determined by

$$\boldsymbol{\sigma}_i = \dot{\boldsymbol{q}}_{ir} - \dot{\boldsymbol{q}}_i = \dot{\boldsymbol{e}}_i + \Lambda e_i \tag{16}$$

and the Lyapunov function is defined as

$$V_i(t) = \frac{1}{2}\boldsymbol{\sigma}_i^T H_i(\boldsymbol{q}_i)\boldsymbol{\sigma}_i. \tag{17}$$

Therefore,

$$\begin{aligned}
\dot{V}_i(t) &= \boldsymbol{\sigma}_i^T H_i(\boldsymbol{q}_i)\dot{\boldsymbol{\sigma}}_i + \frac{1}{2}\boldsymbol{\sigma}_i^T \dot{H}_i(\boldsymbol{q}_i)\boldsymbol{\sigma}_i,\\
&= \boldsymbol{\sigma}_i^T H_i(\boldsymbol{q}_i)\dot{\boldsymbol{\sigma}}_i + \boldsymbol{\sigma}_i^T C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\boldsymbol{\sigma}_i,\\
&= \boldsymbol{\sigma}_i^T [H_i(\boldsymbol{q}_i)(\ddot{\boldsymbol{q}}_{ir} - \ddot{\boldsymbol{q}}_i) + C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)(\dot{\boldsymbol{q}}_{ir} - \dot{\boldsymbol{q}}_i)],\\
&= \boldsymbol{\sigma}_i^T [H_i(\boldsymbol{q}_i)\ddot{\boldsymbol{q}}_{ir} + C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\dot{\boldsymbol{q}}_{ir} + \boldsymbol{g}_i(\boldsymbol{q}_i) - \boldsymbol{\tau}_i]
\end{aligned} \tag{18}$$

with the sliding surface dynamics given by

$$H_i(\boldsymbol{q}_i)\dot{\boldsymbol{\sigma}}_i + C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\boldsymbol{\sigma}_i = Y_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i, \dot{\boldsymbol{q}}_{ir}, \ddot{\boldsymbol{q}}_{ir})\boldsymbol{p}_i - \boldsymbol{\tau}_i. \tag{19}$$

The sliding mode based on the bound of (18) can be written as

$$\begin{aligned}
\dot{V}_i(t) &= -\boldsymbol{\sigma}_i^T [\boldsymbol{\tau}_i - (H_i(\boldsymbol{q}_i)\ddot{\boldsymbol{q}}_{ir} + C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\dot{\boldsymbol{q}}_{ir} + \boldsymbol{g}_i(\boldsymbol{q}_i))],\\
&= -\boldsymbol{\sigma}_i^T [\boldsymbol{\tau}_i - Y_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i, \dot{\boldsymbol{q}}_{ir}, \ddot{\boldsymbol{q}}_{ir})\boldsymbol{p}_i]
\end{aligned} \tag{20}$$

and the control input as

$$\boldsymbol{\tau}_i = \overline{\boldsymbol{k}}_i \; sgn(\boldsymbol{\sigma}_i) = \begin{bmatrix} \overline{k}_{i1} \; sgn(\sigma_{i1}) \\ \overline{k}_{i2} \; sgn(\sigma_{i1}) \end{bmatrix}, \; i = m, s\,. \tag{21}$$

Consider the following region:

$$\mathbb{D}_{\sigma i} = \{\boldsymbol{\sigma}_i \mid \|\boldsymbol{\sigma}_i\| \leq \delta_{\sigma i}\}\,. \tag{22}$$

Next, it will be shown that if $\boldsymbol{\sigma}_i \in \mathbb{D}_{\sigma i}$, then the position and speed errors, as well as the system paths $\boldsymbol{q}_i$ and $\dot{\boldsymbol{q}}_i$, are also bounded. According to (16), we have

$$\boldsymbol{e}_i(t) = e^{-\Lambda_i t}\boldsymbol{e}_i(0) + \int_0^t e^{-\Lambda_i(t-\vartheta)}\boldsymbol{\sigma}_i(\vartheta)d\vartheta. \tag{23}$$

From the previous equation, a position error bound can be calculated as follows:

$$\|\boldsymbol{e}_i\| \leq \|\boldsymbol{e}_i(0)\|e^{-\underline{\lambda}_i t} + \frac{\delta_{\sigma i}}{\underline{\lambda}_i}\left(1 - e^{-\underline{\lambda}_i t}\right) \leq \; \|\boldsymbol{e}_i(0)\| + \frac{\delta_{\sigma i}}{\underline{\lambda}_i}, \tag{24}$$

where $\underline{\lambda}_i \triangleq \lambda_{\min}\{\Lambda_i\}$. From the previous result, the speed error bound is determined by

$$\|\dot{\boldsymbol{e}}_i\| \leq \bar{\lambda}\|\boldsymbol{e}_i(0)\| + \delta_{\sigma i}\left(\frac{\bar{\lambda}_i + \underline{\lambda}_i}{\underline{\lambda}_i}\right), \tag{25}$$

where $\bar{\lambda}_i \triangleq \lambda_{\max}\{\Lambda_i\}$. Since the desired trajectory $\boldsymbol{q}_{\mathrm{d}}(t)$ and its derivatives are bounded functions, the system trajectories $\boldsymbol{q}_i$ and $\dot{\boldsymbol{q}}_i$ are also bounded if $\|\boldsymbol{\sigma}_i\| \leq \delta_{\sigma i}$. From the previous analysis, it is possible to determine the following elements bound for the regressor:

$$Y_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i, \dot{\boldsymbol{q}}_{\mathrm{ir}}, \ddot{\boldsymbol{q}}_{\mathrm{ir}}) = \left[Y_{i(nj)}\right], \left|Y_{i(nj)}\right| \leq \bar{Y}_{i(nj)}. \tag{26}$$

To prove that the state $\boldsymbol{\sigma}_i$ is bounded, consider (17) which satisfies

$$\lambda_{\mathrm{h}i}\|\boldsymbol{\sigma}_i\|^2 \leq V_i \leq \lambda_{\mathrm{H}i}\|\boldsymbol{\sigma}_i\|^2, \tag{27}$$

where $\lambda_{\mathrm{h}i} = \lambda_{\min}\|H_i(\boldsymbol{q}_i)\}$ and $\lambda_{\mathrm{H}i} = \lambda_{\max}\|H_i(\boldsymbol{q}_i)\}$. The derivative of V_i along the system trajectories is given by

$$\begin{aligned} \dot{V}_i &= -\boldsymbol{\sigma}_i^{\mathrm{T}}\bar{\boldsymbol{k}}_i \mathrm{sign}(\boldsymbol{\sigma}_i) + \boldsymbol{\sigma}_i^{\mathrm{T}} Y_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i, \dot{\boldsymbol{q}}_{\mathrm{ir}}, \ddot{\boldsymbol{q}}_{\mathrm{ir}})\boldsymbol{p}_i \\ &= -\sum_{n=1}^{2}\bar{k}_i|\sigma_i| + \sum_{n=1}^{2}\sum_{j=1}^{4}\sigma_i\left[Y_{i(nj)}\right]p_{ij}. \end{aligned}$$

Consider (26) and suppose that the elements of the matrix $\bar{\boldsymbol{k}}_i$ are proposed as

$$\bar{k}_{i(n)} = \sum_{j=1}^{4}\bar{Y}_{i(nj)}\bar{p}_{ij} + \xi, \quad \xi > 0, \quad n = 1, 2\,, \tag{28}$$

where

$$\boldsymbol{p}_i = [p_{i1} \quad p_{i2} \quad p_{i3} \quad p_{i4}]^T, \; |p_{in}| \leq \overline{p_{in}}\;, \; n = 1, 2, 3, 4\,.$$

The derivative of V_i satisfies

$$\dot{V}_i \leq -\sum_{n=1}^{2} |\sigma_i| \left(\sum_{j=1}^{4} \bar{Y}_{i(nj)} \bar{p}_{ij} + \xi \right) + \sum_{n=1}^{2} \sum_{j=1}^{4} |\sigma_i| \bar{Y}_{i(nj)} p_{ij} \leq -\xi \sum_{n=1}^{2} |\sigma_i| < 0. \tag{29}$$

Since the derivative is negative, the closed-loop variables are bounded, and $\boldsymbol{\sigma}_i$ tends to zero. To show the convergence in finite time, the following inequality will be used $\sum_{n=1}^{2} |\sigma_i| \geq \|\boldsymbol{\sigma}_i\|$. With consideration to the previous inequality and (27), an upper bound for (29) is given by

$$\dot{V}_i \leq -\xi \|\boldsymbol{\sigma}_i\| \leq -\alpha \sqrt{V}_i \,, \tag{30}$$

where $\alpha = \xi / \sqrt{\lambda_{\mathrm{h}i}}$. Or

$$D^+ W_i \leq -\alpha \,, \tag{31}$$

where $W_i = 2\sqrt{V_i}$. From the comparison lemma, we have $W_i(\|\boldsymbol{\sigma}(t)\|) \leq W_i(\|\boldsymbol{\sigma}(0)\|) - \alpha t$. Therefore, $\|\boldsymbol{\sigma}_i\| = 0$ in finite time. So, from (23), for a time $t_{\mathrm{R}} \leq W_i(0)/\alpha$, we have

$$\boldsymbol{e}(t) = e^{-\Lambda t} \boldsymbol{e}(0).$$

From the above, the position and velocity errors (16) converge to zero exponentially. Therefore, based on Lyapunov's theory of stability, the master's trajectory tracking and master-slave synchronization are assured.

6 Simulation

The synchronization of the master and slave systems, (9), is validated by a cyclic trajectory (circumference). The master system tracks this trajectory and the slave system tracks the master's end-effector position; both systems start at different initial conditions. Figure 3 shows the master and slave systems, the circular trajectory, the link's idle position (gray lines), the trajectory initial position (dotted lines), and the link's final position (black lines). The master system is represented by the left side links and the slave system is represented by the right side links.

The tracking of the trajectory is done without vertical control of both systems and without considering contact forces with any surface. The Matlab function ODE45 (Dorman-Prince) was used for the simulations with a variable integration step and relative tolerance of 1e-3, with duration of 15 s.

The physical parameters, control values, and initial conditions are presented below.

6.1 Desired trajectory

A circular path was chosen because it is predominantly used in therapeutic exercise equipment. Apply (32) with its center at $x_c = 0.5$ m, $y_c = -1.0$ m, and radius $r = 0.4$ m:

$$\boldsymbol{x}_d = \begin{bmatrix} x_c \\ y_c \end{bmatrix} - r \begin{bmatrix} \cos(\frac{1}{5}\pi t) \\ \sin(\frac{1}{5}\pi t) \end{bmatrix}. \tag{32}$$

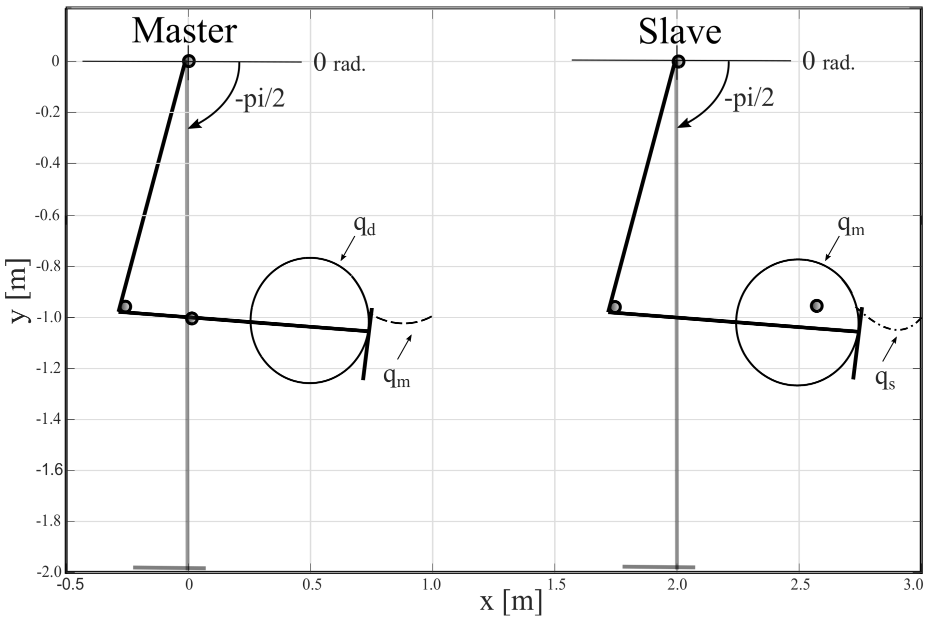


Figure 3: The tracking of the circular trajectory, master (left) and slave (right).

Variable	Value [unit]	Variable	Value [unit]
m_{m1}	1 [kg]	m_{s1}	1 1/5 [kg]
l_{m1}	1 [m]	l_{s1}	9/10 [m]
$l_{m(cl)}$	1/2 [m]	$l_{s(cl)}$	2/3 [m]
I_{m1}	1/12 [kg·m^2]	I_{s1}	0.081 [kg·m^2]
m_{me}	3 [kg]	m_{se}	3 1/4 [kg]
$l_{m(ce)}$	1 [m]	$l_{s(ce)}$	9/10 [m]
I_{me}	2/5 [kg·m^2]	I_{se}	13/30 [kg·m^2]
δ_{me}	0 [rads]	δ_{se}	0 [rads]
ρ_{m1}	$-7/12$	ρ_{s1}	-0.405
ρ_{m2}	9.81	ρ_{s2}	10.9

Table 1: Values of the parameters used.

6.2 Control parameter values

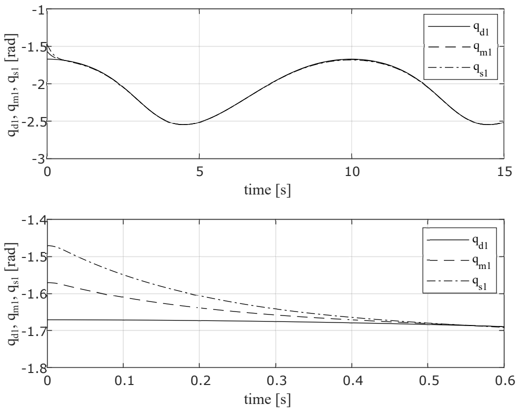
The values used in the simulation for the master and slave systems are shown in Table 1, from which the values for the vector $\boldsymbol{p_i}$ (14) are used in the computation of $\overline{\boldsymbol{k_i}}$ in (28), which in turn, is required for the control input $\boldsymbol{\tau_i}$ in (21).

For sliding mode control simulations, the values of $\Lambda = [5 \quad 0\,; \quad 0 \quad 5]^T$ are chosen, which are applied in (12).

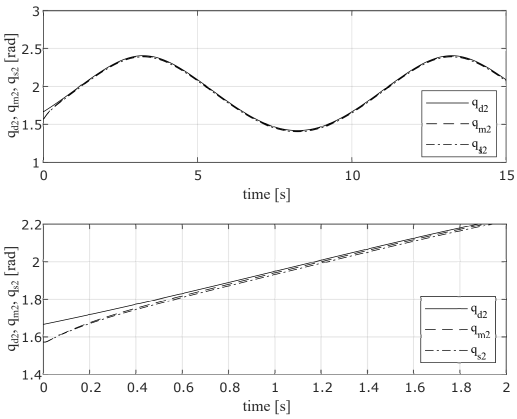
The master's initial conditions are $\boldsymbol{q_m} = [-\pi/2,\ \pi]$, and the slave's initial conditions are $\boldsymbol{q_s} = [-\pi/2+0.1,\ \pi]$, the values of the vector $\overline{\boldsymbol{p_i}}$ in (14) are $\overline{p}_m = |[6.7\ 3.4\ 3.0\ 0.0]^T| + 0.50$ and $\overline{p}_s = |[6.2\ 3.06\ 2.6\ 0.0]^T| + 0.50$, while $\xi = 0.1$. Finally, a saturation function is used instead of a switch function, with $\Delta = 0.05$.

6.3 Results

The master-slave synchronization established in (3) for two dynamic systems described by (9), with different initial positions and applying the control law (21) for a circular path (32), is described below. Applying the *inverse kinematic* by means of (7) and (8) we obtain q_{d1} and q_{d2}. The results in (21) are causing the angular position of the master's first link q_{m1} to move from -1,571 rads. to -1,671 rads., while the angular position of the slave's first link q_{s1} moves from -1,471 rads. to -1,671 rads. The angular position of the master's and slave's second link q_{m2}, q_{s2} moves from 1,571 rads. to 1,791 rads. The graphs in Figure 4 show these values and the time needed to reach the desired path.



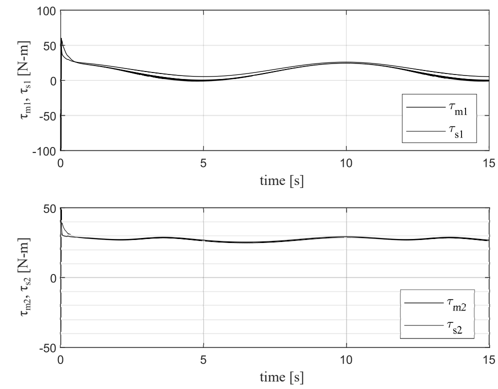
(a)



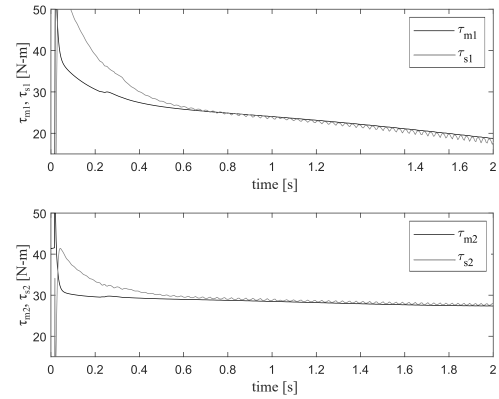
(b)

Figure 4: Link's positions during the desired trajectory tracking (32): a) Upper graphic $\boldsymbol{q_{d1}}$, $\boldsymbol{q_{m1}}$, and $\boldsymbol{q_{s1}}$ with $\boldsymbol{0 < t \leq 15}$, lower graphic with $\boldsymbol{0 < t \leq 0.6}$; b) Upper graphic $\boldsymbol{q_{d2}}$, $\boldsymbol{q_{m2}}$, and $\boldsymbol{q_{s2}}$ with $\boldsymbol{0 < t \leq 15}$, lower graphic with $\boldsymbol{0 < t \leq 2.0}$.

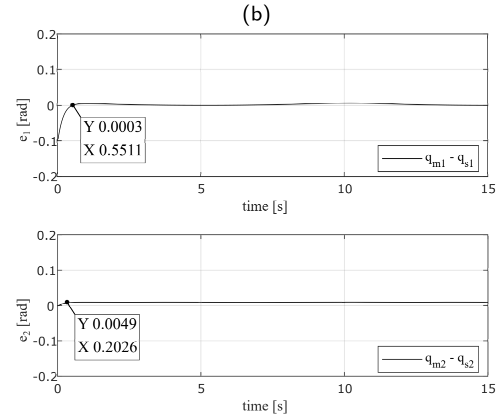
Finally, Figure 5a shows the control input $\boldsymbol{\tau}_i$ applied to q_{i1} and q_{i2} required to follow the desired path and synchronize the slave system with the master system, respectively.



(a)



(b)



(c)

Figure 5: Input signal (21) during the desired trajectory tracking (32): a) Upper graphic $\boldsymbol{\tau_{m1}}$ and $\boldsymbol{\tau_{s1}}$, lower graphic $\boldsymbol{\tau_{m2}}$ y $\boldsymbol{\tau_{s2}}$, both with $\mathbf{0} < \boldsymbol{t} \leq \mathbf{15}$; b) Upper graphic $\boldsymbol{\tau_{m1}}$ and $\boldsymbol{\tau_{s1}}$, lower graphic $\boldsymbol{\tau_{m2}}$ y $\boldsymbol{\tau_{s2}}$, both with $\mathbf{0} < \boldsymbol{t} \leq \mathbf{2}$; c) Upper graphic $\boldsymbol{e_{s1}}$, lower graphic $\boldsymbol{e_{s2}}$.

Figure 5b shows a zoom out of these graphs to appreciate the switching effects *chattering* inherent in the control method by sliding modes, while the graph in Figure 5c shows that the synchronization error is below 1% of the amplitude of the signal $\boldsymbol{x}_d$ in $0.5\,s$ of the slave's first link and approximately $0.25\,s$ for the slave's second link. These results show that the synchronization is reached within the time reported in [17–19], which makes this synchronization scheme a suitable option.

7 Conclusions

Two systems synchronization by the master-slave scheme with the sliding mode control was simulated. It was possible to synchronize them, even with different initial positions, in $0.596\,s$. The graphs presented show that the master and slave tracking errors tend to zero. Therefore, it is feasible to synchronize a mechanical system that emulates a mechanical prosthetic leg with another similar system such as a human leg, where the human leg is the master system and the mechanical prosthetic leg, like a 2-DOF robotic system, is the slave system. The rehabilitation stage and exercises required for a person with a leg amputation are beyond the scope of this work. However, we believe that this proposed synchronization scheme is the basis for those who consider to implement it as part of the rehabilitation, where a circular path is used as a reference trajectory because the therapeutic exercise equipment has it as the main movement.

Acknowledgment

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Existence Result for Positive Solution of a Degenerate Reaction-Diffusion System via a Method of Upper and Lower Solutions

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Abstract: The aim of this paper is to prove the existence of positive maximal and minimal solutions for a class of degenerate elliptic reaction-diffusion systems, including the uniqueness of the positive solution. To answer these questions, we use a technique described by Pao based on the method of upper and lower solutions, its associated monotone interactions and various comparison principles.

Keywords: *reaction-diffusion systems; degenerate elliptic systems; upper and lower solutions.*

Mathematics Subject Classification (2010): 35J62, 35J70, 35K57.

1 Introduction

Reaction-diffusion systems are widely used in biology, ecology, engineering, physics and chemistry. What we observe in modern scientific studies is the great interest of scientists in studying this type of systems; this confirms once again its importance in the development of applied and technological sciences. Various models and real examples can be found in various scientific fields, see Murray [13, 14]. The propagation of epidemics (Coronavirus, Hepatitis, ...), population dynamics, migration of biological species are among many examples of such phenomena. There are many methods and techniques for studying these issues. The reader can see some of them in the works of Alaa and Mesbahi [2,3,11,12], Abbassi et *al.* [1], Lions [10], Raheem [19] and the references therein.

In recent years, special attention has been paid to degenerate systems. However, most of the discussions relate to systems of two equations of the porous reaction medium type

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diffusion and with diffusion coefficients and specific reaction functions. This is because of their wide applications in various sciences. Among the important works on degenerate systems, we mention, for example, Alaa et *al.* [3], Al-Hdaibat et *al.* [4], Anderson [5], where we find techniques and methods of treatment.

The aim of this paper is to show the existence of positive maximal and minimal solutions for a quasilinear elliptic degenerate system, including the uniqueness of the positive solution. The two elliptic operators of the system under consideration can degenerate in the sense that $D_1(0) = 0$ or $D_2(0) = 0$. To answer these questions, we use a technique described by Pao, based on the upper and lower solutions. For more details on this technique, see Deuel and Hess [6], Pao et *al.* [15]- [18]. So, we need to construct suitable upper and lower solutions. We are therefore interested in studying the following system:

$$\begin{cases} -\mathbf{div}\left(D_1(u)\nabla u\right) = f(x,u,v) & \text{in } \Omega, \\ -\mathbf{div}\left(D_2(v)\nabla v\right) = g(x,u,v) & \text{in } \Omega, \\ u(x) = u_0(x) \ , \ v(x) = v_0(x) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with the boundary $\partial\Omega$. D_1, D_2, f and g are prescribed functions satisfying the conditions in hypotheses (H_1) and (H_3). We remark that these two functions f and g verify simple properties, this allows us to choose them from a wide range. Below we will denote $C^{\alpha}(\Omega)$ to the space of Hölder continuous functions in Ω.

The results obtained in this paper can be applied to a large number of reaction-diffusion models, which arise in various fields of the applied science such as theory of shells, Brownian motion theory and many problems of physics and biology. In addition to the classical problems in the fields of mass-heat transfer, chemical reactors, and nuclear reactor dynamics, there are many recently developed models from enzyme kinetics, population growth, nerve axion problems, and others.

The system (1) can model the circulation of an ideal gas in a homogeneous porous medium with an isentropic flow. It can also model the steady state of phenomena such as the heat propagation in a two-components combustible mixture, chemical processes, the interaction of two non-self-limiting biological groups, etc. We send the reader to see many models and applications in Deuel and Hess [6], Friedman [7], Ladyženskaja et *al.* [8], Lei and Zheng [9], especially Pao [16,17] and the references therein. For example, the steady state of the Gas-Liquid Interaction Problem, when considering a dissolved gas **A** and a dissolved reactant **B** that interact in a bounded diffusion medium, is a special case of (1) with the reaction terms $f(u,v) = -\sigma_1 uv$, $g(u,v) = -\sigma_2 uv$, where σ_1 is the rate constant and $\sigma_2 = k_1\sigma_1$. In a more general reaction scheme called the (m,n) order reaction, the resulting equations are given by (1) with $f(x,u,v) = -\sigma_1 u^m v^n + q_1(x)$, $g(x,u,v) = -\sigma_2 u^m v^n + q_2(x)$, $m,n \geq 1$ are constants and $q_1(x), q_2(x) \geq 0$ are possible internal sources.

In the problems of molecular interactions and subsonic flows, a simple model for the density function u is given by (1) with the reaction function $f(u) = \sigma u^p$, with $\sigma > 0$, $p \geq 1$.

This model also describes the temperature in radiating bodies or gases and in nuclear reactors with positive temperature feedback. For more information on this model, and also to see other models, we refer the reader to Pao [17].

The rest of this paper is organized as follows. In the next section, we state our main result. In the third section, we provide some preliminary results on the scalar problem which we need in the proof of the main theorem. Next, we give some results concerning the approximate problem. The fifth section is devoted to proving the main result. Finally, we give an application to the problem under study. The paper ends with a concluding remarks and some perspectives.

2 Statement of the Main Result

In all that follows, we denote $\mathbf{u} \equiv (u, v)$, $\tilde{\mathbf{u}}_s \equiv (\tilde{u}, \tilde{v})$, $\hat{\mathbf{u}}_s \equiv (\hat{u}, \hat{v})$. The inequality $\hat{\mathbf{u}}_s \leq \tilde{\mathbf{u}}_s$ means that $\hat{u} \leq \tilde{u}$ and $\hat{v} \leq \tilde{v}$.

2.1 Assumptions

First, we have to clarify in which sense we want to solve our problem.

Definition 2.1 A pair of functions $\tilde{\mathbf{u}}_s \equiv (\tilde{u}, \tilde{v})$, $\hat{\mathbf{u}}_s \equiv (\hat{u}, \hat{v})$ in $C^2(\Omega) \cap C\left(\bar{\Omega}\right)$ are called ordered upper and lower solutions of (1) if $\hat{\mathbf{u}}_s \leq \tilde{\mathbf{u}}_s$ and

$$\begin{cases} -\mathbf{div}\left(D\left(\hat{u}\right)\nabla\hat{u}\right) \leq f\left(x, \hat{u}, \hat{v}\right) & \text{in } \Omega, \\ -\mathbf{div}\left(D\left(\hat{v}\right)\nabla\hat{v}\right) \leq g\left(x, \hat{u}, \hat{v}\right) & \text{in } \Omega, \\ \hat{u}\left(x\right) \leq u_0\left(x\right) \ , \ \hat{v}\left(x\right) \leq v_0\left(x\right) & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and $\tilde{u}$, $\tilde{v}$ satisfies (2) with inequalities reversed.

For a given pair of ordered upper and lower solutions $\tilde{\mathbf{u}}_s$ and $\hat{\mathbf{u}}_s$, we define

$$\begin{aligned} S_1^* &= \left\{u \in C\left(\bar{\Omega}\right) \mid \hat{u} \leq u \leq \tilde{u}\right\} \quad , \quad S_2^* = \left\{v \in C\left(\bar{\Omega}\right) \mid \hat{v} \leq v \leq \tilde{v}\right\}, \\ S^* &= \left\{\mathbf{u} = (u, v) \in \left(C\left(\bar{\Omega}\right)\right)^2 \mid \hat{\mathbf{u}}_s \leq \mathbf{u} \leq \tilde{\mathbf{u}}_s\right\}. \end{aligned}$$

Now, we make the following assumptions:

(H_1) $f(x, \cdot)$, $g(x, \cdot) \in C^\alpha\left(\bar{\Omega}\right)$ and $u_0(x)$, $v_0(x) \in C^\alpha(\partial\Omega)$.

(H_2) $D_1(u) \in C^2([0, M_1])$, $D_1(u) > 0$ in $(0,\ M_1]$, and $D_1(0) \geq 0$ with $M_1 = \|\tilde{u}\|_{C(\bar{\Omega})}$.
$D_2(v) \in C^2([0, M_2])$, $D_2(v) > 0$ in $(0,\ M_2]$, and $D_2(0) \geq 0$ with $M_2 = \|\tilde{v}\|_{C(\bar{\Omega})}$.

(H_3) $f(\cdot, \mathbf{u})$, $g(\cdot, \mathbf{u}) \in C^1(S^*)$, and

$$\frac{\partial f}{\partial v}(\cdot, \mathbf{u}) \geq 0 \quad \text{and} \quad \frac{\partial g}{\partial u}(\cdot, \mathbf{u}) \geq 0 \quad \text{for all } \mathbf{u} \in S^*.$$

(H_4) There exists a constant $\delta_0 > 0$ such that for any $x_0 \in \partial\Omega$ there exists a ball $\mathbf{K}$ outside of Ω with radius $r \geq \delta_0$ such that $\mathbf{K} \cap \bar{\Omega} = \{x_0\}$.

In the above system, we further assume $D_1(0) = 0$ or $D_2(0) = 0$.
Let $\gamma_1(x)$ and $\gamma_2(x)$ be smooth positive functions satisfying

$$\gamma_1(x) \geq \max\left\{-\frac{\partial f}{\partial u}(x,\ \mathbf{u}) \ ; \ \mathbf{u} \in S^*\right\} \text{ and } \gamma_1(x) \geq C_1(x) + \delta_1, \tag{3}$$

$$\gamma_2(x) \geq \max\left\{-\frac{\partial g}{\partial v}(x,\ \mathbf{u}) \ ; \ \mathbf{u} \in S^*\right\} \text{ and } \gamma_2(x) \geq C_2(x) + \delta_2 \tag{4}$$

for some constants $\delta_1,\ \delta_2 > 0$, where $C_1(x)$ and $C_2(x)$ are analogous to $C(x)$ defined in Section 3 by the relations (11), i.e.,

$$\begin{aligned} C_1(x) &= -\mathbf{div}\nabla(\tilde{u})\, D_1'(\theta_1) - f_u(x,\theta_2), \\ C_2(x) &= -\mathbf{div}\nabla(\tilde{v})\, D_2'(\bar{\theta}_1) - g_v(x,\bar{\theta}_2). \end{aligned}$$

We define for all $\mathbf{u} \in S^*$

$$F(x,\mathbf{u}) = \gamma_1(x)\, u + f(x,\mathbf{u}) \quad \text{and} \quad G(x,\mathbf{u}) = \gamma_2(x)\, v + g(x,\mathbf{u}). \tag{5}$$

A typical example where the result of this paper can be applied is

$$\begin{cases} -\Delta u^{\lambda} = p(x)\, u^j v^k & \text{in } \Omega, \\ -\Delta v^{\mu} = q(x)\, u^{\ell} v^m & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\lambda,\ \mu > 1$, $j,\ k,\ \ell,\ m > 0$ and $p(x),\ q(x) > 0$ in Ω.

It is obvious that the problem (6) is a special case of (1) with

$$\begin{aligned} D_1(u) &= \lambda u^{\lambda-1} \ , \ D_2(v) = \mu u^{\mu-1} \ , \ u_0(x) = v_0(x) = 0, \\ f(x,u,v) &= p(x)\, u^j v^k \ , \ g(x,u,v) = q(x)\, u^{\ell} v^m. \end{aligned}$$

Lemma 2.1 *$F(x,\mathbf{u})$ and $G(x,\mathbf{u})$ are nondecreasing functions in $\mathbf{u}$ for all $\mathbf{u} \in S^*$.*

Proof. According to (H_3) and (5), we have for all $\mathbf{u} \in \mathbf{S}^*$

$$\frac{\partial F}{\partial v}(x,\mathbf{u}) = \frac{\partial f}{\partial v}(x,\mathbf{u}) \geq 0 \ \text{ and } \ \frac{\partial G}{\partial u}(x,\mathbf{u}) = \frac{\partial g}{\partial u}(x,\mathbf{u}) \geq 0.$$

By $(3)-(5)$, we obtain

$$\frac{\partial F}{\partial u}(x,\mathbf{u}) = \gamma_1(x) + \frac{\partial f}{\partial u}(x,\mathbf{u}) \geq 0 \ \text{ and } \ \frac{\partial G}{\partial v}(x,\mathbf{u}) = \gamma_2(x) + \frac{\partial g}{\partial v}(x,\mathbf{u}) \geq 0,$$

which implies the desired result.

2.2 The main result

Now, we can state the main result of this paper.

Theorem 2.1 *Let $\tilde{\mathbf{u}}_s$, $\hat{\mathbf{u}}_s$ be ordered positive upper and lower solutions of (1), and let hypotheses $(H_1)-(H_4)$ hold. Then problem (1) has a minimal solution $\underline{\mathbf{u}}_s$ and a maximal solution $\overline{\mathbf{u}}_s$ such that $\hat{\mathbf{u}}_s \leq \underline{\mathbf{u}}_s \leq \overline{\mathbf{u}}_s \leq \tilde{\mathbf{u}}_s$. If $\underline{\mathbf{u}}_s = \overline{\mathbf{u}}_s\, (\equiv \mathbf{u}_s^*)$, then $\mathbf{u}_s^*$ is the unique positive solution in S^*.*

3 Preliminary Results for the Scalar Problem

To illustrate our basic approach to the coupled system (1), we first consider the following scalar quasilinear elliptic boundary problem:

$$\begin{cases} -\mathbf{div}\,(D(w)\,\nabla w) = h(x,w) & \text{in } \Omega, \\ u(x) = h(x) & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where D and h are prescribed functions satisfying hypotheses $(H_1)-(H_4)$ above.

The following theorem ensures the existence of positive solutions to the scalar problem (7). For the proof, we refer to Friedman [7], Ladyženskaja et *al.* [8], Pao and Ruan [15].

Theorem 3.1 *Let $\tilde{w}_s(x)$, $\hat{w}_s(x)$ be a pair of upper and lower solutions of* (7) *such that $\tilde{w}_s(x) \geq \hat{w}_s(x) > 0$ in Ω, and let hypotheses (H_1) and (H_3) hold. Then problem* (7) *has a classical solution $w_s(x)$ such that $\hat{w}_s(x) \leq w_s(x) \leq \tilde{w}_s(x)$ in $\overline{\Omega}$. Furthermore, there are maximal and minimal solutions $\overline{w}_s(x)$ and $\underline{w}_s(x)$ such that every solution $w_s \in S_0^*$ satisfies $\underline{w}_s(x) \leq w_s(x) \leq \overline{w}_s(x)$.*

Remark 3.1 We consider the scalar problem (7) for w. In this case, we can write

$$-\mathbf{div}\left(D\left(\hat{w}\right)\nabla\hat{w}\right) \leq h\left(x,\hat{w}\right) \text{ in } \Omega, \tag{8}$$

$$-\mathbf{div}\left(D\left(\tilde{w}\right)\nabla\tilde{w}\right) \geq h\left(x,\tilde{w}\right) \text{ in } \Omega. \tag{9}$$

Subtracting (9) from (8), we find

$$\begin{aligned}&-\mathbf{div}\left[D_1\left(\hat{w}\right)\nabla\left(\hat{w}-\tilde{w}\right)+\nabla\tilde{w}\left(\frac{D\left(\hat{w}\right)-D\left(\tilde{w}\right)}{\hat{w}-\tilde{w}}\left(\hat{w}-\tilde{w}\right)\right)\right]\\ \leq\ &\frac{h\left(x,\hat{w}\right)-h\left(x,\tilde{w}\right)}{\hat{w}-\tilde{w}}\left(\hat{w}-\tilde{w}\right).\end{aligned}$$

According to the mean value theorem, there exist $\theta_1, \theta_2 \in [0, M]$, where $M = \|\tilde{w}\|_{C(\bar{\Omega})}$, such that

$$-\mathbf{div}\left[D\left(\hat{w}\right)\nabla z+\nabla\tilde{w}\left(D'\left(\theta_1\right)z\right)\right] \leq h_w\left(x,\theta_2\right)z$$

with $z = \hat{w} - \tilde{w}$, then

$$\begin{gathered}-\mathbf{div}\left(\nabla z\right)\left(D\left(\hat{w}\right)\right)-\nabla\left(D\left(\hat{w}\right)\right)\nabla z\\ -\mathbf{div}\left(\nabla\tilde{w}\left(D'\left(\theta_1\right)z\right)\right)-\nabla\left(\tilde{w}\right)D'\left(\theta_1\right)\nabla z-h_w\left(x,\theta_2\right)z \leq 0.\end{gathered}$$

We get

$$-D\left(\hat{w}\right)\Delta z+\left[-\nabla D\left(\hat{w}\right)-D'\left(\theta_1\right)\nabla\left(\tilde{w}\right)\right]\nabla z+\left[-\nabla.\nabla\left(\tilde{w}\right)D'\left(\theta_1\right)-h_w\left(x,\theta_2\right)\right]z \leq 0.$$

We denote

$$B\left(x\right) = -\nabla D\left(\hat{w}\right)-D'\left(\theta_1\right)\nabla\left(\tilde{w}\right), \tag{10}$$

$$C\left(x\right) = -\mathbf{div}\nabla\left(\tilde{w}\right)D'\left(\theta_1\right)-h_w\left(x,\theta_2\right). \tag{11}$$

To understand the calculations well, see Deuel and Hess [6], Friedman [7], Ladyženskaja et *al.* [8], Pao and Ruan [15].

Another important result is the following.

Lemma 3.1 *If $\underline{z}$, $\overline{z}$ are in $C^2(\Omega) \cap C\left(\bar{\Omega}\right)$ and satisfy the relation*

$$\begin{cases} -\Gamma\left[\underline{z}\right]+\gamma\underline{z} \leq -\Gamma\left[\overline{z}\right]+\gamma\overline{z} & \text{in } \Omega,\\ \underline{z}\left(x\right) \leq \overline{z}\left(x\right) & \text{on } \partial\Omega\end{cases}$$

with $\Gamma[u] = \mathbf{div}\left(D\left(w\right)\nabla w\right)$, then $\underline{z}(x) \leq \overline{z}(x)$ on $\overline{\Omega}$.

Proof. Let $z(x) = \underline{z}(x) - \overline{z}(x)$. Firstly, we have

$$-\Gamma[\underline{z}] + \gamma\underline{z} \leq -\Gamma[\overline{z}] + \gamma\overline{z} = \gamma\overline{z} + h(x, \overline{z}) \equiv F(x, \overline{z}),$$

then

$$-\Gamma[\underline{z}] + \gamma(\underline{z} - \overline{z}) - h(x, \overline{z}) \leq 0. \tag{12}$$

On the other hand, we have

$$\Gamma[\overline{z}] + \gamma(\underline{z} - \overline{z}) + h(x, \underline{z}) \leq 0. \tag{13}$$

Adding (12) and (13), we obtain

$$\begin{gathered} -\mathbf{div}\left[D(\underline{z})\nabla(\underline{z} - \overline{z}) + \nabla\overline{z}\left(\tfrac{D(\underline{z}) - D(\overline{z})}{\underline{z} - \overline{z}}(\underline{z} - \overline{z})\right)\right] \\ +2\gamma(\underline{z} - \overline{z}) + \tfrac{h(x,\underline{z}) - h(x,\overline{z})}{\underline{z} - \overline{z}}(\underline{z} - \overline{z}) \leq 0. \end{gathered}$$

According to the mean value theorem, $\exists\ \theta_1, \theta_2 \in [0, M]$ such that

$$-\mathbf{div}\left[D(\underline{z})\nabla z + \nabla\overline{z}(D'(\theta_1) z)\right] + 2\gamma(\underline{z} - \overline{z}) + \frac{h(x, \underline{z}) - h(x, \overline{z})}{\underline{z} - \overline{z}}(\underline{z} - \overline{z}) \leq 0$$

with $z = \underline{z} - \overline{z}$, then we get

$$\begin{gathered} -\mathbf{div}(\nabla z)(D(\underline{z})) - \nabla(D(\underline{z}))\nabla z \\ -\mathbf{div}(\nabla\overline{z}(D'(\theta_1) z)) - \nabla(\overline{z}) D'(\theta_1)\nabla z + h_w(x, \theta_2) z \leq 0. \end{gathered}$$

We obtain then

$$\begin{gathered} -D(\underline{z})\Delta z - [\nabla D(\underline{z}) - D'(\theta_1)\nabla(\overline{z})]\nabla z \\ -\mathbf{div}\nabla(\overline{z}) D'(\theta_1) z + 2\gamma z + h_w(x, \theta_2) z \leq 0, \\ -D(\underline{z})\Delta z + [-\nabla D(\underline{z}) - D'(\theta_1)\nabla(\overline{z})]\nabla z + \\ [\gamma + \mathbf{div}\nabla(\overline{z}) D'(\theta_1) + h_w(x, \theta_2)] z \leq 0. \end{gathered}$$

We come to

$$-D(\underline{z})\Delta z + (\mathbf{B}(x))\nabla z + (\gamma - \mathbf{C}(x)) z \leq 0,$$

where $\mathbf{B}(x)$ and $\mathbf{C}(x)$ are defined in the same way as $B(x)$ and $C(x)$ of relations (10) and (11), i.e.,

$$\begin{aligned} \mathbf{B}(x) &= -\nabla D(\underline{z}) - D'(\theta_1)\nabla(\overline{z}), \\ \mathbf{C}(x) &= -\mathbf{div}\nabla(\overline{z}) D'(\theta_1) + h_w(x, \theta_2). \end{aligned}$$

Assume, by contradiction, that $z(x)$ has a positive maximum at some point $x_0 \in \overline{\Omega}$. Then $x_0 \in \Omega$ and $\Delta z(x_0) \leq 0$, $\nabla z(x_0) = 0$. This implies that $(\gamma - C) z(x_0) \leq 0$, which is a contradiction because $\gamma - C = \delta > 0$.

4 Approximating Scheme

To prove the main theorem, we use the method of upper and lower solutions and its associated monotonic iteration. The basic idea of this method is that when using an upper solution or a lower solution as the initial iteration in a suitable iterative process, the resulting sequence of iterations is monotone and converges to a solution of the problem. Using then either $\hat{\mathbf{u}}_s$ or $\tilde{\mathbf{u}}_s$ as the initial iteration, we construct a sequence $\left\{\mathbf{u}_s^{(m)}\right\}$ from the iteration process

$$\begin{cases} -\Phi\left[u^{(m)}\right]+\gamma_1 u^{(m)}=F\left(x,\mathbf{u}_s^{(m-1)}\right) & \text{in } \Omega, \\ -\Psi\left[v^{(m)}\right]+\gamma_2 v^{(m)}=G\left(x,\mathbf{u}_s^{(m-1)}\right) & \text{in } \Omega, \\ u^{(m)}(x)=u_0(x)\ ,\ v^{(m)}(x)=v_0(x) & \text{on } \partial\Omega \end{cases} \tag{14}$$

with

$$\Phi[u]=\mathbf{div}\left(D_1(u)\nabla u\right) \quad , \quad \Psi[v]=\mathbf{div}\left(D_2(v)\nabla v\right).$$

We denote the sequence by $\left\{\underline{\mathbf{u}}_s^{(m)}\right\}$ if $\mathbf{u}_s^{(0)}=\hat{\mathbf{u}}_s$, and by $\left\{\overline{\mathbf{u}}_s^{(m)}\right\}$ if $\mathbf{u}_s^{(0)}=\tilde{\mathbf{u}}_s$. We call them minimal and maximal sequences, respectively. The existence of these sequences is ensured by the previous Lemma 3.1.

Lemma 4.1 *The minimal and maximal sequences* $\left\{\underline{\mathbf{u}}_s^{(m)}\right\}$, $\left\{\overline{\mathbf{u}}_s^{(m)}\right\}$ *exist and possess the monotone property*

$$\hat{\mathbf{u}}_s \le \underline{\mathbf{u}}_s^{(m)} \le \underline{\mathbf{u}}_s^{(m+1)} \le \overline{\mathbf{u}}_s^{(m+1)} \le \overline{\mathbf{u}}_s^{(m)} \le \tilde{\mathbf{u}}_s \quad \text{for all } m \ge 1. \tag{15}$$

Proof. Firstly, we consider the scalar problem

$$\begin{cases} -\Phi\left[u^{(m)}\right]+\gamma_1 u^{(m)}=F\left(x,\mathbf{u}_s^{(m-1)}\right) & \text{in } \Omega \\ u^{(m)}(x)=u_0(x) & \text{on } \partial\Omega. \end{cases} \tag{16}$$

We prove by induction. Start from $m=1$ and $\mathbf{u}_s^{(0)}=\hat{\mathbf{u}}_s$. By Definition 2.1, the components $\hat{u}$ of $\hat{\mathbf{u}}_s$ satisfy the relation

$$\begin{cases} -\Phi[\hat{u}]+\gamma_1\hat{u} \le F(x,\hat{\mathbf{u}}_s)=F\left(x,\underline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ \hat{u}(x)\le u_0(x) & \text{on } \partial\Omega \end{cases} \tag{17}$$

and the components $\tilde{u}$ of $\tilde{\mathbf{u}}_s$ satisfy the above inequalities (17) in reverse order, i.e.,

$$\begin{cases} -\Phi[\tilde{u}]+\gamma_1\tilde{u} \ge F(x,\tilde{\mathbf{u}}_s)\ge F\left(x,\underline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ \tilde{u}(x)\ge u_0(x) & \text{on } \partial\Omega. \end{cases}$$

Similarly, by considering the case $m=1$ and $\mathbf{u}_s^{(0)}=\tilde{\mathbf{u}}_s$, we have

$$\begin{cases} -\Phi[\hat{u}]+\gamma_1\hat{u} \le F(x,\hat{\mathbf{u}}_s)\le F(x,\tilde{\mathbf{u}}_s)=F\left(x,\overline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ \hat{u}(x)\le u_0(x) & \text{on } \partial\Omega \end{cases} \tag{18}$$

and the components $\tilde{u}$ of $\tilde{\mathbf{u}}_s$ satisfy the above inequalities (18) in reverse order, i.e.,

$$\begin{cases} -\Phi[\tilde{u}]+\gamma_1\tilde{u} \ge F(x,\tilde{\mathbf{u}}_s)=F\left(x,\overline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ \tilde{u}(x)\ge u_0(x) & \text{on } \partial\Omega. \end{cases}$$

We see that $\tilde{u}$ and $\hat{u}$ are ordered upper and lower solutions of (16) for the case $m = 1$. By Theorem 3.1, problem (16) has also a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ such that $\hat{u} \le \underline{u} \le \overline{u} \le \tilde{u}$. We choose $\underline{u}$ (or $\overline{u}$) as $\underline{u}^{(1)}$ if $\mathbf{u}_s^{(0)} = \hat{\mathbf{u}}_s$ and $\overline{u}$ (or $\underline{u}$) as $\overline{u}^{(1)}$ if $\mathbf{u}_s^{(0)} = \tilde{\mathbf{u}}_s$. So, we get $\hat{u} \le \underline{u}^{(1)} \le \overline{u}^{(1)} \le \tilde{u}$.

The same works if we consider the problem

$$\begin{cases} -\Psi\left[v^{(m)}\right] + \gamma_2 v^{(m)} = G\left(x, \mathbf{u}_s^{(m-1)}\right) & \text{in } \Omega, \\ v^{(m)}(x) = v_0(x) & \text{on } \partial\Omega, \end{cases}$$

which gives $\hat{v} \le \underline{v}^{(1)} \le \overline{v}^{(1)} \le \tilde{v}$.

This shows that $\underline{\mathbf{u}}_s^{(1)} \equiv \left(\underline{u}^{(1)}, \underline{v}^{(1)}\right)$ and $\overline{\mathbf{u}}_s^{(1)} \equiv \left(\overline{u}^{(1)}, \overline{v}^{(1)}\right)$ are solutions of (14) for $m = 1$ and satisfy $\hat{\mathbf{u}}_s \le \underline{\mathbf{u}}_s^{(1)} \le \overline{\mathbf{u}}_s^{(1)} \le \tilde{\mathbf{u}}_s$.

Assume, by induction, that $\underline{\mathbf{u}}_s^{(m-1)} \le \underline{\mathbf{u}}_s^{(m)} \le \overline{\mathbf{u}}_s^{(m)} \le \overline{\mathbf{u}}_s^{(m-1)}$ for some $m > 1$. Then, by the nondecreasing property of $F(., \mathbf{u})$, for $\mathbf{u} \in S^*$, we have

$$\begin{cases} -\Phi\left[\underline{u}^{(m)}\right] + \gamma_1 \underline{u}^{(m)} = F\left(x, \underline{\mathbf{u}}_s^{(m-1)}\right) \le F\left(x, \underline{\mathbf{u}}_s^{(m)}\right), \\ -\Phi\left[\overline{u}^{(m)}\right] + \gamma_1 \overline{u}^{(m)} = F\left(x, \overline{\mathbf{u}}_s^{(m-1)}\right) \ge F\left(x, \overline{\mathbf{u}}_s^{(m)}\right), \\ \underline{u}^{(m)} = \overline{u}^{(m)} = u_0(x). \end{cases}$$

This implies that $\overline{u}^{(m)}$, $\underline{u}^{(m)}$ are ordered upper and lower solutions of (16) when $(m-1)$ is replaced by m and $\mathbf{u}_s^{(m)}$ is either $\underline{\mathbf{u}}_s^{(m)}$ or $\overline{\mathbf{u}}_s^{(m)}$. Again, by Theorem 3.1, problem (16) has a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$. We choose $\underline{u}$ (or $\overline{u}$) as $\underline{u}^{(m+1)}$ if $\mathbf{u}_s^{(m)} = \underline{\mathbf{u}}_s^{(m)}$ and $\underline{u}$ (or $\overline{u}$) as $\overline{u}^{(m+1)}$ if $\mathbf{u}_s^{(m)} = \overline{\mathbf{u}}_s^{(m)}$, which gives us $\underline{u}^{(m)} \le \underline{u}^{(m+1)} \le \overline{u}^{(m+1)} \le \overline{u}^{(m)}$.

This choice ensures that $\underline{\mathbf{u}}_s^{(m+1)} \equiv \left(\underline{u}^{(m+1)}, \underline{v}^{(m+1)}\right)$ and $\overline{\mathbf{u}}_s^{(m+1)} \equiv \left(\overline{u}^{(m+1)}, \overline{v}^{(m+1)}\right)$ are solutions of (14) and possess the monotone property (15), which implies, by induction, the truth of the relation (15).

5 Proof of the Main Result

We are now ready to prove the main result of this work.

Proof. [Proof of Theorem 2.1] In view of Lemma 4.1 the pointwise limits

$$\lim_{m\longrightarrow\infty} \underline{\mathbf{u}}_s^{(m)} = \underline{\mathbf{u}}_s, \quad \lim_{m\longrightarrow\infty} \overline{\mathbf{u}}_s^{(m)} = \overline{\mathbf{u}}_s \tag{19}$$

exist and satisfy $\hat{\mathbf{u}}_s \le \underline{\mathbf{u}}_s \le \overline{\mathbf{u}}_s \le \tilde{\mathbf{u}}_s$. To prove that $\underline{\mathbf{u}}_s$ and $\overline{\mathbf{u}}_s$ are, respectively, the minimal and maximal solutions of (1), we first consider the minimal sequence $\left\{\underline{\mathbf{u}}_s^{(m)}\right\} \equiv \left\{\underline{u}^{(m)}, \underline{v}^{(m)}\right\}$. Define for each m

$$\begin{cases} \underline{w}_1^{(m)}(x) = I_1\left(\underline{u}^{(m)}\right) = \int_0^{\underline{u}^{(m)}} D_1(s)\,ds, \\ \underline{Q}_1^{(m)}(x) = -\gamma_1(x)\,\underline{u}^{(m)} + F\left(x, \underline{\mathbf{u}}^{(m-1)}\right), \end{cases}$$

and

$$\begin{cases} \underline{w}_2^{(m)}(x) = I_2\left(\underline{v}^{(m)}\right) = \int_0^{\underline{v}^{(m)}} D_2(s)\,ds, \\ \underline{Q}_2^{(m)}(x) = -\gamma_2(x)\,\underline{v}^{(m)} + F\left(x, \underline{\mathbf{u}}^{(m-1)}\right). \end{cases}$$

We remark that $I_1'(\underline{u}) = D_1(\underline{u})$ and $I_2'(\underline{v}) = D_2(\underline{v})$. The inverse of $I_1(\underline{u})$ and $I_2(\underline{v})$ exist and are denoted, respectively, by $q_1(\underline{w}_1)$ and $q_2(\underline{w}_2)$.

The quasilinear problem (14) may be written as the scalar linear problem

$$\begin{cases} -\nabla^2 \underline{w}_1^{(m)} = \underline{Q}_1^{(m)}(x) & \text{in } \Omega, \\ -\nabla^2 \underline{w}_2^{(m)} = \underline{Q}_2^{(m)}(x) & \text{in } \Omega, \\ \underline{w}_1^{(m)}(x) = u_0^*(x) \ , \ \underline{w}_2^{(m)}(x) = v_0^*(x) & \text{on } \partial\Omega, \end{cases}$$

where $u_0^*(x) = I_1(u_0) \geq 0$ and $v_0^*(x) = I_2(v_0) \geq 0$. It is clear from (19) and (5) that $\underline{w}_1^{(m)} \to \underline{w}_1 \equiv I_1(\underline{u})$, $\underline{w}_2^{(m)} \to \underline{w}_2 \equiv I_2(\underline{v})$ and $\underline{Q}_1^{(m)} \to f(x, \underline{\mathbf{u}}_s)$, $\underline{Q}_2^{(m)} \to g(x, \underline{\mathbf{u}}_s)$ as $m \to \infty$.

By the argument in the proof for the scalar problem (7), $\underline{w}_1$ is the unique solution of the linear problem

$$\begin{cases} -\nabla^2 \underline{w}_1^{(m)}(x) = \underline{Q}_1^{(m)}(x), \\ \underline{w}_1^{(m)}(x) = u_0^*(x) \end{cases}$$

and $\underline{w}_2$ is the unique solution of the linear problem

$$\begin{cases} -\nabla^2 \underline{w}_2^{(m)}(x) = \underline{Q}_2^{(m)}(x), \\ \underline{w}_2^{(m)}(x) = v_0^*(x). \end{cases}$$

This shows that $\underline{\mathbf{u}}_s \equiv (\underline{u}, \underline{v})$, where $\underline{u} = q_1(\underline{w}_1)$ and $\underline{v} = q_2(\underline{w}_2)$ are solutions of (1) and $\underline{\mathbf{u}}_s \in S^*$.

Now, we show that $\overline{\mathbf{u}}_s$ is a solution of (1) in S^*, for this we consider the maximal sequence $\left\{\overline{\mathbf{u}}_s^{(m)}\right\} \equiv \left\{\overline{u}^{(m)}, \overline{v}^{(m)}\right\}$. Define for each m

$$\begin{cases} \overline{w}_1^{(m)}(x) = I_1\left(\overline{u}^{(m)}\right) = \int_0^{\overline{u}^{(m)}} D_1(s)\, ds, \\ \overline{Q}_1^{(m)}(x) = -\gamma_1(x)\, \overline{u}^{(m)} + F\left(x, \overline{\mathbf{u}}^{(m-1)}\right) \end{cases}$$

and

$$\begin{cases} \overline{w}_2^{(m)}(x) = I_2\left(\overline{v}^{(m)}\right) = \int_0^{\overline{v}^{(m)}} D_2(s)\, ds, \\ \overline{Q}_2^{(m)}(x) = -\gamma_2(x)\, \overline{v}^{(m)} + G\left(x, \overline{\mathbf{u}}^{(m-1)}\right). \end{cases}$$

Then the quasilinear problem (14) may be written as the scalar linear problem

$$\begin{cases} -\nabla^2 \overline{w}_1^{(m)} = \overline{Q}_1^{(m)}(x) & \text{in } \Omega, \\ -\nabla^2 \overline{w}_2^{(m)} = \overline{Q}_2^{(m)}(x) & \text{in } \Omega, \\ \overline{w}_1^{(m)}(x) = u_0^*(x) \ , \ \overline{w}_2^{(m)}(x) = v_0^*(x) & \text{on } \partial\Omega. \end{cases}$$

It is clear from (19) and (5) that $\overline{w}_1^{(m)} \to \overline{w}_1 \equiv I_1(\underline{u})$, $\overline{w}_2^{(m)} \to \overline{w}_2 \equiv I_2(\overline{v})$ and $\overline{Q}_1^{(m)} \to f(x, \overline{\mathbf{u}}_s)$, $\overline{Q}_2^{(m)} \to g(x, \overline{\mathbf{u}}_s)$ as $m \to \infty$.

By the argument in the proof for the scalar problem, $\overline{w}_1$ is the unique solution of the linear problem

$$\begin{cases} -\nabla^2 \overline{w}_1^{(m)}(x) = \overline{Q}_1^{(m)}(x), \\ \overline{w}_1^{(m)}(x) = u_0^*(x) \end{cases}$$

and $\overline{w}_2$ is the unique solution of the linear problem

$$\begin{cases} -\nabla^2 \overline{w}_2^{(m)}(x) = \overline{Q}_2^{(m)}(x), \\ \overline{w}_2^{(m)}(x) = v_0^*(x). \end{cases}$$

This shows that $\overline{\mathbf{u}}_s \equiv (\overline{u}, \overline{v})$, where $\overline{u} = q_1(\overline{w}_1)$ and $\overline{v} = q_2(\overline{w}_2)$ are solutions of (1) and $\overline{\mathbf{u}}_s \in S^*$.

To show that $\underline{\mathbf{u}}_s$ and $\overline{\mathbf{u}}_s$ are, respectively, minimal and maximal solutions of (1) in S^*, we observe that every solution $\mathbf{u} = (u, v)$ of (1) in S^* satisfies

$$\begin{cases} -\Phi[u] + \gamma_1 u = F(x, \mathbf{u}_s) \geq F\left(x, \underline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ u(x) = u_0(x) & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Psi[v] + \gamma_1 v = G(x, \mathbf{u}_s) \geq G\left(x, \underline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ v(x) = v_0(x) & \text{on } \partial\Omega. \end{cases}$$

By (14) (with $m = 1$ and $u^{(1)} = \underline{u}^{(1)}$ and $v^{(1)} = \underline{v}^{(1)}$) we have

$$\begin{aligned} F\left(x, \underline{\mathbf{u}}_s^{(0)}\right) &= -\Phi\left[\underline{u}^{(1)}\right] + \gamma_1 \underline{u}^{(1)}, \\ G\left(x, \underline{\mathbf{u}}_s^{(0)}\right) &= -\Psi\left[\underline{v}^{(1)}\right] + \gamma_1 \underline{v}^{(1)}, \end{aligned}$$

then

$$\begin{aligned} -\Phi[u] + \gamma_1 u &\geq -\Phi\left[\underline{u}^{(1)}\right] + \gamma_1 \underline{u}^{(1)}, \\ -\Psi[v] + \gamma_1 v &\geq -\Psi\left[\underline{v}^{(1)}\right] + \gamma_1 \underline{v}^{(1)}. \end{aligned}$$

By Lemma 3.1 we have $u \geq \underline{u}^{(1)}$ and $v \geq \underline{v}^{(1)}$, i.e. $\mathbf{u} \geq \underline{\mathbf{u}}_s^{(1)}$. This implies, by Lemma 2.1, that $F(x, \mathbf{u}) \geq F\left(x, \mathbf{u}_s^{(1)}\right)$ and $G(x, \mathbf{u}) \geq G\left(x, \mathbf{u}_s^{(1)}\right)$. It follows by an induction argument that

$$\begin{aligned} F(x, \mathbf{u}) &\geq F\left(x, \underline{\mathbf{u}}_s^{(1)}\right) \geq F\left(x, \underline{\mathbf{u}}_s^{(2)}\right) \geq ... \geq F\left(x, \underline{\mathbf{u}}_s^{(m)}\right), \\ G(x, \mathbf{u}) &\geq G\left(x, \underline{\mathbf{u}}_s^{(1)}\right) \geq G\left(x, \underline{\mathbf{u}}_s^{(2)}\right) \geq ... \geq G\left(x, \underline{\mathbf{u}}_s^{(m)}\right), \end{aligned}$$

then $\mathbf{u} \geq \underline{\mathbf{u}}_s^{(m)}$, for every $m \geq 1$.

In the same way, we observe that every solution $\mathbf{u} = (u, v)$ of (1) in S^* satisfies

$$\begin{cases} -\Phi[u] + \gamma_1 u = F(x, \mathbf{u}) \leq F\left(x, \overline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ u(x) = u_0(x) & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Psi[v] + \gamma_1 v = G(x, \mathbf{u}) \leq G\left(x, \overline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ v(x) = v_0(x) & \text{on } \partial\Omega. \end{cases}$$

By (14) (with $m = 1$ and $u^{(1)} = \overline{u}^{(1)}$ and $v^{(1)} = \overline{v}^{(1)}$) we have

$$\begin{aligned} F\left(x, \overline{\mathbf{u}}_s^{(0)}\right) &= -\Phi\left[\overline{u}^{(1)}\right] + \gamma_1 \overline{u}^{(1)}, \\ G\left(x, \overline{\mathbf{u}}_s^{(0)}\right) &= -\Psi\left[\overline{v}^{(1)}\right] + \gamma_1 \overline{v}^{(1)}, \end{aligned}$$

then

$$\begin{aligned} -\Phi[u] + \gamma_1 u &\leq -\Phi\left[\overline{u}^{(1)}\right] + \gamma_1 \overline{u}^{(1)}, \\ -\Psi[v] + \gamma_1 v &\leq -\Psi\left[\overline{v}^{(1)}\right] + \gamma_1 \overline{v}^{(1)}. \end{aligned}$$

By Lemma 3.1, we have $u \leq \overline{u}^{(1)}$ and $v \leq \overline{v}^{(1)}$, i.e., $u_s \leq \overline{\mathbf{u}}_s^{(1)}$. This implies, by Lemma 2.1, that $F(x, \mathbf{u}) \leq F\left(x, \overline{\mathbf{u}}_s^{(1)}\right)$ and $G(x, \mathbf{u}) \leq G\left(x, \overline{\mathbf{u}}_s^{(1)}\right)$. It follows by an induction argument that

$$\begin{aligned} F(x, \mathbf{u}) &\leq F\left(x, \overline{\mathbf{u}}_s^{(1)}\right) \leq F\left(x, \overline{\mathbf{u}}_s^{(2)}\right) \leq \ldots \leq F\left(x, \overline{\mathbf{u}}_s^{(m)}\right), \\ G(x, \mathbf{u}) &\leq G\left(x, \overline{\mathbf{u}}_s^{(1)}\right) \leq G\left(x, \overline{\mathbf{u}}_s^{(2)}\right) \leq \ldots \leq G\left(x, \overline{\mathbf{u}}_s^{(m)}\right), \end{aligned}$$

which implies $u_s \leq \overline{\mathbf{u}}_s^{(m)}$.

Letting $m \to \infty$ and using relation (19) lead to $\underline{\mathbf{u}}_s \leq \mathbf{u} \leq \overline{\mathbf{u}}_s$. This proves the minimal and maximal property of $\underline{\mathbf{u}}_s$and $\overline{\mathbf{u}}_s$. Finally, if $\underline{\mathbf{u}}_s = \overline{\mathbf{u}}_s \ (\equiv \mathbf{u}_s^*)$, then this maximal-minimal property ensures that $\mathbf{u}_s^*$ is the unique positive solution in S^*.

6 Application

As an application of the obtained theorem, we give a model concerning the type of diffusion in porous media, where the diffusion coefficients are degenerate; it is the following two-species Lotka–Volterra competition steady-state model:

$$\begin{cases} -D_1(x)\nabla^2 u^{\alpha} = u(a_1 - b_1 u - c_1 v), \\ -D_2(x)\nabla^2 v^{\beta} = v(a_2 - b_2 u - c_2 v), & t > 0, \ x \in \Omega, \\ u(x) = u_0(x) > 0 \ , \ v(x) = v_0(x) > 0, \end{cases} \tag{20}$$

where for each $i = 1, \ 2, \ \alpha, \ \beta, \ a_i, \ b_i, \ c_i$ are positive constants, and $\alpha > 1, \ \beta > 1$, with $D_i(x) > 0$ on $\bar{\Omega}$. For more details on this model, we refer the reader to Pao in [16, 17].

7 Concluding Remarks and Perspectives

This work has mainly focused on the question of the existence and the uniqueness of positive maximal and minimal solutions for a class of degenerate reaction-diffusion systems. It should be noted that the results obtained can be applied to a number of models arising from biology, ecology and biochemistry as well as to models in several fields of applied sciences and engineering. We have developed original methods to overcome certain difficulties, and despite the complexity of the model studied, we have succeeded in obtaining an existence result.

There are many additional important open problems, which we hope to address in the near future, they are: Numerical simulation, Generalization to the parabolic case, Generalization to the case of a higher order system. This list of questions corresponds to a work in progress or prospective work. Some are a continuation of the work already done, and some are new research projects. This not only makes it possible to delve deeper into the theoretical study, but also goes beyond the theoretical framework by developing models and techniques.

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